COMPLETE PARAMETRIZATION OF PIECEWISE POLYNOMIAL INTERPOLATORS ACCORDING TO DEGREE, SUPPORT, REGULARITY, AND ORDER

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ABSTRACT

The most essential ingredient of interpolation is its basis function. We have shown in previous papers that this basis need not be necessarily interpolating to achieve good results. On the contrary, several recent studies have confirmed that *non-interpolating* bases, such as B-splines and O-moms, perform best. This opens up a much wider choice of basis functions. In this paper, we give to the designer the tools that will allow him to characterize this enlarged space of functions. In particular, he will be able to specify up-front the four most important parameters for image processing: degree, support, regularity, and order. The theorems presented here will then allow him to refine his design by dealing with additional coefficients that can be selected *freely*, without interfering with the main design parameters.

1. INTRODUCTION

Interpolation is a standard operation in image processing. It is usually described by the following equation:

$$f_h(x) = \sum_{k\in\mathbb{Z}} \mathrm{f}_k\, \varphi_{\mathrm{int}}(x/h - k)$$

where f_h is a continuous function reconstructed from discrete samples $\mathrm{f}_k = \mathrm{f}(h\,k)$, where h is the sampling step, and where φ_{int} is the interpolation function. If quality is a key issue—better than commonplace linear interpolation—then the selection of an appropriate φ_{int} becomes very important. For practical reasons, this function is often selected to be a piecewise polynomial of moderate degree and support, with uniform knots.

Over the years, a large body of work has been devoted to the design of interpolators that tend to be sinc-like while offering more practical benefits, in particular a finite support. Beside the requirement that φ_{int} be interpolating (i.e., $\varphi_{\mathrm{int}}(k) = \delta_k$), the aspects that have been emphasized are: 1) its maximal degree N; 2) the width W of its support; 3) its regularity $\varphi_{\mathrm{int}} \in C^R$; and to some extent 4) its order of approximation L. This search for adequate interpolators is still active today; recent contributions include those of Schaum [1], Appledorn [2], German [3], Dodgson [4], or Meijering [5]. Unfortunately, it appears that the improvements of each new proposal have been less and less substantial. Recently, we showed that one reason for this saturation of design is that the interpolation constraint is too strong; only by relaxing it altogether were we able to achieve significant gains in performance [6]. The corresponding generalized interpolation model is

$$f_h(x) = \sum_{k\in\mathbb{Z}} c_k\, \varphi(x/h - k)$$

where the function φ is not necessarily interpolating anymore, and where the coefficients c_k are determined from the samples f_k such that f_h fits the sample values exactly: $\mathrm{f}_k = \mathrm{f}(h\,k) = f_h(h\,k)$.

The traditional design of functions φ_{int} imposes the interpolation constraint from the start on, and thereafter builds on it. Here instead, we propose to let the designer proceed by first imposing the four other characteristics: degree N, support W, regularity R, and order L. The main contribution of this paper is to be able to express in a finite-dimension vector space the *complete* class of piecewise polynomials that satisfy these four characteristics. The designer may then freely select among them, or may perhaps throw in additional constraints for good measure, like the symmetry and interpolation constraint if he so chooses.

Degree—The maximal degree of a piecewise polynomial is, in some sense, an index of the complexity of what can be achieved with the polynomial. In particular, a raise in the degree N results in more parameters—in this case, coefficients—to play with.

To formulate our results, we shall extend the range of possible N to negative values in the following way: the Dirac distribution δ is considered a piecewise polynomial of degree -1 while its n-th derivative $\delta^{(n)}$ is a piecewise polynomial of degree $(-n-1)$. Derivatives have to be understood in the sense of distributions.

Support—Without loss of generality, we consider that the support of φ or φ_{int} is $[0, W]$. The corresponding interpolation constraint becomes $\varphi_{\mathrm{int}}(W/2 + k) = \delta_k$. Outside this interval, we have that $\varphi = \varphi_{\mathrm{int}} = 0$. The value of W is the most critical parameter to determine the computational cost of interpolation. In p dimensions, this cost grows like W^p. Distributions may have a support concentrated on the origin, with $W = 0$.

Regularity—In the traditional design of interpolators, regularity has often been maximized so as to give the designer a criterion to help him reject solutions, by want of better design criteria. In the context of image processing, less-than-maximal regularity is often sufficient, because only the image and its gradient need be continuously defined. Reclaiming degrees of freedom by reducing the requirement on the regularity of φ from $R_{\max}$ to R can be put to good use towards a better design. To formulate our results, we shall extend to negative values the range to which R belongs: a function u is said to be of regularity at least C^{-1} provided it is bounded; a Dirac distribution δ is said to be of regularity C^{-2} while its n-th derivative $\delta^{(n)}$ is of regularity C^{-2-n}.

Order—One aspect often overlooked in the traditional design of a function φ is its order of approximation L, which is an essential index of its intrinsic quality [7]. It is defined by the rate of decrease of the error between the original function f and the reconstructed f_h when the sampling step h vanishes

$$\|\mathrm{f} - f_h\| \propto h^L \quad \text{as} \quad h \to 0$$

From approximation theory, we know that the order L can be determined from φ or φ_{int} only, no matter what the sampled function f may be [8]. The order of approximation is particularly relevant to image processing because the frequency content of most images is essentially low-pass, which is equivalent to say that the sampling step h is small relatively to the image content. Thus, the continuous image f_h reconstructed from the samples f_k will be closer to the original f when the approximation order L associated to φ is high than when it is low. The importance of the order has been confirmed by all our experiments [6].

2. DEFINITIONS

Let φ be a uniform piecewise polynomial distribution of unit integral characterized by a maximal degree N, a maximal support $W \geq 0$, a minimal regularity R, and a minimal approximation order $L \geq 0$. The distribution φ is a true function when $R \geq -1$ and $N \geq 0$. With some notational abuse, we write

$$\varphi \in \{N, W, R, L\}$$

Let $(x)_+^n$ be the causal n-th power function defined by

$$(x)_+^n = \begin{cases} \frac{1}{2}\,(1 + \operatorname{sign}(x)) & n = 0 \\ (\max(0,x))^n & n \neq 0 \end{cases}$$

Let $\beta^n(x)$ be the causal B-spline of degree n

$$\beta^n(x) = \frac{1}{n!} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x-k)_+^n$$

Let $\gamma_k^n(x)$ be a family of functions defined for $k \in [1, n]$ by

$$\begin{aligned} \gamma_k^n(x) \quad &= \quad \frac{1}{\lambda_{k,n-k+1}} \left(\frac{(x)_+^{n-k}}{(n-k)!} \right. \\ &\left. -\frac{1}{n!} \sum_{l=0}^{n-k} \lambda_{k,l} \sum_{m=0}^{k+l} (-1)^m \binom{k+l}{m} (x-m)_+^n \right) \end{aligned}$$

where the coefficients $\lambda_{k,l}$ are defined by the Mac-Laurin development of the function $(\frac{1}{z}\,\log\frac{1}{1-z})^k = \sum_{l \in \mathbb{N}} \lambda_{k,l}\, z^l$.

3. DECOMPOSITION THEOREMS

We first state a theorem that links the four most important design parameters of the function φ which is expressed as the convolution of two components, the first one carrying the full approximation order and the second one carrying no order at all.

Theorem 1 *Let $\varphi \in \{N, W, R, L\}$ be a uniform piecewise polynomial function with $L \geq 1$. Then, there exists a distribution $u \in \{N-L, W-L, R-L, 0\}$ such that*

$$\varphi(x) = \left(\beta^{L-1} * u\right)(x)$$

Since the support of u has to be non-negative, a direct consequence of Theorem 1 is that the condition $L \leq W$ must be satisfied for φ to exist. In addition, since since $\int u(x)\,\mathrm{d}x = 1$, then u must also satisfy $N - L \geq -1$.

The most important contribution of this paper is to develop an explicit form for the distribution u which is made of two additive terms: the first one consists in a true function, while the second one carries the distributional aspect of u.

Theorem 2 *Let $u \in \{n, w, r, 0\}$ be a uniform piecewise polynomial distribution. Then, there exists a unique function $\psi \in \{n, w, \rho, 0\}$ and a unique set of coefficients $d_{k,l}$ such that*

$$u(x) = \psi(x) + \sum_{k=0}^{-r-2} \sum_{l=0}^{w} d_{k,l}\, \delta^{(k)}(x-l)$$

where $\rho = \max(-1, r)$

Note that the distribution $u - \psi$ may exist whenever $r \leq -2$; otherwise, u is itself a true function and we have that $u = \psi$.

Theorem 3 *Let $\psi \in \{n, w, \rho, 0\}$ be a bounded uniform piecewise polynomial function with $\rho \geq -1$. Then, there exists a unique set of coefficients $\{b_{k,l}, g_{k,l}\}$ such that*

$$\begin{aligned} \psi(x) \quad &= \quad \sum_{k=0}^{n-\rho-1} \sum_{l=0}^{w-n+k-1} b_{k,l}\, \beta^{n-k}(x-l) \\ &+ \sum_{k=1}^{n-\rho-1} \sum_{l=0}^{n-k} g_{k,l}\, (\gamma_k^{n-l} * \beta^{l-1})(x) \end{aligned}$$

Conversely, when $n \leq w$, any arbitrary choice of unit-sum coefficients $\{b_{k,l}, g_{k,l}\}$ results in $\psi \in \{n, w, \rho, 0\}$. However, when $n \geq w+1$, we have in general that $\psi \in \{n, n, \rho, 0\}$, and the coefficients $\{b_{k,l}, g_{k,l}\}$ must satisfy additional constraints to ensure that $\psi \in \{n, w, \rho, 0\}$.

After examination of the limits of summation in Theorem 3, we conclude that the condition $\rho \leq n-1$ must hold true to avoid $\psi = 0$. Since the function φ of Theorem 1 is at least of order 0, we deduce from Theorem 3 that the condition $R \leq N-1$ must be satisfied for φ to exist. In addition, we deduce from Theorem 3 that every function ψ of the largest possible regularity $\rho = n-1$ is a linear sum of shifted B-splines of degree n.

We now combine into a single result the conclusions of Theorems 1, 2, and 3. The proof of these theorems will be presented in a forthcoming paper.

Theorem 4 *The function $\varphi \in \{N, W, R, L\}$ with $L \leq W$, $-1 \leq N-L$, and $R \leq N-1$, is uniquely defined by a set of coefficients $\{d_{k,l}, g_{k,l}, b_{k,l}\}$ such that*

$$\begin{aligned} \varphi(x) \quad &= \quad \sum_{k=0}^{N-R-1} \sum_{l=0}^{W-N+k-1} b_{k,l}\, \beta^{N-k}(x-l) \\ &+ \sum_{k=1}^{N-R-1} \sum_{l=0}^{N-L-k} g_{k,l}\, (\gamma_k^{N-L-l} * \beta^{L+l-1})(x) \\ 1 \quad &= \quad \sum_{k=0}^{N-R-1} \sum_{l=0}^{W-N+k-1} b_{k,l} + \sum_{k=1}^{N-R-1} \sum_{l=0}^{N-L-k} g_{k,l} \end{aligned}$$

for $R \geq L-1$, and

$$\begin{aligned} \varphi(x) \quad &= \quad \sum_{k=0}^{N-L} \sum_{l=0}^{W-N+k-1} b_{k,l}\, \beta^{N-k}(x-l) \\ &+ \sum_{k=1}^{N-L} \sum_{l=0}^{N-L-k} g_{k,l}\, (\gamma_k^{N-L-l} * \beta^{L+l-1})(x) \end{aligned}$$

Table 1. Partition of the degrees of freedom

	$R \geq L-1$
$\text{card}\{b_{k,l}\}$	$\frac{1}{2}\,(N-R)\,(2W-(N+R)-1)$
$\text{card}\{g_{k,l}\}$	$\frac{1}{2}\,(N-R-1)\,(N+R-2L+2)$
$\text{card}\{d_{k,l}\}$	0
	$R \leq L-2$
$\text{card}\{b_{k,l}\}$	$\frac{1}{2}\,(N-L+1)\,(2W-(L+N))$
$\text{card}\{g_{k,l}\}$	$\frac{1}{2}\,(N-L+1)\,(N-L)$
$\text{card}\{d_{k,l}\}$	$(L-R-1)\,(W-L+1)$

$$
\begin{aligned}
& + \sum_{k=0}^{L-R-2} \sum_{l=0}^{W-L} d_{k,l}\, \frac{\mathrm{d}^k \beta^{L-1}(x-l)}{\mathrm{d}x^k} \\
1 \;&=\; \sum_{k=0}^{N-L} \sum_{l=0}^{W-N+k-1} b_{k,l} + \sum_{k=1}^{N-L} \sum_{l=0}^{N-L-k} g_{k,l} + \sum_{l=0}^{W-L} d_{0,l}
\end{aligned}
$$

for $R \leq L-2$.

Conversely, any choice of coefficients $\{d_{k,l}, g_{k,l}, b_{k,l}\}$ satisfying the normalization constraints above results in $\varphi \in \{N, W, R, L\}$ when $N \leq W$.

3.1. Degrees of Freedom

When a degree larger than the support is desired, more constraints than those mentioned in Theorem 4 must be satisfied by the design coefficients $\{d_{k,l}, g_{k,l}, b_{k,l}\}$. We note however that the existence of a basis function φ that satisfies $N \geq W+1$ is often impossible because desirable external constraints (e.g., symmetry, interpolation) may be incompatible with the design parameters. To the best of our knowledge, no φ that would satisfy $N \geq W+1$ has ever been found useful in the literature. For this reason, from now on we concentrate on the case $N \leq W$.

Taking the normalization condition of the coefficients into account, there are $(N-R)\,(W-L)+L-R-2$ degrees of freedom, irrespective of whether $R \geq L-1$ or $R \leq L-2$. Table 1 explains how to split them. Finally, for a function $\varphi \in \{N, W, R, L\}$ to exist in the case $N \leq W$, the design parameters must satisfy

$$
L \leq W \qquad -1 \leq N-L \qquad R \leq N-1
$$
$$
0 \leq (N-R)\,(W-L)+L-R-2
$$

The attractiveness of this result is that the designer can address at an early stage the aspects of the design that are the most important in the context of image processing—particularly, support and order—while he can defer to later stages the fulfilling of less important constraints. Of most relevance is that the coefficients $\{d_{k,l}, g_{k,l}, b_{k,l}\}$ are essentially free (except when $N \geq W+1$), so that they do not interfere with the characteristics $\{N, W, R, L\}$.

4. EXAMPLES

4.1. MOMS

Since u satisfies $L \leq W$, we see that no function φ has a smaller support than that of β^{L-1} for the same order L (or, for that matter, a larger order for the same support). However, there exist functions other than B-splines with $L = W$; those are called MOMS (Maximal Order Minimal Support) and all are members of the family

$$
\varphi(x) = \sum_{m=0}^{L-1} a_m\, \frac{\mathrm{d}^m}{\mathrm{d}x^m}\, \beta^{L-1}(x)
$$

where $\{a_m\}$ is a set of constants [7].

Members of this family include the symmetric and interpolating functions discussed by Schaum [1]. Let us apply the design methodology proposed in this paper to rederive the Lagrange-like quadratic Schaum interpolator $\varphi_{\mathrm{S}} \in \{2, 3, -1, 3\}$. We first verify that the existence conditions are satisfied, and determine that there are two degrees of freedom. Since $N \leq W$, these two degrees of freedom need not be further reduced. Since $R \leq L-2$, we verify from Table 1 that the degrees of freedom are carried exclusively by the coefficients $d_{k,l}$ when $L = N+1$. In the present case, these coefficients are $d_{1,0}$ and $d_{2,0}$ ($d_{0,0} = 1$ because of the normalization condition). The symmetry condition removes a first degree of freedom by imposing that $d_{1,0} = 0$; the interpolation property removes the last one and results in

$$
\varphi_{\mathrm{S}}(x) = \beta^2(x) - \frac{1}{8}\, \frac{\mathrm{d}^2}{\mathrm{d}x^2} \beta^2(x)
$$

Other members of the family of MOMS include the so-called O-moms functions discussed by Blu [7]. Such functions share the same parameters $\{N, W, R, L\}$ than those of Schaum, are symmetric too, but are not interpolating. For example, the actual design methodology implies that every symmetric $\varphi \in \{2, 3, -1, 3\}$ is uniquely defined by the single free coefficient $d_{2,0}$. By asking that φ be interpolating, we already built φ_{S}. Alternatively, by optimizing some mathematically well-defined criterion that expresses the intrinsic quality of φ, we can also get the optimal function

$$
\varphi_{\mathrm{O}}(x) = \beta^2(x) + \frac{1}{60}\, \frac{\mathrm{d}^2}{\mathrm{d}x^2} \beta^2(x)
$$

Last but not least, the B-splines themselves are members of the family of MOMS. In addition to having the best order of approximation for a given support ($L = W$), they are also the most regular functions for a given degree ($R = N-1$). Moreover, they exhibit the lowest possible degree for a given order ($N = L-1$). Like the O-moms functions, they are not interpolating. We gain two orders of regularity over φ_{S} or φ_{O} by imposing $d_{2,0} = 0$ in the derivation above, or, equivalently, by setting $u = \delta$ in Theorem 1, which results in $\varphi_{\mathrm{B}} \in \{2, 3, 1, 3\}$.

4.2. Dodgson

As another simple design example, we now rederive the Dodgson function φ_{D} characterized by $\{N, W, R, L\} = \{2, 3, 0, 2\}$ which is interpolating and symmetric [4]. We first verify that φ_{D} satisfies the existence conditions. Moreover, the design has exactly two degrees of freedom since $N \leq W$. From their partition in the case $R \leq L-2$, and from the normalization condition, we conclude that $1 = b_{0,0} + d_{0,0} + d_{0,1}$. The symmetry imposes $d_{0,0} = d_{0,1} = d_0$; the symmetry and interpolation imposes $\varphi_{\mathrm{D}}(\frac{1}{2}) = \varphi_{\mathrm{D}}(\frac{5}{2}) = 0$ and $\varphi_{\mathrm{D}}(\frac{3}{2}) = 1$. From $\beta^2(\frac{3}{2}) = \frac{3}{4}$, $\beta^2(\frac{1}{2}) = \beta^2(\frac{5}{2}) = \frac{1}{8}$, $\beta^1(-\frac{1}{2}) = 0$, and $\beta^1(\frac{1}{2}) = \beta^1(\frac{3}{2}) = \frac{1}{2}$, we get that

$$
\left\{
\begin{array}{l}
\varphi_{\mathrm{D}}(x) = b_{0,0}\, \beta^2(x) + d_0\, (\beta^1(x) + \beta^1(x-1)) \\
\varphi_{\mathrm{D}}(\frac{1}{2}) = \frac{1}{8}\, b_{0,0} + d_0(\frac{1}{2} + 0) = 0 \\
\varphi_{\mathrm{D}}(\frac{3}{2}) = \frac{3}{4}\, b_{0,0} + d_0(\frac{1}{2} + \frac{1}{2}) = 1
\end{array}
\right.
$$

which results in $b_{0,0} = 2$ and $d_0 = -\frac{1}{2}$. We finally get that

$$\varphi_{\mathrm{D}}(x) = 2\,\beta^2(x) - \frac{1}{2}\left(\beta^1(x) + \beta^1(x-1)\right)$$

The important message of this derivation is that the crucial design parameters $\{N, W, R\}$, and especially $\{L\}$, have been specified up-front. We could have assigned random numbers to $\{b_{0,0}, d_{0,0}, d_{0,1}\}$ and we would still have satisfied the design parameters (up to the normalization condition $\int \varphi(x)\,\mathrm{d}x = 1$).

5. EXPERIMENTS

Comparing the design parameters of $\varphi_{\mathrm{S}} \in \{2, 3, -1, 3\}$ to those of $\varphi_{\mathrm{D}} \in \{2, 3, 0, 2\}$, we see that the approximation order of the former has been traded for the regularity of the latter, the other design parameters (i.e., degree, support, symmetry, and interpolation constraint) having been kept constant. Comparing the design parameters of $\varphi_{\mathrm{O}} \in \{2, 3, -1, 3\}$ to those of $\varphi_{\mathrm{B}} \in \{2, 3, 1, 3\}$, we see that only their regularity differs, the other design parameters (i.e., degree, support, symmetry, and non-interpolating basis) having been kept constant.

We now perform a compound-rotation experiment in which we rotate the standard Lena image in 18 incremental steps of 20° each, which tends to amplify the imperfections of the interpolators so that they become clearly perceptible. The top-left and bottom-left of Figure 1 shows the result of this process for the O-moms and for the Schaum quadratic interpolator which are both characterized by $\{2, 3, -1, 3\}$, respectively, while the bottom-right of Figure 1 corresponds to the more regular Dodgson interpolator of same support and degree characterized by $\{2, 3, 0, 2\}$. Even more regularity applies to the B-spline characterized by $\{2, 3, 1, 3\}$ and shown at top-right.

After a full 360° turn, the difference between the original Lena and the one obtained from the compounded rotation can be expressed as a signal-to-noise ratio in dB; we get $\mathrm{SNR_O} = 20.32$, $\mathrm{SNR_B} = 20.15$, $\mathrm{SNR_S} = 19.13$, and $\mathrm{SNR_D} = 16.98$, which tends to show that the order of approximation L is more important than the regularity R in the context of image processing. This claim is supported from the visual inspection of Figure 1.

Fig. 1. Compound rotation. Top-left: O-moms $\{2, 3, -1, 3\}$. Top-right: B-spline $\{2, 3, 1, 3\}$. Bottom-left: Schaum $\{2, 3, -1, 3\}$. Bottom-right: Dodgson $\{2, 3, 0, 2\}$. Dodgson (less order) results in somewhat more blurring than the other cases (more order). Further benefit results from relaxing the interpolation constraint (O-moms, B-spline). The regularity plays no role.

6. CONCLUSION

The main contribution of this paper is a unique and complete representation of the functions φ or φ_{int} of given degree, support, regularity, and approximation order, in terms of essentially free coefficients. This result can be used as a tool which allows the designer to specify—from the onset—an arbitrary degree N, an arbitrary support size W, an arbitrary regularity R, and an arbitrary order L. The remaining degrees of freedom are decoupled from these four design parameters and can be tuned to specific applications.

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