Matérn B-Splines and the Optimal Reconstruction of Signals

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***Abstract*—Starting from the power spectral density of Matérn stochastic processes, we introduce a new family of splines that is defined in terms of the whitening operator of such processes. We show that these Matérn splines admit a stable representation in a B-spline-like basis. We specify the Matérn B-splines (causal and symmetric) and identify their key properties; in particular, we prove that these generate a Riesz basis and that they can be written as a product of an exponential with a fractional polynomial B-spline. We also indicate how these new functions bridge the gap between the fractional polynomial splines and the cardinal exponential ones. We then show that these splines provide the optimal reconstruction space for the minimum mean-squared error estimation of Matérn signals from their noisy samples. We also propose a digital Wiener-filter-like algorithm for the efficient determination of the optimal B-spline coefficients.**

***Index Terms*—Fractional splines, Matérn processes, Matérn splines, minimum mean-squared error (MMSE) estimation, power spectral density (PSD), Riesz bases.**

I. Introduction

The Matérn class is a parametric family of power spectral densities (PSDs) that was introduced by Bertil Matérn [1] to model a large variety of naturally occurring stationary processes in one dimension. This model is commonly used in geostatistics for the prediction/estimation of spatial data and is also well known in the Kriging literature [1], [2]. The general form of the Matérn PSD is

$$\hat{\phi}(\omega) = \frac{\sigma_0^2}{|\alpha^2+\omega^2|^{\gamma+1}} = \frac{\sigma_0}{(\alpha+j\omega)^{\gamma+1}} \cdot \frac{\sigma_0}{(\alpha-j\omega)^{\gamma+1}}. \tag{1}$$

It is parametrized by three real-valued quantities: the amplitude $\sigma_0 > 0$, the exponential factor $\alpha > 0$, and the degree $\gamma > -(1/2)$ (which is one less than the order). A generic whitening operator of this process is denoted by $\mathrm{L}_\alpha^{\gamma+1}$. Specific instances are the causal and symmetric versions of the operator, which are best defined via their Fourier transforms

$$\text{causal}: \mathrm{L}_{\alpha,+}^{\gamma+1} \overset{\mathcal{F}}{\longleftrightarrow} \hat{\mathrm{L}}_{\alpha,+}^{\gamma+1}(\omega) = (\alpha+j\omega)^{\gamma+1} \tag{2}$$

$$\text{symmetric}: \mathrm{L}_{\alpha,*}^{\gamma+1} \overset{\mathcal{F}}{\longleftrightarrow} \hat{\mathrm{L}}_{\alpha,*}^{\gamma+1}(\omega) = |\alpha^2+\omega^2|^{\frac{\gamma+1}{2}}. \tag{3}$$

We define a cardinal Matérn spline with parameters (α,γ) as a function $s(x)$ such that

$$\mathbf{L}_\alpha^{\gamma+1}\{s(x)\} = \sum_{k\in\mathbb{Z}} a[k]\delta(x-k) \tag{4}$$

where $\delta(x)$ is the Dirac distribution. Such a spline has singularities of order $\gamma+1$ at the integers.

Our goal in this letter is to construct a Riesz basis for the stable representation of these Matérn splines. The basis is generated from the integer-shifts of a so-called Matérn B-spline, which is a pulse-shaped spline that is well localized in the sense of its essential support being the shortest possible. We develop our formulation for the causal operator, keeping in mind that a parallel development can be made for the symmetric operator as well.

This letter is organized as follows. In Section II, we construct the Matérn B-splines (both causal and symmetric) and present their main properties. We then show, in Section III, how to use these splines, and the corresponding B-spline representation, for the optimal interpolation/estimation of Matérn stochastic processes from their noisy samples at the integers.

II. Matérn B-Spline

A. Green Function of the Matérn Whitening Operator

Using distributional calculus [3], it can be shown that the (causal) Green function of $L_{\alpha,+}^{\gamma+1}$ for noninteger $\gamma > -1$ is

$$\rho_{\alpha,+}^{\gamma}(x) = \frac{e^{-\alpha x}x_+^{\gamma}}{\Gamma(\gamma+1)} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(\alpha+j\omega)^{\gamma+1}} \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function, and x_+^γ is the one-sided power function

$$x_+^\gamma = \begin{cases} x^\gamma, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Using this result, we can solve (i.e., integrate) (4) to get an explicit Matérn spline representation

$$s(x) = \sum_{k\in\mathbb{Z}} a[k]\rho_{\alpha,+}^{\gamma}(x-k) + p_0(x) \tag{7}$$

where $p_0(x)$ is an exponential polynomial in the null-space of $\mathrm{L}_\alpha^{\gamma+1}$, which still needs to be specified by imposing some boundary conditions. In the present cardinal setting, the presence of $p_0(x)$ is not a problem because we can show that it is also in the span of $\{\rho_{\alpha,+}^{\gamma}(x-k)\}_{k\in\mathbb{Z}}$, which singles out these functions as the fundamental building blocks of Matérn splines.

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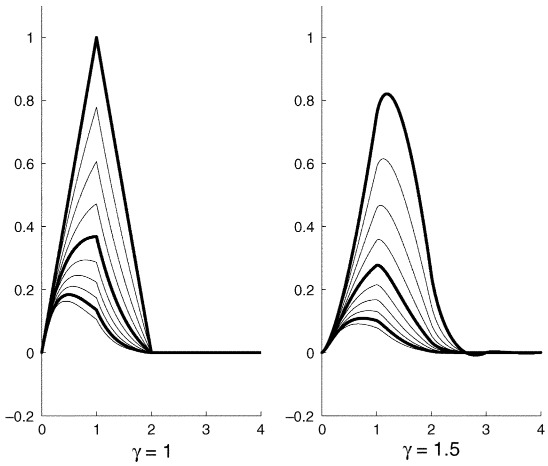

Fig. 1. Causal Matérn B-splines for $\alpha = 0$ to 2.5 in steps of 0.25.

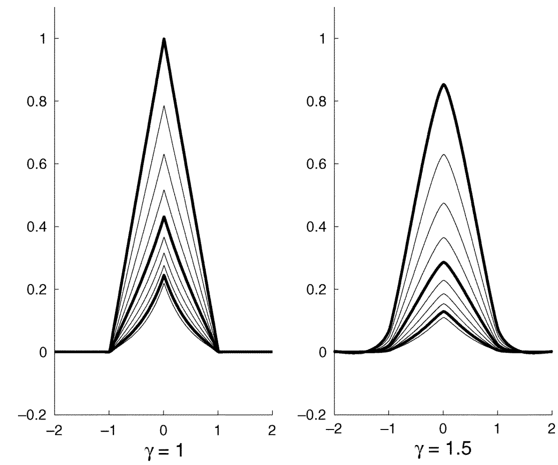

Fig. 2. Symmetric Matérn B-splines for $\alpha = 0$ to 2.5 in steps of 0.25.

Their downside, however, is that they are not well localized and potentially nonstable, unlike their B-spline counterparts.

B. Causal Matérn B-spline

We propose to construct the Matérn B-splines of real degree $\gamma > -1$ by taking appropriate (fractional) weighted differences of the Green function (5). Here, we consider a complex α but restrict ourselves to the case $\mathrm{Re}\{\alpha\} \geq 0$.

The forward weighted-differences operator (localization operator) $\Delta_{\alpha,+}^{\gamma}$ is defined as

$$\Delta_{\alpha,+}^{\gamma}\{f(x)\} = \sum_{k\geq 0}(-1)^k \binom{\gamma}{k} e^{-\alpha k} f(x-k). \tag{8}$$

Its frequency response is

$$\begin{aligned}\Delta_{\alpha,+}^{\gamma}(e^{j\omega}) &= \left(1 - e^{-(\alpha+j\omega)}\right)^{\gamma} \\ &= \sum_{k\geq 0}(-1)^k \binom{\gamma}{k} e^{-\alpha k} e^{-j\omega} k \end{aligned} \tag{9}$$

$$\text{where} \quad \binom{\gamma}{k} = \frac{\Gamma(\gamma+1)}{\Gamma(k+1)\Gamma(\gamma-k+1)}. \tag{10}$$

Definition 1: The causal Matérn B-spline $\beta_{\alpha,+}^{\gamma}(x)$ of degree γ is defined as

$$\begin{aligned}\beta_{\alpha,+}^{\gamma} &:= \Delta_{\alpha,+}^{\gamma+1} p\left\{\rho_{\alpha,+}^{\gamma}(x)\right\} = \Delta_{\alpha,+}^{\gamma+1}\left\{\frac{e^{-\alpha x} x_+^{\gamma}}{\Gamma(\gamma+1)}\right\} \qquad (11) \\ &= \sum_{k\geq 0} \frac{(-1)^k e^{-\alpha k}}{\Gamma(\gamma+1)} \binom{\gamma+1}{k} e^{-\alpha(x-k)} (x-k)_+^{\gamma} \\ &= \frac{e^{-\alpha x}}{\Gamma(\gamma+1)} \sum_{k\geq 0}(-1)^k \binom{\gamma+1}{k} (x-k)_+^{\gamma} \\ &= e^{-\alpha x} \beta_+^{\gamma}(x) \qquad (12)\end{aligned}$$

where $\beta_+^{\gamma}(x)$ is the fractional polynomial B-spline of degree γ defined in [4].

This implies that the Fourier transform of the B-spline is

$$\hat{\beta}_{\alpha,+}^{\gamma}(\omega) = \frac{\Delta_{\alpha,+}^{\gamma+1}(\omega)}{\hat{\mathrm{L}}_{\alpha,+}^{\gamma+1}(\omega)} = \left(\frac{1 - e^{-(\alpha+j\omega)}}{\alpha + j\omega}\right)^{\gamma+1}. \tag{13}$$

C. Symmetric Matérn B-Spline

The symmetric Matérn B-spline can be constructed using a similar fractional weighted difference of the Green function of $\mathrm{L}_{\alpha,*}^{\gamma+1}$. Here, we propose a simpler approach, which is to define the symmetric Matérn B-spline of degree γ as the autocorrelation of the causal Matérn B-spline of degree $(\gamma-1)/2$

$$\beta_{\alpha,*}^{\gamma}(x) := \beta_{\alpha,+}^{\frac{\gamma-1}{2}}(x) * \beta_{\alpha,+}^{\frac{\gamma-1}{2}}(-x). \tag{14}$$

By considering the Fourier equivalent of this definition and making use of (13), we readily calculate the Fourier transform of the symmetric Matérn B-spline

$$\hat{\beta}_{\alpha,*}^{\gamma}(\omega) = \left|\frac{1 + e^{-2\alpha} - 2e^{-\alpha}\cos(\omega)}{|\alpha|^2 + \omega^2}\right|^{\frac{\gamma+1}{2}}. \tag{15}$$

Note that if we had considered a B-spline of twice the order, we would have gotten a denominator that is the same as that of the Matérn PSD (1). We will further justify this link in Section III.

Corresponding examples of causal and symmetric Matérn B-splines with varying α are plotted in Figs. 1 and 2, respectively.

D. Properties

We use β_{α}^{γ} to refer to both the causal and the symmetric Matérn B-spline, unless indicated explicitly. These functions satisfy a number of interesting properties, which we only prove for the causal case.

Convolution: The following convolution rule holds:

$$\left(\beta_{\alpha}^{\gamma_1} * \beta_{\alpha}^{\gamma_2}\right)(x) = \beta_{\alpha}^{\gamma_1+\gamma_2+1}(x). \tag{16}$$

This can be easily established in the Fourier domain by considering (13) or (15).

Fractional differentiation using weighted differences: We can apply a lower order Matérn operator to the B-spline and get an explicit formula for the result: $\mathrm{L}_{\alpha,+}^{\gamma_1}\{\beta_{\alpha,+}^{\gamma_2}(x)\} = \Delta_{\alpha,+}^{\gamma_1}\{\beta_{\alpha,+}^{\gamma_2-\gamma_1}(x)\}$, with $\gamma_1 \leq \gamma_2$. This too is best proven in the frequency domain.

Support: From (11), we see that for $\mathrm{Re}\{\alpha\} \geq 0$, $|\beta_{\alpha,+}^{\gamma}(x)| = |e^{-ax}||\beta_+^{\gamma}(x)| \leq |\beta_+^{\gamma}(x)|$. Since $\beta_+^{\gamma}(x)$ is not compactly supported for noninteger γ, the same holds true for $\beta_{\alpha,+}^{\gamma}(x)$ (see Fig. 4). However, $\beta_{\alpha}^{\gamma}(x)$ has exponential decay for $\mathrm{Re}\{\alpha\} > 0$,

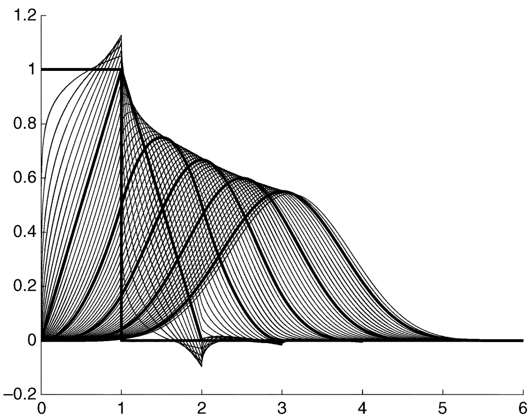

Fig. 3. Causal Matérn B-splines: $\gamma = 0$ to 5.2 in steps of 0.1, $\alpha = 0$ (the fractional polynomial case).

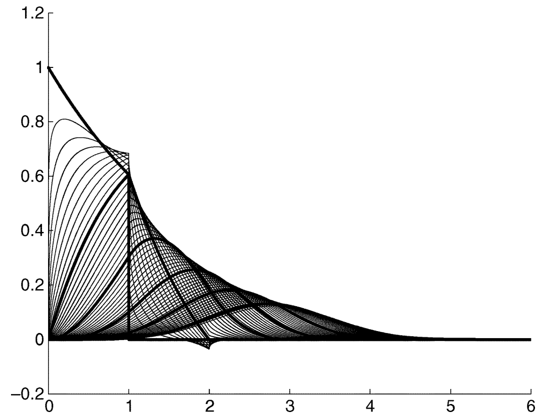

Fig. 4. Causal Matérn B-splines: $\gamma = 0$ to 5.2 in steps of 0.1, $\alpha = 0.5$.

unlike its polynomial counterpart $\beta_+^\gamma(x)$, which only decays as $O(1/|x|^{\gamma+2})$.

Riesz bases: The Matérn B-splines generate Riesz bases for the space of Matérn splines with knots at the integers.

Theorem 1: For $\mathrm{Re}\{\alpha\} \geq 0$, and $\forall \gamma > -(1/2)$, the functions $\{\beta_\alpha^\gamma(x-k)\}_{k\in\mathbb{Z}}$ form a Riesz basis of the Matérn spline space

$$V_\alpha^\gamma = \left\{ f(x) = \sum_{k\in\mathbb{Z}} c[k]\beta_\alpha^\gamma(x-k) : c \in \ell_2 \right\}$$

with the Riesz bounds : $$A^2_{\beta_\alpha^\gamma} \geq \left(\frac{|1-e^{-\alpha}|^2}{|\alpha|^2+4\pi^2}\right)^{\gamma+1}$$

$$B^2_{\beta_\alpha^\gamma} \leq \left|\frac{1+e^{-\alpha}}{\alpha}\right|^{2\gamma+2} \cdot \left(1 + \frac{2\zeta(2\gamma+2)}{\left(\frac{2\pi}{|\alpha|}\right)^{2\gamma+2}}\right)$$

where $\zeta(\cdot)$ is the Riemann-zeta function.

Proof: The necessary and sufficient condition for $\beta_\alpha^\gamma(x)$ to generate a Riesz basis is that the 2π-periodic function

$$\begin{aligned} G(\omega) &= \sum_{k\in\mathbb{Z}} \left|\hat{\beta}_\alpha^\gamma(\omega+2k\pi)\right|^2 \\ &= \sum_{k\in\mathbb{Z}} \left|\frac{1-e^{-(\alpha+j\omega)}}{\alpha+j(\omega+2\pi k)}\right|^{2\gamma+2} \end{aligned} \quad (17)$$

be strictly bounded from above and below [5]. We estimate the lower bound $A^2_{\beta^\gamma_{\alpha,+}}$ by using the dominant term $k=0$ in (17)

$$G(\omega) > \left|\frac{1-e^{-(\alpha+j\omega)}}{\alpha+j\omega}\right|^{2\gamma+2} \geq \left(\frac{|1-e^{-\alpha}|^2}{|\alpha|^2+4\pi^2}\right)^{\gamma+1} > 0.$$

To get the upper bound $B^2_{\beta^\gamma_{\alpha,+}}$, we isolate the dominant term and perform the following manipulation on (17):

$$\begin{aligned} G(\omega) &\leq \frac{|1+e^{-\alpha}|^{2\gamma+2}}{|\alpha|^{2\gamma+2}} + \sum_{k\neq 0} \frac{|1+e^{-\alpha}|^{2\gamma+2}}{(\omega+2k\pi)^{2\gamma+2}} \\ &\leq \frac{|1+e^{-\alpha}|^{2\gamma+2}}{|\alpha|^{2\gamma+2}} + \frac{|1+e^{-\alpha}|^{2\gamma+2}}{(2\pi)^{2\gamma+2}} \sum_{k\geq 1} \frac{2}{k^{2\gamma+2}} \\ &= \left|\frac{1+e^{-\alpha}}{\alpha}\right|^{2\gamma+2} \cdot \left(1 + \frac{2\zeta(2\gamma+2)}{\left(\frac{2\pi}{|\alpha|}\right)^{2\gamma+2}}\right) < +\infty \end{aligned}$$

which yields the desired result.■

Note that for $\alpha = 0$, the Riesz bounds are the same as those of the fractional polynomial B-splines [4].

Functional properties: For $\mathrm{Re}\{\alpha\} \geq 0$, we have $|\beta_\alpha^\gamma(x)| \leq |\beta_+^\gamma(x)|$. This is illustrated in Figs. 1 and 2, where the Matérn B-splines are plotted for varying values of α, starting from $\alpha = 0$, which corresponds to the polynomial B-splines. Since $\beta_+^\gamma(x) \in L_1(\mathbb{R})$ for $\gamma > -1$, and $\beta_+^\gamma(x) \in L_2(\mathbb{R})$ for $\gamma > -(1/2)$, the same applies to the Matérn B-splines $\beta_\alpha^\gamma(x)$. Also, the Matérn B-splines are infinite and discontinuous at the integers for $\gamma \leq 0$ and continuous $\forall\, \gamma > 0$, just like their fractional polynomial counterparts.

***L_p*-stability**: For $\gamma > 0$, the Matérn B-splines are L_p-stable in the sense that $\sup_{x\in[0,1]} \sum_{k\in\mathbb{Z}} |\beta_\alpha^\gamma(x+k)| < \infty$ [5]. Indeed, the sum $\sum_{k\in\mathbb{Z}} |\beta_\alpha^\gamma(x+k)|$ is uniformly convergent because of the B-spline decay. It is therefore continuous and bounded on any finite interval because of the continuity of the individual terms. A limit case is $\beta^0_{\alpha,+}(x)$, which is bounded and compactly supported and therefore obviously L_p-stable as well. The L_p-stability condition is a strong one that implies that the L_p-norm of these B-splines is finite for $1 \leq p \leq \infty$.

Generalization: The Matérn B-splines fill the gap between the fractional polynomial B-splines [4] and the exponential ones [6]–[8]. Specifically, we recover the former by taking $\alpha = 0$. Conversely, when γ is an integer, we obtain an exponential B-spline with a root α of multiplicity γ. This generalization is illustrated in Figs. 3 and 4, which displays the Matérn B-splines for a range of γ for $\alpha = 0$ (the fractional polynomial case) and for $\alpha = 0.5$. The exponential ones (integer values of γ) are compactly supported and are represented using thicker lines.

Exponential Polynomial Reproduction: The Matérn B-spline $\beta^\gamma_{\alpha,+}(x)$ reproduces exponential monomials $e^{-ax}x^m$ up to degree $n = \lceil\gamma\rceil$. The result follows from the polynomial reproduction property of its fractional polynomial counterpart. Specifically, there exist some coefficients $c_m[k]$ such that

$$x^m = \sum_{k\in\mathbb{Z}} c_m[k]\beta_+^\alpha(x-k), \quad m = 0,\ldots,n.$$

Multiplying both sides by e^{-ax} and using (12), we ultimately get

$$e^{-\alpha x}x^m = \sum_{k\in\mathbb{Z}} e^{-\alpha k} c_m[k]\beta^\gamma_{\alpha,+}(x-k), \quad m = 0,\ldots,n \quad (18)$$

which formally proves the result.

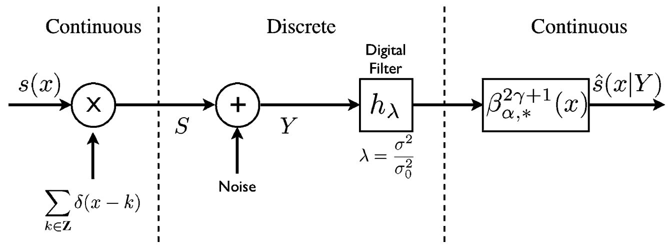

Fig. 5. Reconstruction setup.

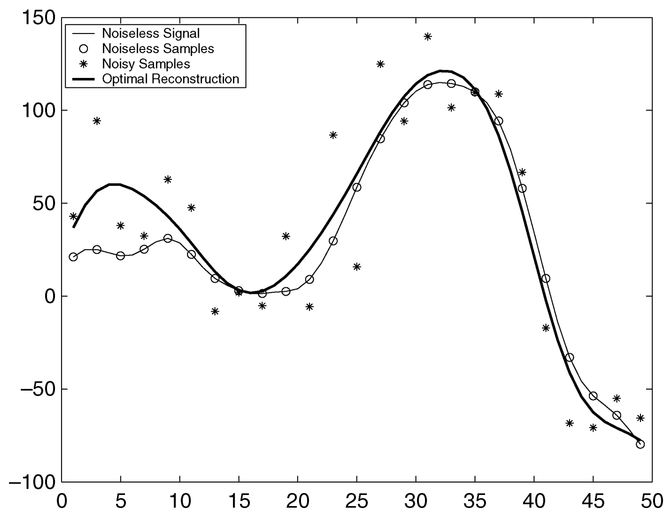

Fig. 6. Optimal reconstruction of a Matérn signal from noise corrupted samples: $\sigma_0 = 1, \alpha = 0.25, \nu = 3.66$, and $\sigma = 24.35$.

III. Optimal Reconstruction of Matérn Signals

A strong statistical justification for using the proposed spline framework is that the minimum mean-square error (MMSE) estimate of a Matérn process $s(x)$, given its noisy samples $Y = S + N$, lies in a corresponding spline space. Moreover, the reconstruction can be performed efficiently according to the filtering procedure outlined in Fig. 5. Note that this block diagram is compatible with the popular framework of sampling in shift-invariant spaces [9], [10]. In fact, our formulation leads to a globally optimal solution to the sampling problem: it yields both the optimal space and the optimal reconstruction filter.

To establish this result, we consider the measurement model $Y = \{y[k] = s(k)+n[k]\}_{k\in\mathbb{Z}}$ that involves the integer samples S of a realization $s(x)$ of a Matérn process corrupted by discrete additive white noise $N = \{n[k]\}_{k\in\mathbb{Z}}$. The hypotheses are that the signal has a PSD given by (1) with $\alpha > 0$, that the noise is zero-mean with variance σ^2, and that both are Gaussian and independent from one another.

Proposition 1: The MMSE estimate of $s(x)$ at $x = x_0$ is given as

$$\tilde{s}(x_0|Y) = \sum_{k\in\mathbb{Z}} \underbrace{(y * h_\lambda)[k]}_{c[k]} \beta_{\alpha,*}^{2\gamma+1}(x_0 - k) \tag{19}$$

where $\beta_{\alpha,*}^{2\gamma+1}$ is the symmetric Matérn B-spline of degree $2\gamma+1$, and $h_\lambda[k]$ is the digital filter whose frequency response is

$$H_\lambda(e^{j\omega}) = \frac{1}{B_\alpha^{2\gamma+1}(e^{j\omega}) + \lambda \left|1 - e^{-(\alpha+j\omega)}\right|^{2\gamma+2}} \tag{20}$$

with $B_\alpha^{2\gamma+1}(e^{j\omega}) = \sum_{k\in\mathbb{Z}} \hat{\beta}_{\alpha,*}^{2\gamma+1}(\omega + 2k\pi)$ and $\lambda = \sigma^2/\sigma_0^2$.■

The result follows from the application of Theorem 5 [11], Theorem 1 (for "spline-admissibity"), and the observation that $(\mathrm{L}_\alpha^{\gamma+1})^* \circ \mathrm{L}_\alpha^{\gamma+1} = \mathrm{L}_{\alpha,*}^{2\gamma+2}$, where $\mathrm{L}_\alpha^{\gamma+1}$ is the whitening operator of the process with innovation variance σ_0^2. Fig. 6 gives an illustration of optimal reconstruction of a Matérn signal from noisy samples.

The direct implication is that the MMSE estimator $\tilde{s}(x|Y)$, as x varies over $\mathbb{R}$, is a cardinal Matérn spline of degree $2\gamma + 1$. Proposition 1 yields the optimal reconstruction space as well as a practical algorithm for computing the B-spline coefficients $c[k]$ of the reconstructed signal by simple filtering of the noisy input signal with the digital restoration filter $h_\lambda[k]$.

IV. Summary and Conclusions

In this letter, we have defined Matérn splines and have shown that these could be represented in terms of B-spline basis functions. In doing so, we have come up with a fractional generalization of the exponential B-splines with a multiple root α. We have derived the main properties of the new Matérn B-splines. We also have shown that the spline spaces spanned by those B-splines are the optimal reconstruction spaces for stochastic signals in the Matérn class and that the reconstruction could be achieved by suitable filtering.

References

[1] M. L. Stein, *Interpolation of Spatial Data*. New York: Springer, 1999.
[2] P. J. Diggle, P. J. Ribeiro, and O. F. Christensen, "An introduction to model-based geostatistics," *Spatial Statist. Comput. Methods*, pp. 43–86, 2002.
[3] L. Schwartz, *Théorie des distributions*. Paris, France: Hermann, 1966.
[4] M. Unser and T. Blu, "Fractional splines and wavelets," *SIAM Rev.*, vol. 42, no. 1, pp. 43–67, 2000.
[5] A. Aldroubi and K. Gröchenig, "Nonuniform sampling and reconstruction in shift invariant spaces," *SIAM Rev.*, vol. 43, pp. 585–620, 2001.
[6] A. Ron, "Exponential box splines," *Construct. Approx.*, vol. 4, pp. 357–378, 1988.
[7] M. Unser and T. Blu, "Cardinal exponential splines: part I—theory and filtering algorithms," *IEEE Trans. Signal Process.*, vol. 53, no. 4, pp. 1425–1438, Apr. 2005.
[8] T. Asahi, K. Ichige, and R. Ishii, "A computationally efficient algorithm for exponential b-splines based on difference/IIR filter approach," *IEICE Trans. Fundam. Electron. Commun. Comput. Sci.*, vol. E85A, no. 6, pp. 1265–1273, 2002.
[9] W. Chen and S. Itoh, "A sampling theorem for shift-invariant subspace," *IEEE Trans. Signal Process.*, vol. 46, no. 10, pp. 2822–2824, Oct. 1998.
[10] Y. C. Eldar and U. M. , "Non-ideal sampling and interpolation from noisy observations in shift-invariant spaces," *IEEE Trans. Signal Process.*, to be published.
[11] M. Unser and T. Blu, "Generalized smoothing splines and the optimal discretization of the wiener filter," *IEEE Trans. Signal Process.*, vol. 53, no. 6, pp. 2146–2159, Jun. 2005.