SELF-SIMILARITY, FRACTAL MODELS AND SPLINE APPROXIMATION

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ABSTRACT

Invariance is an attractive principle for specifying image processing algorithms. In this presentation, we concentrate on self-similarity—more precisely, shift, scale and rotation invariance—and identify the corresponding class of operators in d dimensions. For $d = 1$, there is no rotation and the scale-invariant differential operators are the fractional derivatives: $\partial_\tau^\gamma \longleftrightarrow (-j\omega)^{\frac{\gamma}{2}-\tau}(j\omega)^{\frac{\gamma}{2}+\tau}$, which are indexed by the order γ and a phase factor τ [1]. In higher dimensions, the family reduces to the fractional Laplacians: $(-\Delta)^{\gamma/2} \longleftrightarrow \|\boldsymbol{\omega}\|^\gamma$ with $\gamma \in \mathbb{R}^+$ [2].

We then specify some corresponding differential equations:

$$\partial_\tau^\gamma s(x) = r(x) \quad \text{or} \quad (-\Delta)^{\gamma/2} s(\boldsymbol{x}) = r(\boldsymbol{x})$$

and show that the solution s (in the distributional sense) is either a fractional Brownian motion [3] or a fractional spline [4], depending on the nature of the system input r (driving term): stochastic (white noise) or deterministic (stream of Dirac impulses). For dimensions higher than one, the relevant stochastic entities are fractional Brownian fields which are in direct functional correspondence with the deterministic polyharmonic splines [2].

The scale-invariance of the operator has two remarkable consequences: (1) the statistical self-similarity of the fractional Brownian motion [5], and (2) the fact that the fractional/polyharmonic splines specify a multiresolution analysis of L_2 and lend themselves to the construction of wavelet bases. We prove that these wavelets essentially behave like the operator from which they are derived, and that they are ideally suited for the analysis of signals with fractal characteristics (fractional differentiation, and whitening property) [6]. Another interesting property is that the corresponding spline families are closed with respect to fractional differentiation. In particular, this allows us to specify to a new dual-tree B-spline wavelet transform where the wavelets in the two branches are exact Hilbert transforms of each other [7].

The differential operators ∂_τ^γ or $(-\Delta)^{\gamma/2}$ may also be used to define a scale-invariant energy measure, which provides a regularization functional for interpolating or fitting the noisy samples of a signal [1, 8]. Specifically, we prove that the corresponding variational (or smoothing) spline estimator,

$$s_\lambda(\boldsymbol{x}|f) = \arg \min_{s \in L_2(\mathbb{R}^d)} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} |f[\boldsymbol{k}] - s(\boldsymbol{k})|^2 + \lambda \|(-\Delta)^{\gamma/2} s\|^2_{L_2(\mathbb{R}^d)},$$

is a cardinal fractional spline of order 2γ, which admits a stable representation in a B-spline basis. As $\lambda \to 0$, it yields the spline interpolator of the input sequence $f[\boldsymbol{k}]$. We also show that, for an adequate choice of λ, the smoothing spline $s_\lambda(\boldsymbol{x}|f)$ provides the hybrid Wiener (MMSE) estimator of a fractional Brownian field of order γ corrupted by additive Gaussian white noise [5]. Another remarkable feature of the above spline estimator, beside its stochastic stochastic optimality, is that the reconstruction algorithm commutes with any affine transformation of the coordinate system. This is a fundamental universality property that distinguishes the present splines from any other interpolation algorithms.

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