SPLINE MULTIRESOLUTIONS AND WAVELET TRANSFORMS

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ABSTRACT

We present an extension of the family of Battle/Lemarié spline wavelet transforms. By relaxing the intra-level orthogonality constraint, we show how to construct generalized polynomial spline scaling functions and wavelets that span the same multiresolution spaces, but can also exhibit very distinct properties. Particular examples in this family include the B-spline wavelets of compact support that are optimally localized in time-frequency; the cardinal spline wavelets that have the fundamental interpolation property; and the dual spline wavelets (the biorthogonal complement of the B-spline wavelets). We provide a full characterization of the digital filters for the corresponding fast wavelet transform algorithms. We also discuss the asymptotic properties of these representations, and indicate the link with Shannon's sampling theory and the Gabor transform.

1. INTRODUCTION

Polynomial splines have a number of attractive properties that make them useful in a variety of applications. These features include good smoothness properties, a simple analytical form (piecewise polynomial), and the fact that they have convenient representations in terms of simple basis functions (B-splines).

Due to these properties, polynomial splines were among the first functions used to construct orthogonal wavelet transforms [1, 2, 3]. More recently, Chui and Wang, as well as our group, independently introduced the B-spline wavelets of compact support that are the natural counterpart of the classical B-spline functions [4, 5].

The purpose of this presentation is to unify those results by introducing an extended class of polynomial representations and wavelet transforms. We will emphasize a signal processing formulation and also relate these approaches to some recent filter-based algorithms for polynomial spline approximation [6].

One of the main points of this work will be to show that it is possible to construct a variety of equivalent scaling functions and wavelets with some very specific properties by simply relaxing the standard intra-level orthogonality constraint. The design or selection criteria considered here are : (a) simplicity of implementation (FIR or recursive filter banks), (b) near-optimal time-frequency localization, and (c) good bandpass characteristics of the equivalent filter bank. Since it is not possible to enforce all of these properties simultaneously (for instance, property (b) and (c) are contradictory), the selection of the most appropriate representation ultimately depends on the application.

Note that the present family of spline wavelet transforms falls into the general framework of biorthogonal wavelet transforms. However, the present method of construction has the distinctive feature that it preserves the orthogonality of the wavelets across scales, a property that is usually lost in other biorthogonal schemes.

2. POLYNOMIAL SPLINE PYRAMIDS

2.1 Preliminary definitions

The basic function space $V_{(0)}$ considered here is the space of polynomial splines of order n (n odd) with knots at the integers. Specifically, $V_{(0)}$ is the sub-set of functions in L_2 that are of class C^{n-1} and are equal to a polynomial of degree n on each interval $[k, (k+1)]$ with $k \in Z$. A classical result in approximation theory is that this function space can be defined as (c.f. [7, 8])

$$V_{(0)} = \left\{ g_{(0)}(x) = \sum_{k=-\infty}^{+\infty} c(k)\varphi_b^n(x-k),\ c \in l_2 \right\} \tag{1}$$

where $\varphi_b^n(x)$ is Schoenberg's central B-spline of order n; this scaling function is generated by repeated convolution of a B-spline of order 0:

$$\varphi_b^n(x) = \varphi_b^0 * \varphi_b^{n-1}(x) \tag{2}$$

where $\varphi_b^0(x)$ is the characteristic function in the interval $[-\frac{1}{2}, +\frac{1}{2})$. This definition states that any polynomial spline can be represented by a weighted sum of shifted B-splines and is therefore entirely characterized by its sequence of B-spline coefficients. A fundamental characteristic of B-splines is their compact support, the property that makes them useful in a variety of applications [9].

We define the discrete B-spline kernels by sampling the B-splines at the integers

$$b^n(k) := \varphi_b^n(x)\big|_{x=k}. \tag{3}$$

We will also use some special notations to represent certain operators that act on discrete signals. A list of these operations is given in Table I.

TABLE I
DISCRETE OPERATORS AND THEIR EFFECT IN THE Z-TRANSFORM DOMAIN

Operator	signal domain	z-transform
convolution inverse	$(a)^{-1}(k)$	$1/A(z)$
square-root inverse	$(a)^{-1/2}(k)$	$1/\sqrt{A(z)}$
modulation	$\tilde{a}(k)=(-1)^k a(k)$	$A(-z)$
time reversal	$a'(k)=a(-k)$	$A(1/z)$
up-sampling	$[a]_{\uparrow 2}(k)$	$A(z^2)$
down-sampling	$[a]_{\downarrow 2}(k)$	$\frac{1}{2}\left(A(z^{\frac{1}{2}})+A(-z^{\frac{1}{2}})\right)$

2.2 Multiresolution spline approximation

It is not difficult to verify that the B-splines of order n (n odd) satisfy the two-scale relation

$$\varphi_b^n(x/2)=\sum_{k=-\infty}^{+\infty}u_2^n(k)\varphi_b^n(x-k), \tag{4}$$

where u_2^n is the symmetrical binomial kernel of order n

$$u_2^n(k) \xleftrightarrow{\;Fourier\;} 2\cos^{n+1}(\pi f). \tag{5}$$

We can therefore consider a sequence of nested polynomial spline subspaces that forms a multiresolution analysis of L_2 in the sense defined by Mallat: $\cdots\supset V_{(-1)}\supset V_{(0)}\supset V_{(1)}\cdots\supset V_{(i)}\cdots$ [3]. $V_{(i)}$ is the subspace of polynomial splines of order n with knot points $k\cdot 2^i$, $k\in Z$.

A minimum error (L_2-norm) polynomial spline approximation of a function $g\in L_2$ at a given resolution (i) is obtained by orthogonal projection on $V_{(i)}$. This approximation, which we denote by $g_{(i)}$, can be expressed as

$$g_{(i)}=2^{-i}\sum_{i\in Z}\langle g,\mathring{\varphi}_{i,k}\rangle\varphi_{i,k}, \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the standard L_2 inner product. The basis functions $\varphi_{i,k}=\varphi^n(2^{-i}x-k)$ and $\mathring{\varphi}_{i,k}=\mathring{\varphi}^n(2^{-i}x-k)$ are biorthogonal. They are defined through the following formulas:

$$\varphi^n(x)=\sum_{k=-\infty}^{+\infty}p(k)\varphi_b^n(x-k) \tag{7}$$

$$\mathring{\varphi}^n(x)=\sum_{k=-\infty}^{+\infty}\left(p'*b^{2n+1}\right)^{-1}(k)\varphi_b^n(x-k), \tag{8}$$

where the operators $(\cdot)^{-1}$ and ' are defined in Table I. This representation is parametrized by the sequence p which can be any invertible convolution operator from l_2 into itself. φ^n is a generalized polynomial spline scaling function.

Some examples of scaling functions for $n=3$ (cubic splines) corresponding to different choice of the parameter p are shown in Fig. 1. These functions all have some specific properties : (a) orthogonality, (b) compact support, (c) interpolation property, and (d) biorthogonal complement of the B-spline. The corresponding values of the parameter p are given in Table II.

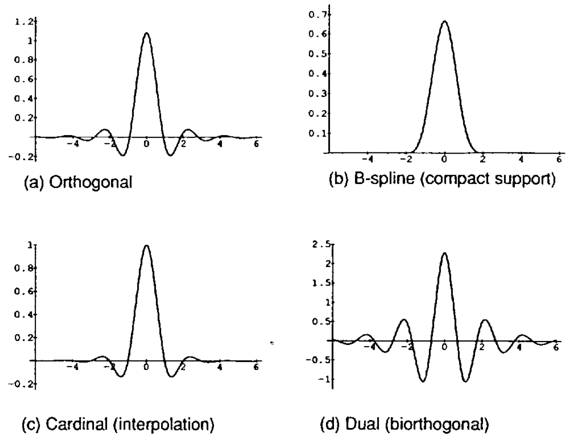


Fig. 1. Examples of cubic spline scaling functions.

We have shown previously that the interpolating and orthogonal spline scaling functions, as well as their dual, converge to sinc(x) as the order of the spline goes to infinity [6, 10]. These asymptotic results provide an explicit link with the classical procedure dictated by Shannon's sampling theorem. Note that the dual basis function in (6) acts in a way that is analogous to the anti-aliasing lowpass filter required in conventional sampling theory.

3. POLYNOMIAL SPLINE WAVELETS

3.1 The B-spline wavelet

To obtain the wavelet transform, we consider the sequence of orthogonal complementary spaces $\cdots,W_{(-1)},W_{(0)},W_{(1)}\cdots,W_{(i)},\cdots$ with $V_{(i-1)}=W_{(i)}\oplus V_{(i)}$. A fundamental result is that the basic residual space $W_{(0)}$ is generated from the integer translations of a basic wavelet $\psi(x)$; for example, the B-spline wavelet.

The B-spline wavelets of compact support are the wavelet analogue of the classical B-splines; they can be defined as follows

$$\psi_b^n(x/2)=\sum_{k=-\infty}^{+\infty}\tilde{u}_2^n*\tilde{b}^{2n+1}(k+1)\varphi_b^n(x-k), \tag{9}$$

where the symbol "~" is the modulation operator (cf. Table I) and where u_2^n is the binomial kernel defined by (5). An attractive feature of these wavelets is their excellent time/frequency localization [5]. In fact, we have shown that these functions converge to modulated Gaussians (real part of Gabor functions) as the order of the spline goes to infinity. Based on this asymptotic result, we get the following approximation

$$\psi_b^n(x)\cong\frac{4a^{n+1}}{\sqrt{2\pi(n+1)}\sigma}\cos\left(2\pi f_0(2x-1)\right)\exp\left(-\frac{(2x-1)^2}{2\sigma^2(n+1)}\right)$$

with a=0.697066, f_0=0.409177 and σ^2=0.561145. For n=3 (cubic splines), the relative L_2 approximation error is 2.6% and the product of the time-frequency uncertainties is within 2% of the limit specified by the uncertainty principle.

TABLE II
SPECIFIC PARAMETERS FOR VARIOUS SETS OF SPLINE SCALING FUNCTIONS AND WAVELETS

Representations	p	q
orthogonal	$\left(b^{2n+1}\right)^{-1/2}$	$\left(\left[\tilde{b}^{2n+1} * b^{2n+1}\right]_{\downarrow 2} * b^{2n+1}\right)^{-1/2}$
basic (B-splines)	δ_0 (identity)	δ_0 (identity)
cardinal (C-splines)	$\left(b^{n}\right)^{-1}$	$\left(\left[b^{n} * \tilde{u}_2^{n} * \tilde{b}^{2n+1}\right]_{\downarrow 2}\right)^{-1}$
dual (D-splines)	$\left(b^{2n+1}\right)^{-1}$	$\left(\left[\tilde{b}^{2n+1} * b^{2n+1}\right]_{\downarrow 2} * b^{2n+1}\right)^{-1}$

3.2 Generalized spline wavelets

A full polynomial spline wavelet expansion of a function $g \in L_2$ can be obtained as follows:

$$g = \sum_{(i,k)\in Z^2} 2^{-i} \langle g, \mathring{\psi}_{i,k} \rangle \psi_{i,k} \quad (10)$$

where the generalized spline wavelets $\psi_{i,k} = \psi^n(2^{-i}x-k)$ and $\mathring{\psi}_{i,k} = \mathring{\psi}^n(2^{-i}x-k)$ are obtained from the dilation (index i) and translation (index k) of the basic wavelets :

$$\psi^n(x) = \sum_{k=-\infty}^{+\infty} q(k)\varphi_b^n(x-k) \quad (11)$$

$$\mathring{\psi}^n(x) = \sum_{k=-\infty}^{+\infty} \left(q' * b^{2n+1} * \left[\tilde{b}^{2n+1} * b^{2n+1}\right]_{\downarrow 2}\right)^{-1}(k)\varphi_b^n(x-k), \quad (12)$$

which form a biorthogonal pair. These functions are defined in terms of the B-spline wavelet (9), and a parameter sequence q which is an invertible convolution operator.

3.3 Specific wavelet transforms and their properties

The polynomial spline wavelets that are the counterpart of the scaling functions in Fig. 1 are shown in Fig. 2. Each case corresponds to a specific choice of the parameter q dictated by our desire to satisfy a certain property. The corresponding parameter values can be found in Table II.

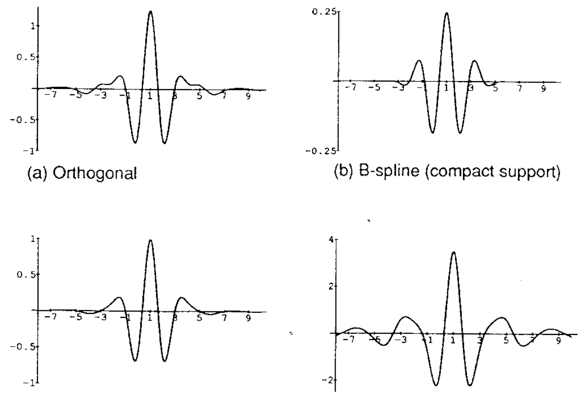
3.3.1 Battle-Lemarié wavelet transform (O-splines)

The sequence p and q can be selected so that the basis functions are orthogonal. If we also impose symmetry, we get the well known Battle-Lemarié spline wavelets [1, 2].

3.3.2 Wavelet transform of compact support (B-splines)

The main property of this representation is the compact support of the basis functions. In most applications, this property will translate in systems of equations that are band-diagonal and can be solved very efficiently. In terms of numerical stability and ease of computation, this representation appears to offer the same advantages as the classical B-splines used routinely in a variety of engineering and applied mathematics applications [9].

The other attractive property of this transform is the near optimal time-frequency localization of the basis functions. The B-spline wavelet transform is in this sense very similar to a hierarchical Gabor transform with the advantage that it has a fast algorithm associated with it. It is therefore also well suited for the analysis of non-stationary signals.



(a) Orthogonal (b) B-spline (compact support) (c) Cardinal (interpolation) (d) Dual (biorthogonal)

Fig. 2. Examples of cubic spline wavelets at level (1) (step size Δ=2).

3.3.3 Cardinal wavelet transform (C-splines)

The cardinal representation has the unique property that the wavelet coefficients are precisely the samples of the underlying residual signal $g_{(i-1)} - g_{(i)} \in W_{(i)}$. This is only possible because the cardinal scaling functions and wavelets vanish at all the integers except at their origin where they take the value one. This representation is therefore the most appropriate for visualizing the underlying signals. Furthermore, the complexity of the reconstruction algorithm can be reduced by a factor of two since it is only necessary to compute the finer level coefficients that are between knot points (interpolation). The cardinal spline representation is very similar to the orthogonal one. In fact, the orthogonal and cardinal spline wavelets all tend to the modulated sinc-wavelet (ideal bandpass filter) as the order of the spline goes to infinity [11].

3.3.4 Dual wavelet transform (D-splines)

This function is the biorthogonal complement of the B-spline wavelet. The dual representation therefore corresponds to the flow graph transpose of the B-spline case. The corresponding analysis functions are the B-spline wavelets which are also very similar to Gabor functions. This property together with the simple FIR form of the decomposition algorithm may turn out to be quite useful for certain computer vision applications such as edge detection or texture segmentation.

4. FAST WAVELET ALGORITHM

Let us define the quantities

$$c_{(i)}(k) = 2^{-i}\langle g, \mathring{\varphi}_{i,k}\rangle \quad (13)$$

$$d_{(i)}(k) = 2^{-i}\langle g, \mathring{\psi}_{i,k}\rangle, \quad (14)$$

which correspond to the expansion coefficients in (6) and (10), respectively. We will now show that these quantities can be computed efficiently by extending Mallat's quadrature mirror filter algorithm to the case of non-orthogonal basis functions.

In practice, the signal to be analyzed is specified by a sequence of sample values $\{g(k)\}$. The starting point of the analysis will be to map this sequence into a continuous-time signal $g_{(0)}(x)$ that provides our signal representation at the finer resolution level (i=0). This interpolation problem can be solved efficiently by digital filtering [12]. In the present context, this leads to the initialization procedure

$$c_{(0)}(k) = \left(p * b^n\right)^{-1} * g(k) \tag{15}$$

where $(p*b^n)^{-1}$ is the convolution inverse of the sampled scaling function $\varphi^n(x)\big|_{x=k}$ (cf. Eq. (7)).

The multiresolution spline and wavelet coefficients down to a certain resolution level I are then computed by filtering and down-sampling by a factor of two :

$$\begin{cases} c_{(i+1)}(k) = [\mathring{v} * c_{(i)}]_{\downarrow 2}(k) \\ d_{(i+1)}(k) = [\mathring{w} * c_{(i)}]_{\downarrow 2}(k) \end{cases} \tag{16}$$

The procedure is applied iteratively starting from the finer resolution representation $c_{(0)}$. After I iterations of this process, we obtain the following wavelet representation of our signal

$$g_{(0)}(x) = \sum_{k \in Z} c_{(I)}(k)\varphi_{I,k} + \sum_{i=1}^{I}\sum_{k \in Z} d_{(i)}(k)\psi_{i,k} \tag{17}$$

The quantities $\left\{d_{(1)}, d_{(2)}, \cdots d_{(I)}\right\}$ are the so-called wavelet coefficients; the sequence $\left\{c_{(I)}\right\}$, on the other hand, codes for the lower resolution signal at resolution (I).

The indirect wavelet transform (reconstruction) is implemented in a similar fashion (up-sampling and post-filtering) by successively reconstructing the spline coefficients starting at the bottom of the pyramid

$$c_{(i)}(k) = v * \left[c_{(i+1)}\right]_{\uparrow 2}(k) + w * \left[d_{(i+1)}\right]_{\uparrow 2}(k). \tag{18}$$

The digital filters in (16) and (18) are given by

$$\begin{cases} \mathring{v}(k) = \frac{1}{2}\left[\left(p * b^{2n+1}\right)^{-1}\right]_{\uparrow 2} * p * b^{2n+1} * u_2^n(k) \\ \mathring{w}(k+1) = \frac{1}{2}\left[\left(q * b^{2n+1}\right)^{-1}\right]_{\uparrow 2} * p * \tilde{u}_2^n(k) \\ v(k) = [p]_{\uparrow 2} * (p)^{-1} * u_2^n(k) \\ w(k-1) = [q]_{\uparrow 2} * (p)^{-1} * \tilde{u}_2^n * \tilde{b}^{2n+1}(k) \end{cases} \tag{19}$$

and define a perfect reconstruction filter bank. The analysis and synthesis filters for the polynomial spline basis functions and wavelets displayed in Fig. 1 and 2 can be obtained by substitution of the corresponding values of the parameters p and q in Table II.

5. CONCLUSION

In this paper, we have described an extended class of polynomial spline transforms. Beside having all the advantages usually associated with the wavelet transform (hierarchical decomposition, fast algorithm, etc...), these representations are the only ones for which it is possible to obtain explicit formulas (piecewise polynomials) for the corresponding scaling functions and wavelets in the signal domain. The regularity of these functions is simply controlled by the order of the spline n.

In addition, we have identified several important properties that should make these representations useful in a variety of applications. The B-spline wavelet transform appears to be the representation of choice if the determining factor is the time-frequency localization of the basis functions. The cardinal and orthogonal representation, on the other hand, have good bandpass characteristics and are therefore well suited for coding; the quality of the approximation of an ideal bandpass filter can be improved by simply increasing n. The cardinal spline wavelet is the sole function in this family to provide a signal decomposition in terms of the sample values of the underlying continuous functions and is therefore most appropriate for visualization and conventional signal processing. Finally, the dual spline representation has the advantage of a very simple (FIR) decomposition algorithm, while the impulse response of the equivalent analysis filter is very similar to a Gabor function.

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