ÉCOLE POLYTECHNIQUE
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Splines, noise, fractals, and optimal signal reconstruction

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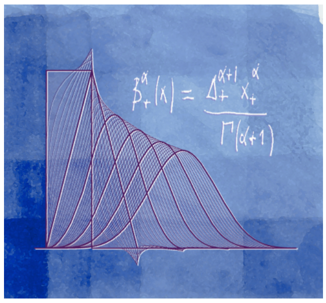

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Generalized sampling: roadmap $(T = 1)$

- Nonideal acquisition system

$f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^d$ → acquisition device → sampling → (+ noise) →

Measurements:

$$\tilde{y}[\boldsymbol{k}] = (h * f)(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}} + n[\boldsymbol{k}]$$

↓

Reconstruction algorithm

↓

signal coefficients

Goal: Specify φ and the reconstruction algorithm so that $\tilde{f}(\boldsymbol{x})$ is a "good" approximation of $f(\boldsymbol{x})$

Continuous-domain reconstruction

$$\tilde{f}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c[\boldsymbol{k}] \varphi(\boldsymbol{x} - \boldsymbol{k})$$

⟷ *Riesz-basis property* ⟷ $\{c[\boldsymbol{k}]\}_{\boldsymbol{k} \in \mathbb{Z}^d}$

↕ *Interpolation*

Discrete signal

$$\{\tilde{f}[\boldsymbol{k}]\}_{\boldsymbol{k} \in \mathbb{Z}^d}$$

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Integer-shift-invariant spaces $(T = 1)$

Integer-shift-invariant subspace associated with a generating function φ (e.g., B-spline):

$$V(\varphi) = \left\{ f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) : c \in \ell_2(\mathbb{Z}^d) \right\}$$

Generating function: $\varphi(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}(\boldsymbol{\omega}) = \int_{\boldsymbol{x}\in\mathbb{R}^d} \varphi(\boldsymbol{x}) e^{-j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x}$

- Autocorrelation (or Gram) sequence

$$a_\varphi[\boldsymbol{k}] \stackrel{\Delta}{=} \langle \varphi(\cdot), \varphi(\cdot - \boldsymbol{k})\rangle \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad A_\varphi(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |\hat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2$$

- Riesz basis property

$$0 < C_{\min} \, \|c\|_{\ell_2(Z^d)} \leq \overbrace{\left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \right\|_{L_2}}^{\|f\|_{L_2(\mathbb{R}^d)}} \leq C_{\max} \, \|c\|_{\ell_2(Z^d)}$$

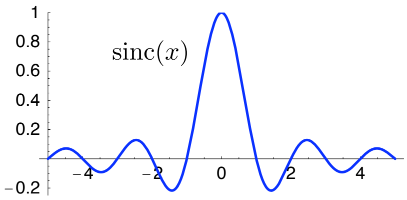
$$\Updownarrow$$

$$0 < C_{\min}^2 \leq A_\varphi(e^{j\boldsymbol{\omega}}) \leq C_{\max}^2 < +\infty$$

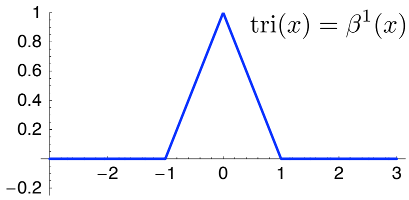
Orthonormal basis $\Leftrightarrow \quad a_\varphi[\boldsymbol{k}] = \delta[\boldsymbol{k}] \quad \Leftrightarrow \quad \|c\|_{\ell_2} = \|f\|_{L_2}$ (Parseval)

Examples of popular generating functions

- Bandlimited (Shannon, 1948)



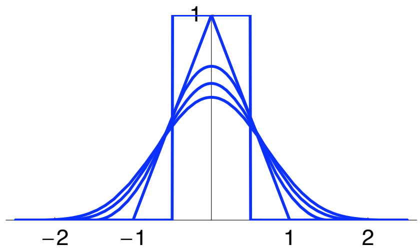
- Piecewise linear



- Centered B-spline of degree n (Schoenberg, 1946)

$$\beta^n(x) = \underbrace{\beta^0 * \beta^0 * \cdots * \beta^0}_{(n+1) \text{ times}}(x)$$

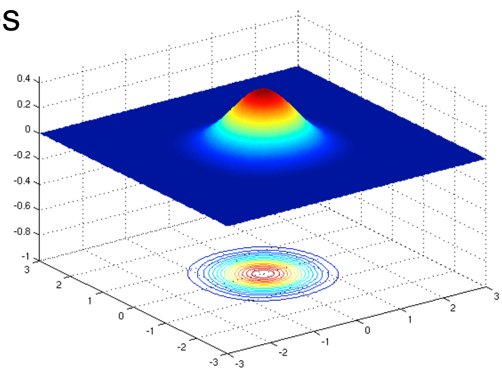
$$\beta^0(x) = \begin{cases} 1, & x \in [-\frac{1}{2}, \frac{1}{2}) \\ 0, & \text{otherwise.} \end{cases}$$



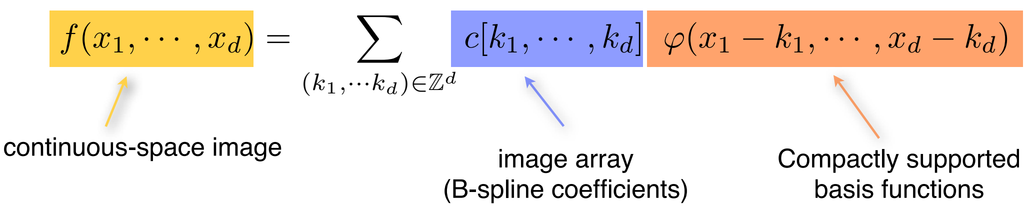
B-spline representation of images

- Symmetric, tensor-product B-splines

$$\varphi(x_1, \cdots, x_d) = \beta^n(x_1) \times \cdots \times \beta^n(x_d)$$

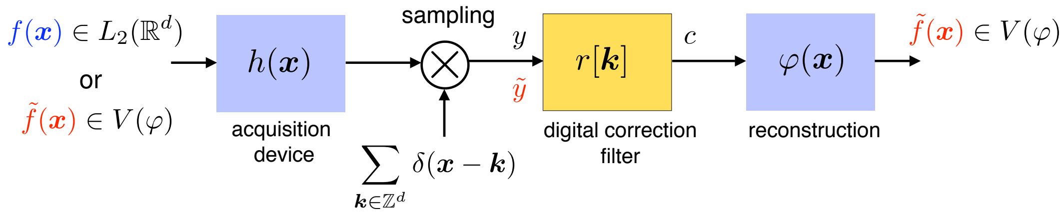


- Multidimensional spline function



$$f(x_1, \cdots, x_d) = \sum_{(k_1, \cdots k_d) \in \mathbb{Z}^d} c[k_1, \cdots, k_d]\, \varphi(x_1 - k_1, \cdots, x_d - k_d)$$

Consistent signal reconstruction



Justification: to an observer, the reconstruction $\tilde{f}(\boldsymbol{x})$ is undistinguishable from $f(\boldsymbol{x})$

- **Consistent-sampling theorem** (U.-Aldroubi, 1994)

  Let $|C_{12}(e^{j\boldsymbol{\omega}})| \geq m > 0$. Then, there exists a unique function $\tilde{f} \in V(\varphi)$ that is consistent with f in the sense that $y[\boldsymbol{k}] = \langle f, \varphi(\cdot - \boldsymbol{k})\rangle = \langle \tilde{f}, \varphi(\cdot - \boldsymbol{k})\rangle = \tilde{y}[\boldsymbol{k}]$

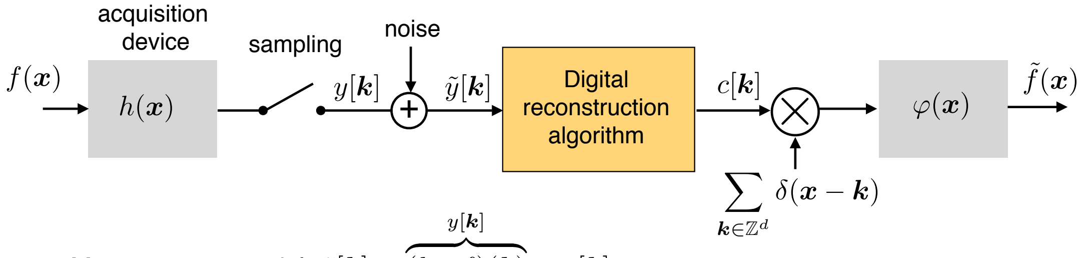
  $$\tilde{f}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (r_0 * y)[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \quad \text{with} \quad R_0(e^{j\boldsymbol{\omega}}) = \frac{1}{C_{12}(e^{j\boldsymbol{\omega}})}$$

Cross-correlation sequence: $c_{12}[\boldsymbol{k}] = \langle h, \varphi(\boldsymbol{k} - \cdot)\rangle \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad C_{12}(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{h}(\boldsymbol{\omega}+2\pi\boldsymbol{k})\hat{\varphi}(\boldsymbol{\omega}+2\pi\boldsymbol{k})$

SAMPLING IN THE PRESENCE OF NOISE

- Context: fixed reconstruction space $V(\varphi)$
  - spline, bandlimited or other
- Use of prior knowledge
  - Signal: deterministic vs. stochastic
  - Noise distribution: stationary Gaussian
    $\Rightarrow$ weighted least-squares data term
- Three alternative formulations
  - Variational: regularized-least squares (Tikhonov)
  - Minimax reconstruction
  - Minimum mean-square error (MMSE)

Statement of generalized sampling problem



- Measurement model: $\tilde{y}[\boldsymbol{k}] = \overbrace{(h * f)(\boldsymbol{k})}^{y[\boldsymbol{k}]} + n[\boldsymbol{k}]$
- Noise component: $n[\boldsymbol{k}]$ (stationary Gaussian with spectral power density $C_n(e^{j\boldsymbol{\omega}})$)
- Reconstruction formula: $\tilde{f}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k})$

Signal reconstruction problem

Given the noisy measurements $\{\tilde{y}[\boldsymbol{k}]\}$ and the reconstruction space $V(\varphi)$, determine the signal coefficients $\{c[\boldsymbol{k}]\}$ such that $\tilde{f}(\boldsymbol{x})$ is the "closest" to $f(\boldsymbol{x})$

Variational solution (Tikhonov)

- Simulated measurements: $y[\boldsymbol{k}] = (h * f)(\boldsymbol{k})$
- Least-squares data term: $J_{\text{data}}(\tilde{y}, y) = \int_{[\pi,\pi]^d} C_n(e^{j\boldsymbol{\omega}})^{-1} |\tilde{Y}(e^{j\boldsymbol{\omega}}) - Y(e^{j\boldsymbol{\omega}})|^2 \mathrm{d}\boldsymbol{\omega}$

  (frequency-weighted sum of square differences = Gaussian likelihood function)
- Regularization constraint: $R(f) = \|\mathrm{L}f\|^2_{L_2(\mathbb{R}^d)} \leq \sigma_0^2$, L: differential operator

Regularized least-squares reconstruction: $\min_{f\in V(\varphi)} \{J_{\text{data}}(\tilde{y}, y) + \lambda\, R(f)\}$

- Regularization parameter: $\lambda = \lambda(\sigma_0^2) \geq 0$ (Lagrange multiplier)

$\Rightarrow$ Digital-filter solution: $\tilde{f}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (r_{\text{RLS}} * \tilde{y})[\boldsymbol{k}]\ \varphi(\boldsymbol{x}-\boldsymbol{k})$

where $R_{\text{RLS}}(e^{j\boldsymbol{\omega}}) = \dfrac{C_{12}^*(e^{j\boldsymbol{\omega}})}{|C_{12}(e^{j\boldsymbol{\omega}})|^2 + \lambda C_n(e^{j\boldsymbol{\omega}}) \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |\hat{L}(\boldsymbol{\omega}+2\pi\boldsymbol{n})|^2 |\hat{\varphi}(\boldsymbol{\omega}+2\pi\boldsymbol{n})|^2}$

Minimax solution

- MMSE estimation of a deterministic signal: worst-case optimization
  - Admissible solution space: $\mathcal{S} = \left\{ f(\boldsymbol{x}) : \|\mathrm{L}f\|^2_{L_2(\mathbb{R}^d)} \leq \sigma_0^2 \right\}$
  - Reference deterministic solution: $f_V(\boldsymbol{x}) = \arg \min_{s \in V(\varphi)} \|f - s\|^2_{L_2(\mathbb{R}^d)}$
  - Digital-filtering reconstruction algorithm: $c[\boldsymbol{k}] = (r * \tilde{y})[\boldsymbol{k}]$
  - Minimax solution at given location $\boldsymbol{x}_0$: $\min_r \max_{\tilde{f} \in V(\varphi) \cap \mathcal{S}} E\left\{ |\tilde{f}(\boldsymbol{x}_0) - f_V(\boldsymbol{x}_0)|^2 \right\}$
- Optimal reconstruction filter

$$R_{\mathrm{MX}}(e^{j\boldsymbol{\omega}}) = \frac{\sigma_0^2 \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \frac{\hat{h}^*(\boldsymbol{\omega} + 2\pi\boldsymbol{n}) \hat{\varphi}^*(\boldsymbol{\omega} + 2\pi\boldsymbol{n})}{|\hat{L}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2}}{A_\varphi(e^{j\boldsymbol{\omega}}) \left(C_n(e^{j\boldsymbol{\omega}}) + \sigma_0^2 \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \frac{|\hat{h}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2}{|\hat{L}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2} \right)}$$

Minimum mean-square-error solution

- Hypotheses
  - Signal = samples of a stationary process with known power spectrum: $\Phi_f(\boldsymbol{\omega})$
  - Discrete stationary noise with power spectrum: $C_n(e^{j\boldsymbol{\omega}})$
  - Reconstruction by digital filtering: $c[\boldsymbol{k}] = (r * \tilde{y})[\boldsymbol{k}]$
- Minimum-error reconstruction
  - Reference noise-free reconstruction: $f_V(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \langle \tilde{\varphi}(\cdot - \boldsymbol{k}), f \rangle \, \varphi(\boldsymbol{x} - \boldsymbol{k})$
  - MMSE solution at given location $\boldsymbol{x}_0$: $\min_r E\left\{ |\tilde{f}(\boldsymbol{x}_0) - f_V(\boldsymbol{x}_0)|^2 \right\}$
- Projected Wiener filter

$$R_{\mathrm{W}}(e^{j\boldsymbol{\omega}}) = \frac{\sum_{\boldsymbol{n} \in \mathbb{Z}^d} \Phi_f(\boldsymbol{\omega} + 2\pi\boldsymbol{n}) \hat{h}^*(\boldsymbol{\omega} + 2\pi\boldsymbol{n}) \hat{\varphi}^*(\boldsymbol{\omega} + 2\pi\boldsymbol{n})}{A_\varphi(e^{j\boldsymbol{\omega}}) \left(C_n(e^{j\boldsymbol{\omega}}) + \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \Phi_f(\boldsymbol{\omega} + 2\pi\boldsymbol{n}) |\hat{h}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2 \right)}$$

Comparison of signal-recovery methods

	Signal model	Noise model	Criterion
Least-squares filter	no constraint	irrelevant	Data term
Tikhonov filter	Deterministic: $f(\boldsymbol{x}) \in \mathcal{S}$	not explicit	Data term + regularization
Projected minimax filter	Deterministic: $f(\boldsymbol{x}) \in \mathcal{S}$	stationary process	Worst-case projected MSE at $\boldsymbol{x} = \boldsymbol{x}_0$
Projected Wiener filter	stationary process	stationary process	projected MSE at $\boldsymbol{x} = \boldsymbol{x}_0$

$\lambda \to 0$

Equivalence

$$\Phi_f(\omega) = \frac{\sigma_0^2}{|\hat{L}(\omega)|^2}$$

(Eldar-U., *IEEE-SP*, 2006)

OPTIMAL RECONSTRUCTION SPACE

Simplifying assumption: ideal sampling

$$\Leftrightarrow \quad h(\boldsymbol{x}) = \delta(\boldsymbol{x})$$

- Optimal spline generator
- Globally optimum variational solution
- Hybrid Wiener filter
- Equivalence of solutions

Optimal spline generator

- Generalized Sobolev space

$$W_2^{\mathrm{L}}(\mathbb{R}^d) = \left\{ f(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^d : \|f\|^2_{L_2(\mathbb{R}^d)} + \|\mathrm{L}f\|^2_{L_2(\mathbb{R}^d)} < +\infty \right\}$$

Definition: $\varphi_{\mathrm{L}} \in W_2^{\mathrm{L}}(\mathbb{R}^d)$ is an **optimal generator** with respect to L iff

- it generates a shift-invariant Riesz basis $\{\varphi_{\mathrm{L}}(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k} \in \mathbb{Z}^d}$
- φ_{L} is a cardinal $\mathrm{L}^*\mathrm{L}$-spline; i.e., there exists a sequence $q[\boldsymbol{k}]$ s.t. $\mathrm{L}^*\mathrm{L}\{\varphi_{\mathrm{L}}\}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} q[\boldsymbol{k}]\delta(\boldsymbol{x} - \boldsymbol{k})$.

[U.-Blu, IEEE-SP, 2005]

- Optimality property

$$\forall f \in W_2^{\mathrm{L}}(\mathbb{R}^d), \quad \|\mathrm{L}\{f\}\|^2_{L_2(\mathbb{R}^d)} = \|\mathrm{L}\{s_{\mathrm{int}}\}\|^2_{L_2(\mathbb{R}^d)} + \|\mathrm{L}\{f - s_{\mathrm{int}}\}\|^2_{L_2(\mathbb{R}^d)}$$

where s_{int} is the unique interpolator of f in $V(\varphi_{\mathrm{L}})$; i.e., $f(\boldsymbol{k}) = s_{\mathrm{int}}(\boldsymbol{k}), \forall \boldsymbol{k} \in \mathbb{Z}^d$

Application: Optimality of cubic splines

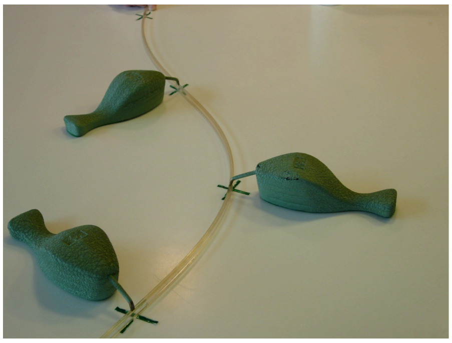
- Solution of minimum-curvature interpolation problem
  - Interpolators of $\{f[k]\}_{k\in\mathbb{Z}}$: $\mathcal{S}_f = \{s(x) : x \in \mathbb{R}, s(k) = f[k], \forall k \in \mathbb{Z}\}$
  - The optimal interpolant is a cubic spline

$$s_{\mathrm{int}}(x) = \arg \min_{s(x) \in \mathcal{S}_f} \|\mathrm{D}^2 s\|^2_{L_2(\mathbb{R})}$$

$$\Updownarrow$$

$$s_{\mathrm{int}}(x) \in V(\beta^3)$$

$\beta^3(x)$: Schoenberg's cubic B-spline



Proof: $\beta^3(x)$ is optimal with respect to $\mathrm{L} = \mathrm{D}^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$,

$$\mathrm{L}^*\mathrm{L}\{\beta^3(x)\} \overset{\mathcal{F}}{\longleftrightarrow} |\omega|^4 \left(\frac{\sin(\omega/2)}{\omega/2}\right)^4 = (-e^{j\omega} + 2 - e^{-j\omega})^2 = Q(e^{j\omega}), \quad (2\pi\text{-periodic})$$

which implies that: $\forall s \in \mathcal{S}_f, \|\mathrm{D}^2 s\|^2 = \|\mathrm{D}^2 s_{\mathrm{int}}\|^2 + \|\mathrm{D}^2(s - s_{\mathrm{int}})\|^2 \geq \|\mathrm{D}^2 s_{\mathrm{int}}\|^2$ (QED)

Globally optimum variational solution

- General cost function with quadratic regularization

$$J(f,y) = J_{\text{data}}(f,y) + \lambda \|\mathrm{L}f\|^2_{L_2(\mathbb{R}^d)}$$

$J_{\text{data}}(f,y)$: arbitrary, but depends on the input data $y[\boldsymbol{k}]$ and the samples $\{f(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ only

Theorem. If φ_{L} is optimum with respect to L and a solution exists, then the optimum reconstruction over ALL continuously-defined functions f is such that

$$\min_f J(f,y) = \min_{f\in V(\varphi_{\mathrm{L}})} J(f,y).$$

Hence, there is an optimal solution of the form $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi_{\mathrm{L}}(\boldsymbol{x}-\boldsymbol{k})$ that can be found by DISCRETE optimization.

If $J_{\text{data}}(f,y) = \|y-f\|^2_{\ell_2}$, then the estimator is called a **smoothing spline**.

Sketch of proof: $J(f,y) = J_{\text{data}}(s_{\text{int}},y) + \lambda\|\mathrm{L}\{s_{\text{int}}\}\|^2_{L_2(\mathbb{R}^d)} + \lambda\|\mathrm{L}\{f-s_{\text{int}}\}\|^2_{L_2(\mathbb{R}^d)}$

- Data term depends on the sample values only
- Regularization term is further minimized by taking $f(\boldsymbol{x}) = s_{\text{int}}(\boldsymbol{x})$

Smoothing-spline estimator

- Quadratic cost function (RLS)

$$J(f,y) = \|f-y\|^2_{\ell_2(\mathbb{Z}^d)} + \lambda\|\mathrm{L}f\|^2_{L_2(\mathbb{R}^d)}$$

- Optimal generator: generalized B-spline

$\rho(\boldsymbol{x})$: Green function of $\mathrm{L}^*\mathrm{L}$ $\Leftrightarrow$ $\mathrm{L}^*\mathrm{L}\{\rho\} = \delta(\boldsymbol{x})$

"localization" filter that cancels singularities of $\frac{1}{|\hat{L}(\boldsymbol{\omega})|^2}$

$$\varphi_{\mathrm{L}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} q[\boldsymbol{k}]\,\rho(\boldsymbol{x}-\boldsymbol{k}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}_{\mathrm{L}}(\boldsymbol{\omega}) = \frac{Q(e^{j\boldsymbol{\omega}})}{|\hat{L}(\boldsymbol{\omega})|^2}$$

- Efficient digital-filter-based algorithm

$$f_\lambda(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (h_\lambda * y)[\boldsymbol{k}]\varphi_{\mathrm{L}}(\boldsymbol{x}-\boldsymbol{k}) \quad \text{with} \quad H_\lambda(e^{j\boldsymbol{\omega}}) = \frac{B^*_{\mathrm{L}}(e^{j\boldsymbol{\omega}})}{|B_{\mathrm{L}}(e^{j\boldsymbol{\omega}})|^2 + \lambda Q(e^{j\boldsymbol{\omega}})B_{\mathrm{L}}(e^{j\boldsymbol{\omega}})}$$

where $B_{\mathrm{L}}(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \hat{\varphi}_{\mathrm{L}}(\boldsymbol{\omega}+2\pi\boldsymbol{n}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_{\mathrm{L}}(\boldsymbol{k})e^{-j\langle\boldsymbol{\omega},\boldsymbol{k}\rangle}$

Hybrid Wiener filter

Hypotheses

- Measurement model: $y[\boldsymbol{k}] = f(\boldsymbol{k}) + n[\boldsymbol{k}]$
- Signal = samples from a stationary process with spectral density function:
  $$\Phi_f(\boldsymbol{\omega}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad c_f(\boldsymbol{x}) = E\{f(\cdot)f(\cdot+\boldsymbol{x})\}$$
- Discrete stationary noise with power spectrum: $C_n(e^{j\boldsymbol{\omega}}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad E\{n[\cdot]n[\cdot+\boldsymbol{k}]\}$

Optimal Wiener solution

Minimum mean-square-error (MMSE) estimator: $\tilde{f}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} y[\boldsymbol{k}]\varphi_W(\boldsymbol{x}-\boldsymbol{k})$

$$\varphi_{\mathrm{W}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} r[\boldsymbol{k}]c_f(\boldsymbol{x}-\boldsymbol{k}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad R(e^{j\boldsymbol{\omega}})\Phi_f(\boldsymbol{\omega}) = \frac{\Phi_f(\boldsymbol{\omega})}{C_n(e^{j\boldsymbol{\omega}}) + \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\Phi_f(\boldsymbol{\omega}+2\pi\boldsymbol{k})}$$

Interpretation: optimal estimator included in $V(\varphi_{\mathrm{W}}) = \mathrm{span}\{c_f(\boldsymbol{x}-\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$

Equivalence of solutions

Context: Samples of a signal $f(\boldsymbol{x})$ corrupted by white noise with variance σ^2: $y[\boldsymbol{k}] = f(\boldsymbol{k}) + n[\boldsymbol{k}]$

- Smoothing spline estimator: $f_\lambda(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (h_\lambda * y)[\boldsymbol{k}]\varphi_{\mathrm{L}}(\boldsymbol{x}-\boldsymbol{k})$
- Hybrid Wiener filter: $f_{\mathrm{W}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} y[\boldsymbol{k}]\varphi_W(\boldsymbol{x}-\boldsymbol{k}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{\Phi_f(\boldsymbol{\omega})}{\sigma^2 + \sum_{\boldsymbol{k}\in\mathbb{Z}^d}\Phi_f(\boldsymbol{\omega}+2\pi\boldsymbol{k})} \cdot Y(e^{j\boldsymbol{\omega}})$
- Minimax estimator

$$f_{\mathrm{MX}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (r_{\mathrm{MX}} * y)[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{\sigma_0^2 \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \frac{\hat{\varphi}^*(\boldsymbol{\omega}+2\pi\boldsymbol{n})}{|\hat{L}(\boldsymbol{\omega}+2\pi\boldsymbol{n})|^2}}{A_\varphi(e^{j\boldsymbol{\omega}})\left(\sigma^2 + \sum_{\boldsymbol{n}\in\mathbb{Z}^d}\frac{\sigma_0^2}{|\hat{L}(\boldsymbol{\omega}+2\pi\boldsymbol{n})|^2}\right)} \hat{\varphi}(\boldsymbol{\omega}) \cdot Y(e^{j\boldsymbol{\omega}})$$

Proposition: The smoothing spline, hybrid Wiener and Minimax filters are equivalent if:

(i) $\Phi_f(\boldsymbol{\omega}) = \frac{\sigma_0^2}{|\hat{L}(\boldsymbol{\omega})|^2}$; that is, L is the whitening operator of the stochastic process f,

(ii) $\lambda = \frac{\sigma^2}{\sigma_0^2}$; i.e., the regularization parameter is inversely proportional to SNR

(iii) $\varphi \in V(\varphi_{\mathrm{L}})$ (optimal Minimax approximation space)

SELECTING THE OPERATOR

- Splines, stochastic processes, and differential operators
- Scale-invariant operators
- Fractional B-splines
- Fractal processes (fBm)
- Stochastic optimality of splines

Splines and stochastic processes

- Differential equation: $\mathrm{L}\{s\}(\boldsymbol{x}) = r(\boldsymbol{x})$

  $r(\boldsymbol{x})$: system input or excitation

  $s(\boldsymbol{x})$: output (spline or stochastic process)

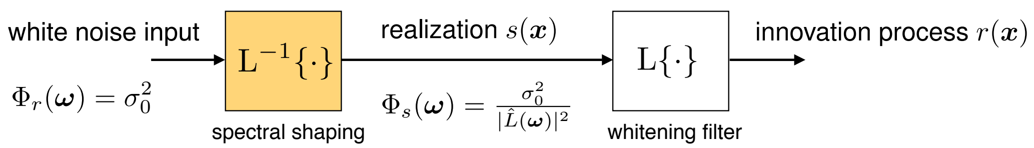
  L: Differential operator (e.g., $\mathrm{L} = \mathrm{D}^n = \frac{\mathrm{d}^n}{\mathrm{d}x^n}$)

> Formal solution:
>
> $s(\boldsymbol{x}) = \mathrm{L}^{-1}\{r\}(\boldsymbol{x})$
>
> (with appropriate boundary conditions)

- Spline generator: $r(\boldsymbol{x}) = \sum_{\boldsymbol{k}} d[\boldsymbol{k}]\delta(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{k}})$ (sum of Dirac impulses)

  $d[\boldsymbol{k}]$: appropriate weights; $\{x_{\boldsymbol{k}}\}$: spline knots or singularities

- Stochastic-process generator: white Gaussian noise input



Scale-invariant operators

Definition: A convolution operator L is scale-invariant iff it commutes with dilation: i.e., $\forall s(x), \mathrm{L}\{s(\cdot)\}(x/a) = C_a \, \mathrm{L}\{s(\cdot/a)\}(x)$.

Theorem
The complete family of real scale-invariant 1D convolution operators is given by the fractional derivatives ∂_τ^γ, whose frequency response is

$$\hat{L}(\omega) = (-j\omega)^{\frac{\gamma}{2}-\tau}(j\omega)^{\frac{\gamma}{2}+\tau}$$

γ: order of the derivative (i.e., $|\hat{L}(\omega)| = |\omega|^\gamma$)

τ: phase (or asymmetry) factor ($\tau \in \mathbb{R}$)

Construction of B-splines

Derivative operator: $\mathrm{D} = \partial_{\frac{1}{2}}^1 \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad j\omega$

Finite difference: $\Delta_+ \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad 1 - e^{j\omega}$

Liouville's fractional derivative: $\mathrm{D}^\gamma = \partial_{\gamma/2}^\gamma \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (j\omega)^\gamma$

Fractional finite differences: $\Delta_+^\gamma \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (1 - e^{j\omega})^\gamma$

- Causal fractional B-splines

Discrete operator: localization filter $Q(e^{j\omega})$

Spline degree: $\alpha = \gamma - 1$

$$\frac{(1 - e^{-j\omega})^{\alpha+1}}{(j\omega)^{\alpha+1}} \quad \stackrel{\mathcal{F}^{-1}}{\longrightarrow} \quad \beta_+^\alpha(x)$$

Continuous-domain operator: $\hat{L}(\omega)$

Causal fractional B-splines

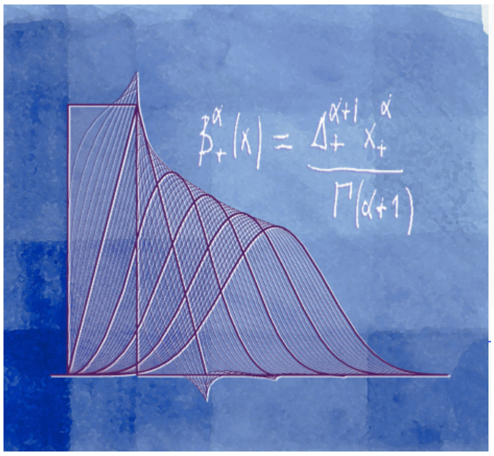
- Causal B-splines

$$\beta_+^0(x) = \Delta_+ x_+^0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1-e^{-j\omega}}{j\omega}$$

$$\vdots \qquad\qquad \vdots$$

$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1}$$

One-sided power function: $x_+^\alpha = \begin{cases} x^\alpha, & x \geq 0 \\ 0, & x < 0 \end{cases}$



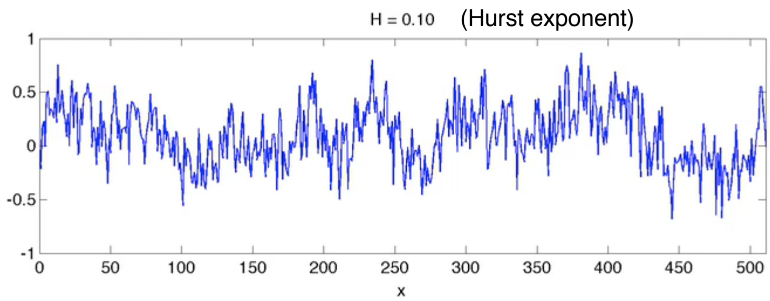
- B-spline generating signal

$$r(x) = \Delta_+^{\alpha+1}\{\delta\} = \sum_{k\in\mathbb{Z}} (-1)^k \binom{\alpha+1}{k} \delta(x-k)$$

(Unser & Blu, *SIAM Rev*, 2000)

fractional Brownian motion (fBm)

- fBm is a self-similar process of great interest for the modeling of natural signals [Mandelbrot, Van Ness, 1968]
- fBms are nonstationary, meaning that the Wiener formalism is not applicable (their power spectrum is not defined!)
- Yet, using Gelfand's distributional theory of generalized stochastic processes, we can show that these are whitened by fractional derivatives of order $\gamma = H + \frac{1}{2}$



fractional integration of white noise

$$s_H(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{j\omega x}-1}{(j\omega)^{H+\frac{1}{2}}} \mathrm{d}W(\omega) \quad \Leftrightarrow \quad Z_{s_H}(\phi) = \exp\left(-\frac{\epsilon_H^2}{4\pi}\int \frac{|\hat{\phi}(\omega)-\hat{\phi}(0)|^2}{|\omega|^{2H+1}} \mathrm{d}\omega\right)$$

Stochastic optimality of splines

- Stationary processes
  - A smoothing-spline estimator provides the MMSE estimation of a continuously defined signal $f(\boldsymbol{x})$ given its noisy samples iff L is the whitening operator of the process and $\lambda = \frac{\sigma^2}{\sigma_0^2}$ [U.-Blu, 2005]
  - Advantages: the spline machinery often yields a most efficient implementation, such as shortest basis functions (B-splines) and recursive algorithms (in 1D)

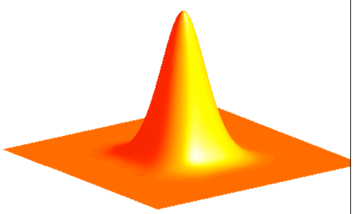
- Fractal processes
  - MMSE estimate of a fBm with Hurst exponent H is a fractional smoothing spline of order $\gamma = 2H + 1$: $\hat{L}(\omega) = (j\omega)^{\gamma/2}$ [Blu-U., 2007]
  - Special case: MMSE estimate of Wiener process (Brownian motion) is a linear spline ($\gamma = 2$)

Multidimensional extension

- Operator (rotation and scale-invariant): fractional Laplacian

$$\Delta^{\gamma/2} \overset{\mathcal{F}}{\longleftrightarrow} \|\boldsymbol{\omega}\|^{\gamma}$$

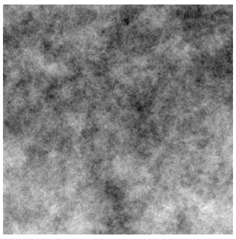
- Function spaces: thin-plate splines (Duchon, 1979)



- Basis functions: polyharmonic B-splines (Rabut, 1992)

$$\hat{\varphi}_\gamma(\boldsymbol{\omega}) = \left(\frac{\|2\sin(\boldsymbol{\omega}/2)\|}{\|\boldsymbol{\omega}\|}\right)^{\gamma}$$

- Approximation algorithm: polyharmonic smoothing splines (Tirosh, 2006)

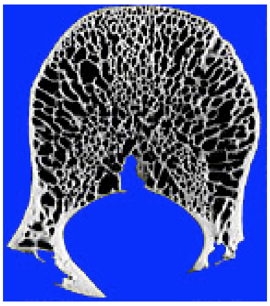


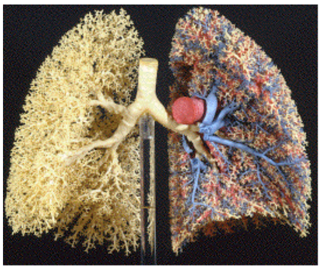
- Stochastic process: fractional Brownian field (Adler, 1985)

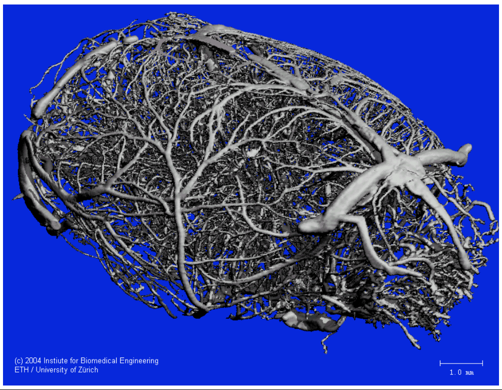
$$Z_{s_H}(\phi) = \exp\left(-\frac{\varepsilon_H^2}{4\pi}\int \frac{|\hat{\phi}(\boldsymbol{\omega}) - \hat{\phi}(\mathbf{0})|^2}{\|\boldsymbol{\omega}\|^{2H+d}}\,\mathrm{d}\boldsymbol{\omega}\right)$$

Relevance of self-similarity for bio-imaging

- Fractals and physiology







BEYOND TRADITIONAL SAMPLING

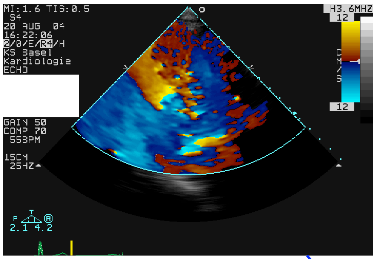
- Previous formulations are (in principle) also applicable for:
  - Nonuniform sampling
  - Multidimensional sampling
  - Multicomponent, multichannel signals
  - Generalized measurements

  ... but the “details” have to be worked out!

- A concrete vector-sampling problem:
  “Full motion and flow-field recovery from incomplete Doppler data”

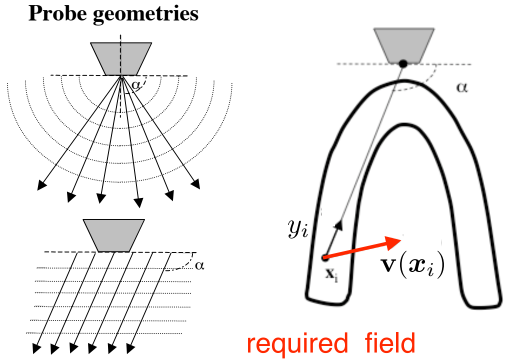
Echo-Doppler imaging system

Scan converted and color coded



axial velocity (scalar)

Probe geometries



beam direction

required field (unknown)

Measurement equation: $y_i = \langle \mathbf{d}(\boldsymbol{x}_i), \mathbf{v}(\boldsymbol{x}_i) \rangle + n_i$

noise

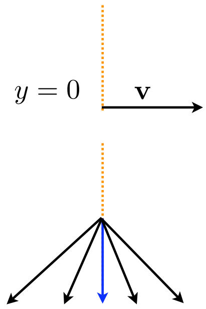
Vector-image reconstruction problem:

Recover the full, continuously defined vector field $\mathbf{v}(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^d$

Doppler reconstruction is ill-posed!

- Limiting factors
  - Partial information
    $\Rightarrow$ vector fields cannot be unambiguously recovered, even for simple motion models (e.g., rigid rotation)
  - Non-uniform data: polar geometry, hand-held probe
  - Noise

blind to orthogonal motion

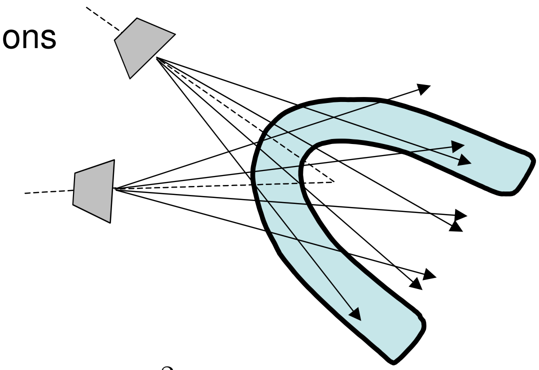


undistinguishable motion vectors

- Proposed solution
  - Integrate the information from multiple views
  - Constrain the solution via problem-specific regularization
  - Develop new vector splines by imposing invariance properties

Formulation of reconstruction problem

- Combine multiview acquisitions



- Variational criterion

$$J(y, \mathbf{v}) = \sum_{i=1}^{N} |y_i - \langle \mathbf{d}(\boldsymbol{x}_i), \mathbf{v}(\boldsymbol{x}_i)\rangle|^2 + \lambda\, R(\mathbf{v})$$

- Quadratic regularization functional: $L_2^d(\mathbb{R}^d) \to \mathbb{R}^+$

$$R(\mathbf{v}) = \langle \mathbf{v}, \mathrm{U}\mathbf{v}\rangle_{L_2^d(\mathbb{R}^d)}$$

U: suitable differential matrix operator

Vector-field regularization

- Shift-invariant regularization functional (via Parseval identity)

$$R(\mathbf{v}) = \langle \mathbf{v}, \mathrm{U}\mathbf{v}\rangle_{L_2^d(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} \int_{\boldsymbol{\omega}\in\mathbb{R}^d} \hat{\mathbf{v}}^H(\boldsymbol{\omega})\hat{\mathbf{U}}(\boldsymbol{\omega})\hat{\mathbf{v}}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$$

- **Invariance theorem**
  The solution of the vector-field reconstruction problem is (sub-space) rotation- and scale-invariant iff:

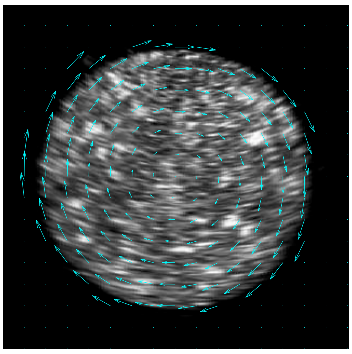
$$\hat{\mathbf{U}}(\boldsymbol{\omega}) = \lambda_{\mathrm{d}} \left(\|\boldsymbol{\omega}\|^{4\alpha_d}\boldsymbol{\omega}\boldsymbol{\omega}^T\right) + \lambda_{\mathrm{r}} \left(\|\boldsymbol{\omega}\|^{4\alpha_r}(\|\boldsymbol{\omega}\|^2\mathbf{I} - \boldsymbol{\omega}\boldsymbol{\omega}^T)\right)$$

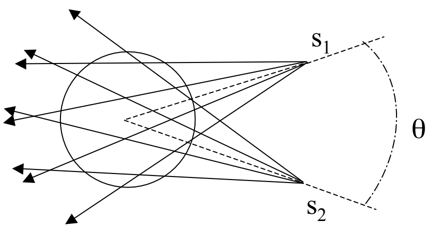
Interpretation: $R(\mathbf{v}) = \lambda_{\mathrm{d}}\, \|\Delta^{\alpha_{\mathrm{d}}}\mathrm{div}(\mathbf{v})\|^2_{L_2(\mathbb{R}^d)} + \lambda_{\mathrm{r}}\, \|\Delta^{\alpha_{\mathrm{r}}}\mathrm{rot}(\mathbf{v})\|^2_{L_2(\mathbb{R}^d)}$

$\Rightarrow$ independent control of irrotational and solenoidal components;
i.e., $\mathbf{v}(\boldsymbol{x}) = \mathbf{v}_{\mathrm{irr}}(x) + \mathbf{v}_{\mathrm{sol}}(x)$ with $\mathrm{rot}(\mathbf{v}_{\mathrm{irr}}) = 0$ and $\mathrm{div}(\mathbf{v}_{\mathrm{sol}}) = 0$

Experimental results

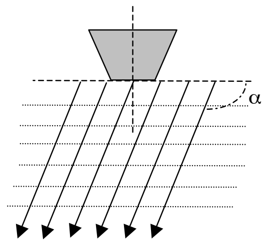
- Numerical procedure (Arigovindan et al., *IEEE-TMI*, 2007)
  - Regularization: $R(\mathbf{v}) = \lambda_{\mathrm{d}} \, \|\Delta^{\frac{1}{2}} \mathrm{div}(\mathbf{v})\|^2_{L_2(\mathbb{R}^2)} + \lambda_{\mathrm{r}} \, \|\Delta^{\frac{1}{2}} \mathrm{rot}(\mathbf{v})\|^2_{L_2(\mathbb{R}^2)}$
  - Discretization of 2D reconstruction problem in a uniform B-spline basis
    $\Rightarrow$ sparse, band-diagonal system of equations
  - Efficient matrix solver (Matlab)
- Physical phantom experiment: rotating sponge



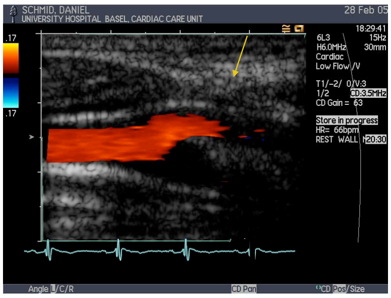


Patient data: carotid artery

Siemens 6L3 probe

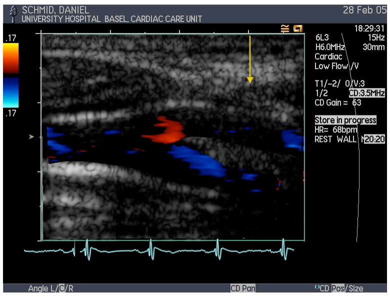


α=110°



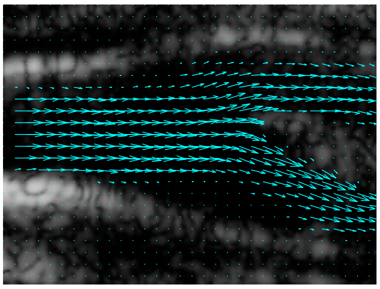
α=70°

α=90°

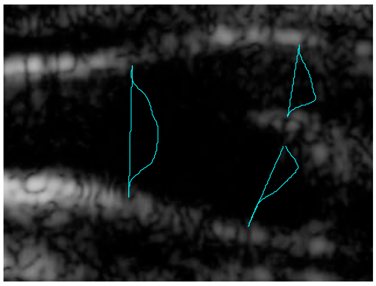


(Arigovindan et al., IEEE-TMI, 2007)

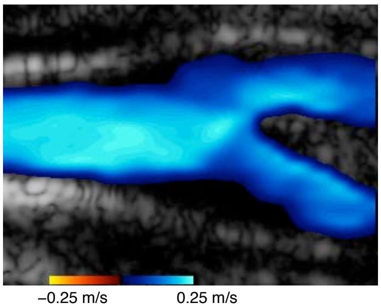
Reconstruction results: carotid bifurcation



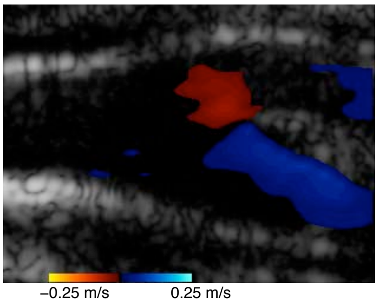
reconstructed flow field



velocity profiles



horizontal velocity



vertical velocity

(Collaboration with Dr. Hunziker, Univ. Hospital, Basel)

CONCLUSIONS

- Alternative sampling formulations with noise: deterministic vs. stochastic
  - Variational, minimax, or MMSE signal reconstructions
  - Efficient computational solutions: digital filters
- Globally optimal solution lives in integer-shift-invariant subspace
  - Optimal space tied to regularization (resp., whitening) operator
  - Optimal reconstruction is generally not bandlimited
  - Optimal estimator is a generalized smoothing spline
  - All formulations lead to the same spline-based reconstruction algorithm
- Fractional/polynomial splines are optimal for the estimation of fractal processes (fBm)
  - Self-similar defining operator
  - Solution of same type of differential equation
- Further directions for sampling research
  - Nonuniform sampling, multidimensional
  - Vector fields, ...

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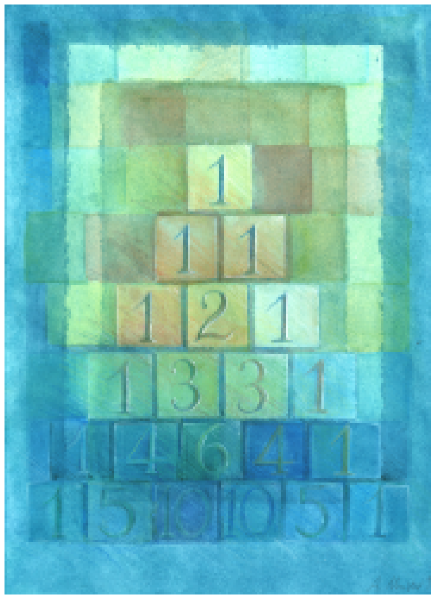
- Sathish Ramani
- Annette Unser, Artist

+ many other researchers and graduate students



Preprints and demos: **http://bigwww.epfl.ch/**

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