POLYHARMONIC SMOOTHING SPLINES FOR MULTI-DIMENSIONAL SIGNALS WITH $1/\|\omega\|^\tau$ - LIKE SPECTRA

Shai Tirosh, Dimitri Van De Ville and Michael Unser

Biomedical Imaging Group
Swiss Federal Institute of Technology (EPFL)
CH-1015 Lausanne, Switzerland
e-mail: {Shai.Tirosh, Dimitri.VanDeVille, Michael.Unser}@epfl.ch

ABSTRACT

Motivated by the fractal-like behavior of natural images, we propose a new smoothing technique that uses a regularization functional which is a fractional iterate of the Laplacian. This type of functional has previously been introduced by Duchon in the context of radial basis functions (RBFs) for the approximation of non-uniform data. Here, we introduce a new solution to Duchon's smoothing problem in multiple dimensions using non-separable fractional polyharmonic B-splines. The smoothing is performed in the Fourier domain by filtering, thereby making the algorithm fast enough for most multi-dimensional real-time applications.

1. INTRODUCTION

In many signal processing systems, denoising algorithms are applied. Denoising, or smoothing, which are qualitatively the same thing, can be done in many ways. Traditionally, there have been two communities dealing with this problem: the signal processing community and the statistics community. The first one deploys popular algorithms such as filtering (e.g., Wiener filtering) and wavelet thresholding with its variations [1]. The statistics community, which frequently uses non-uniform grids, adheres to Bayesian restoration (e.g. [2]) and smoothing splines [3] (particularly thin-plate splines [4]).

In fact, one can think about denoising as taking some information out of a signal. The information we want to take out depends on *a-priori* information or assumptions about the noiseless signal. A common assumption is that the energy proportion of the signal is most important in the low frequencies, while the high-frequency components contain mainly noise.

Multi-dimensional data, such as images, usually have high local correlation in all directions. Finding a suitable smoothing algorithm that uses this fact is challenging. The simplest method is to apply a separable algorithm (e.g. by using tensor product functions), which does not address this correlation. However, their implementation tends to be fast, and the mathematics are straightforward.

Another idea is to use RBFs, such as Duchon's (m, s) splines [5], providing us with a span of isotropic power functions. The problem is formulated in variational terms with a Tikhonov-like regularizer. However, the RBFs are poorly conditioned, thus making it very difficult to implement when there are many data points as is typical in signal processing. In addition, the method is computationally very expensive.

Under the Gaussian assumption, there is a well-known equivalence between the Bayesian formulation and regularization techniques. The appropriate regularization for fractal-like signals—signals with a spectra like $O(1/\|\omega\|^\tau)$—is a fractional iterate of the Laplacian which whitens the signal. In fact, it has recently been demonstrated that many natural images have this kind of spectral behavior [6, 7].

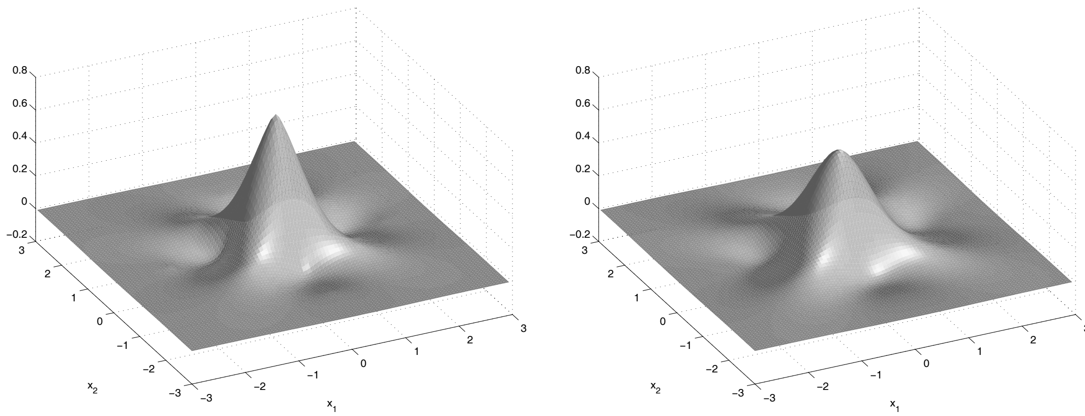
Here, we propose a new algorithm to solve Duchon's problem on a uniform grid using non-separable fractional polyharmonic B-splines basis functions. These functions are localized versions of RBFs, and thus span the same space. We work out the solution for the fractional case, obtaining a suitable algorithm for fractal-like signals.

The fact that we are dealing with a uniform grid allows us to develop a fast Fourier-based filtering algorithm. In addition, our algorithm can work in any number of dimensions without any special modifications. Thus, we are able to solve the statistics formulation (Duchon's splines) with signal-processing techniques (Fourier filtering), achieving a fast algorithm.

2. MATHEMATICAL PRELIMINARIES

2.1. Polyharmonic B-Splines

Our multi-dimensional basis functions are the polyharmonic B-splines [8–10]. We denote this function of order α as



(a) Order 1.5 **(b) Order 2.6**

Fig. 1. **Example of Polyharmonic B-splines in 2D.** Note that they are not compactly supported and decay like $O\left(\|\boldsymbol{x}\|^{-d-2}\right)$.

$\beta_\alpha(\boldsymbol{x})$, and specify its Fourier transform:

$$\mathcal{F}\{\beta_\alpha(\boldsymbol{x})\} = \hat{\beta}_\alpha(\boldsymbol{\omega}) = \frac{\|\sin(\boldsymbol{\omega}/2)\|^\alpha}{\|\boldsymbol{\omega}/2\|^\alpha}, \text{for } \alpha > \frac{d}{2}, \quad (1)$$

with $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_d) \in \mathbb{R}^d$, where $\sin(\boldsymbol{\omega}/2) = (\sin(\omega_1/2), \cdots, \sin(\omega_d/2))$. Note that the order of those functions can also be fractional (see also [11]). Figure 1 shows an example of two polyharmonic B-splines in 2D.

An important property of these B-splines is the convolution relation $\beta_{\alpha_1} * \beta_{\alpha_2} = \beta_{\alpha_1+\alpha_2}$, which follows directly from their definition in the Fourier domain.

2.2. Fractional Differentiation

In multiple dimensions, it is convenient to consider the isotropic fractional differential operator:

$$\partial_*^s f(\boldsymbol{x}) \longleftrightarrow \|\boldsymbol{\omega}\|^s \hat{f}(\boldsymbol{\omega}), \quad (2)$$

which is defined in the sense of distributions. The discrete counterpart is the (Laplacian-like) finite difference operator:

$$\triangle_*^s \longleftrightarrow \|2\sin(\boldsymbol{\omega}/2)\|^s = 2^s \left(\sum_{i=1}^{d} \sin(\omega_i/2)^2\right)^{s/2}. \quad (3)$$

It is easy to prove the following formula that allows us to easily differentiate polyharmonic B-splines:

$$\partial_*^s \beta_\alpha(\boldsymbol{x}) = \triangle_*^s \beta_{\alpha-s}(\boldsymbol{x}). \quad (4)$$

3. SMOOTHING FORMULATION

Duchon's smoothing formulation is variational with a Tikhonov-like regularization [5]. The solution of the smoothing operator, $\tilde{f}$, is defined by: $\tilde{f} = \arg\min_{f \in L_2} \{\epsilon_s^2\}$ where:

$$\epsilon_s^2 = \sum_{\boldsymbol{x_i} \in \mathrm{S}} (g(\boldsymbol{x_i}) - f(\boldsymbol{x_i}))^2 + \lambda \|\partial_*^s f(\boldsymbol{x})\|_{L_2}^2. \quad (5)$$

The first, *signal term* quantifies the distance between our solution and the given measurements $g(\boldsymbol{x_i})$, on S — a discrete localization of the continuous space, i.e. the discrete points at which we are observing the signal. The second, *smoothness term* penalizes the lack of smoothness of the solution. λ is a regularization parameter, making a balance between the two terms — higher λ means more smoothing and less fidelity to the data, and vice versa.

Duchon has shown that the solution of (5) in the general case is:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{x_i} \in \mathrm{S}} a_k \rho(\boldsymbol{x} - \boldsymbol{x_i}) + polynomial, \quad (6)$$

where $\rho(\boldsymbol{x})$ is the Green function of ∂_*^{2s}: $\partial_*^{2s}\rho(\boldsymbol{x}) = \delta(\boldsymbol{x})$. In the Fourier domain, this leads to: $\rho(\boldsymbol{x}) \longleftrightarrow \hat{\rho}(\boldsymbol{\omega}) = \frac{1}{\|\boldsymbol{\omega}\|^{2s}}$. One problem with $\hat{\rho}(\boldsymbol{\omega})$ is its singularity at zero frequency. In the frequency domain, this function looks very similar to the polyharmonic B-splines (except for the sin term). By adding the sin term, the function is tempered at zero.

We can show that these functions span the same space iff $\boldsymbol{x_i} \in \mathbb{Z}^d$, which is mostly the case in signal processing applications (It is easy to see that $\frac{\|\sin(\boldsymbol{\omega}/2)\|^{2s}}{\|\boldsymbol{\omega}/2\|^{2s}} = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} d(\boldsymbol{k})\,\rho(\boldsymbol{x} - \boldsymbol{k})$). This assumption also allows us to use fast Fourier techniques. Furthermore, the polyharmonic B-splines of order $2s$ reproduce polynomials of degree $\lceil 2s \rceil - 1$, which allows us to also take care of the second polynomial term in (6).

We can now solve Eq. (5) using a spline representation for our signals:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c(\boldsymbol{k})\,\beta_{2s}(\boldsymbol{x}-\boldsymbol{k}). \qquad (7)$$

It can be proven that this choice of order of spline gives a globally optimal solution among all possible function spaces. The proof is a generalization of the one in [11].

By substituting (7) in (5) and using the differentiation property of the splines (4) we obtain:

$$\begin{aligned}\epsilon_s^2 &= \langle g,g\rangle - 2\langle g,\beta_{2s}*c\rangle + \langle\beta_{2s}*c,\beta_{2s}*c\rangle \\ &+\lambda\langle\triangle_*^s * c, \triangle_*^s * c * \beta_{2s}\rangle. \qquad (8)\end{aligned}$$

Taking the partial derivative with respect to c (partial derivative of a vector) and equating it to zero, we get:

$$\begin{aligned}\left((\beta_{2s})' * g\right)(\boldsymbol{x}) &= \left((\beta_{2s})' * (\beta_{2s}) * c\right)(\boldsymbol{x}) \\ &+ \left(\lambda\,(\triangle_*^s)' * \triangle_*^s * (\beta_{2s})' * c\right)(\boldsymbol{x}) \qquad (9)\end{aligned}$$

Going to the Fourier domain, we find the solution:

$$\hat{f}(\boldsymbol{\omega}) = \frac{B_{2s}\left(e^{j\boldsymbol{\omega}}\right)}{B_{2s}\left(e^{j\boldsymbol{\omega}}\right) + \lambda\,\|2\sin(\boldsymbol{\omega}/2)\|^{2s}}\,\hat{g}(\boldsymbol{\omega}), \qquad (10)$$

where the polyharmonic B-spline periodization sequence $B_{2s}(e^{j\boldsymbol{\omega}})$ is the Fourier transform of a sampled polyharmonic spline. It can be expressed as the periodized version of $\hat{\beta}_{2s}(\boldsymbol{\omega})$ (using Poisson summation formula):

$$\begin{aligned}B_{2s}\left(e^{j\boldsymbol{\omega}}\right) &= \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\hat{\beta}_{2s}(\boldsymbol{\omega}+2\pi\boldsymbol{k}) \\ &= \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{\|\sin(\boldsymbol{\omega}/2+\pi\boldsymbol{k})\|^{2s}}{\|\boldsymbol{\omega}/2+\pi\boldsymbol{k}\|^{2s}}. \qquad (11)\end{aligned}$$

This sequence can be also regarded as the autocorrelation sequence of β_s. We have found an efficient way to calculate it, by applying a two-scale relation in the Fourier domain. (More details about this will be published in a future paper).

4. FRACTAL-LIKE BEHAVIOR OF NATURAL IMAGES

Natural phenomena are widely presumed to be self-similar from a statistical perspective. As a consequence, natural images tend to be scale invariant — seeing an object from two yards or one yard will result in very similar images transmitted by our visual system [7].

The notion of self-similarity is reminiscent of fractals, which possess this property. This leads to a fractal-like model for natural images — a spectral density function that behaves like $O(1/\|\boldsymbol{\omega}\|^{\tau})$, where τ is the fractal order [6]. Experimental results show that most natural images fit well to this model, with τ being usually between 1 to 1.5.

Assuming that our prior signal model is fractal, we can then apply a fractional derivative of power $s=\tau$ to whiten this signal. This suggests that choosing s according to the fractal power of a signal should yield better results.

This value is calculated by taking the radial frequency response of the image, transforming it into log-log coordinates and fitting a straight line, whose slope yields τ. Since experimental results show that this value is usually fractional, it is a good inducement for using fractional splines.

5. RESULTS

We demonstrate our approach on a slice of an MRI T2* volume, as used in functional analysis of brain activity (this image is in fact the brain of one of the former Ph.D students of our group). The original image is given in Fig. 2(a). In Fig. 2(c), we show the results of smoothing the noisy image of Fig. 2(b). In this case, s was estimated from the spectrum of the image, and we chose the value of λ that achieves the best SNR.

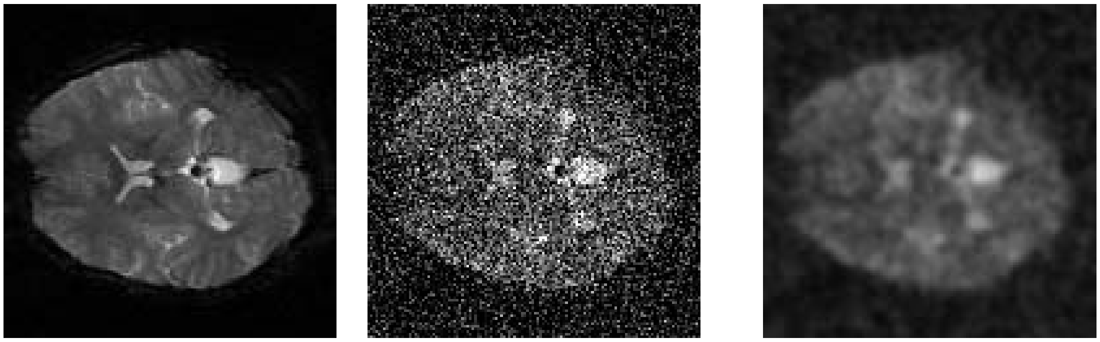
There are known methods for selecting the value of λ, but it is beyond the scope of this paper. Note that, when λ is small, the result fits the observation more closely, and edges are preserved better; however, this comes at the expense of more residual noise. On the other hand, when λ is too large, the image is over smoothed, and edges are not well preserved; however, we suppress most of the noise. The right balance will depend on the particular application.

Figure 3(a) shows the radial frequency response of the image in Fig. 2, and the regressed fractal model (on a log-log scale). Note that the value of s for the smoothing was chosen according to this analysis. In Fig. 3(b), we show that this way of selecting s is indeed optimal.

6. CONCLUSIONS

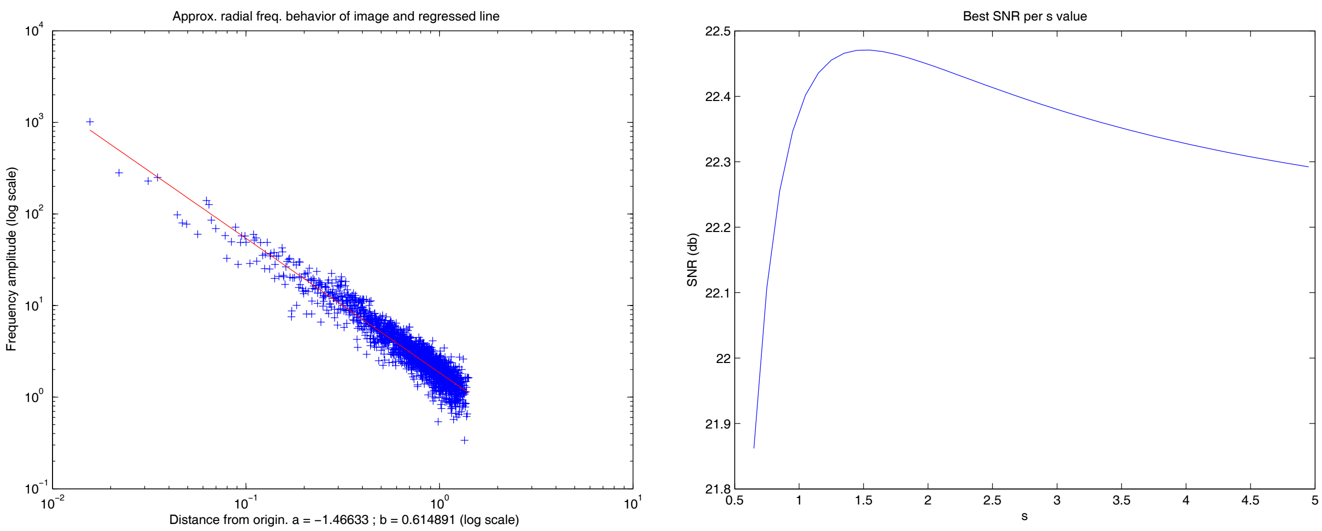
We proposed a multi-dimensional smoothing algorithm using non-separable fractional polyharmonic B-splines. This method is a solution of Duchon's smoothing formulation, and combines the statistics community's methods with the signal processing community's tools, allowing for fast smoothing of multi-dimensional signals using Fourier-based filtering. This is only possible because we are working on a uniform grid and because we found a fast algorithm to compute (11).

Secondly, we use the fact that natural images behave in a fractal-like way with respect to their radial spectrum, which in turn allows us to tune the value of s — the fractional derivative order in the variational smoothing formulation. Additionally, preliminary experimental results confirm that this choice is optimal. Our approach is especially suitable



(a) Original image (b) Noisy image (c) Optimal least squares smoothing

Fig. 2. **Smoothing results.** SNR: (b) 14.7804 (c) 22.4707, with $\lambda = 3.07$, $s = 1.46633$.



(a) Radial frequency behavior (log-log scale) **(b) Best SNR vs. value of s**

Fig. 3. **Fractal-like properties.** (a) The radial frequency behavior of the image, with the corresponding regressed line (on a log-log scale). (b) The effect of choosing the correct value of s on the optimal denoising SNR.

for fractal-like signals as well as real world images thanks to our fractional extension of smoothing splines.

7. REFERENCES

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