On the Optimality of Ideal Filters for Pyramid and Wavelet Signal Approximation

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Abstract—**The reconstructed lowpass component in a quadrature mirror filter (QMF) bank provides a coarser resolution approximation of the input signal. Since the outputs of the two QMF branches are orthogonal, the transformation that provides the maximum energy compaction in the lowpass channel is also the one that results in the minimum approximation error. This property is used as a common strategy for the optimization of QMF banks, orthogonal wavelet transforms, and least squares pyramids. A general solution is derived for the QMF bank that provides the optimal decomposition of an arbitrary wide sense stationary process. This approach is extended to the continuous case to obtain the minimum error approximation of a signal at a given sampling rate. In particular, it is shown that the sinc-wavelet transform is optimal for the representation at all scales of signals with non-increasing spectral density.**

I. Introduction

The equivalence between the wavelet transform, quadrature mirror filters (QMF), and perfect reconstruction filter banks in general is now well recognized [1]–[3]. The wavelet transform provides a vector space interpretation of a subband decomposition, while the filter bank formulation is a convenient description of the fast wavelet transform algorithm. Traditionally, the design of QMF banks has been guided by the desire to approximate a perfect half-band filter in the best possible way [4]–[6]. More recently, the tendency has shifted toward designing signal-adapted filters and wavelets [7], [8]. Interestingly, there are many cases in which both strategies yield very similar results. For instance, Delsarte *et al.* observed that the ideal half-band filter bank is optimal in the sense that it minimizes the variance of the highpass component for stationary signals with non-increasing spectral density [7].

Another commonly used class of multiresolution techniques for signal analysis and compression is the pyramid decomposition, introduced by Burt and Adelson [9]. The transmitted information is the difference between successive levels of the pyramid. To improve coding performance, it is desirable to minimize the variances of those residues, which naturally leads to the concept of a least squares pyramid [10]. A signal-dependent approach was considered in [11]. In fact, there is a direct relationship between such least squares pyramids and QMF banks [2], [10].

This note deals with the problem of finding optimal wavelet or pyramid decompositions for certain families of signals. Unlike most other studies, no special constraints are imposed on the class of admissible filters.

II. QMF Banks, Wavelet Transforms, and Pyramids

The problem of defining an optimal wavelet transform can be entirely formulated in terms of the design of a QMF bank. The goal of this section is therefore to review the most important properties of such filter banks and also indicate the link with least squares pyramids. This will justify our optimization approach, which is to maximize the signal contribution in the lowpass channel. The results presented here apply to the deterministic case in which all signals are assumed to have finite energy (i.e., $x \in l_2$).

A. Quadrature Mirror Filter Bank

A perfect reconstruction quadrature mirror filter bank is represented in Fig. 1. It is entirely specified by the filter $H(f)$ which has to satisfy the property [12]

$$|H(f)|^2 + |H(f + \tfrac{1}{2})|^2 = 2. \tag{1}$$

The corresponding highpass filter $G(f)$ is given by

$$G(f) = e^{-j2\pi f} H^*(f + \tfrac{1}{2}). \tag{2}$$

The block diagram in Fig. 1 also provides the basis for the fast wavelet transform [1]. The wavelet coefficients at the finer scale correspond to the coefficients x_g in the highpass branch. Similarly, the wavelet coefficients at coarser scales can be obtained by reapplying the analysis procedure iteratively on the decimated sequence x_h in the lowpass channel.

While an orthogonal wavelet transform always corresponds to a perfect reconstruction QMF bank, the converse statement is only true if $H(f)$ satisfies some additional constraints [1]. The simplest of these conditions is $H(1/2) = 0$. The number of zeros of $H(f)$ at $f = 1/2$ also determines the regularity (or smoothness) of the underlying basis functions [13].

B. Energy Analysis

It is not difficult to verify that the QMF constraint (1) has the following implications:

Proposition 1: A perfect reconstruction QMF bank has the energy (or squared l_2-norm) preserving properties:

$$\|x_h\|^2 = \|x_{hh}\|^2 \tag{3}$$

$$\|x_g\|^2 = \|x_{gg}\|^2 \tag{4}$$

$$\|x\|^2 = \|x_h\|^2 + \|x_g\|^2 = \|x_{hh}\|^2 + \|x_{gg}\|^2 \tag{5}$$

where $x \in l_2$ is the input signal and where x_h, x_{hh}, x_g, and x_{gg} are defined in Fig. 1.

These properties are better known in the context of the wavelet transform. They express the fact that the basis functions associated with the synthesis filters define an orthogonal set (see [3], for an interpretation of the wavelet transform in l_2). In particular, the system is such that the outputs of the lowpass and highpass channels are orthogonal.

Proposition 1 suggests a number of alternative approaches to the optimization of such wavelet filters. For instance, finding the filter that minimizes the approximation error $\|e\|^2 = \|x - x_{hh}\|^2$ is equivalent to maximizing $\|x_h\|^2$. The advantage of this latter approach is that it lends itself quite naturally to a number of generalizations; for example, the case of a P-band system, or the sampling of a continuous-time signal considered in Section IV.

Such an optimization could in principle be based on the following formula for the energy in the lowpass channel

$$\|x_h\|^2 = \tfrac{1}{2}\int_0^1 |H(f)|^2 X(f)|^2\, df + \tfrac{1}{2}\int_0^1 H^*(f)H(f+\tfrac{1}{2})X(f)X^*(f+\tfrac{1}{2})\, df. \tag{6}$$

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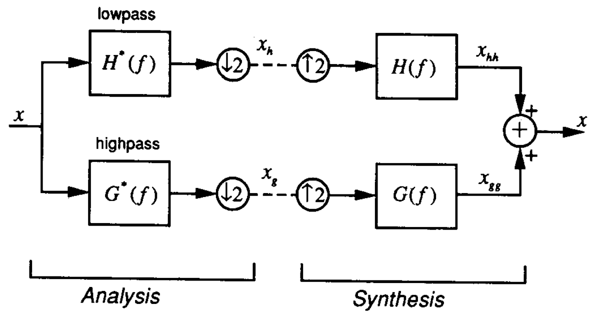

Fig. 1. Quadrature mirror filter bank.

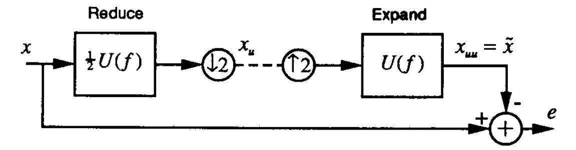

Fig. 2. Schematic representation of the pyramid decomposition algorithm. The input signal is first reduced (filtering and downsampling) and then expanded (upsampling and filtering) to provide the coarser resolution approximation ($\tilde{x}$). $\tilde{x}$ is then subtracted from the input x to yield the error signal e, the information transmitted to the coder.

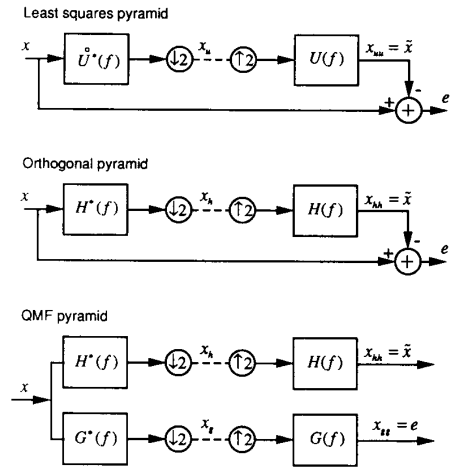

Fig. 3. Equivalent interpretations of the least squares pyramid decomposition algorithm.

However, the purely deterministic case is not so interesting because this error can usually be made zero by selecting the approximation space generated by the signal itself, as is briefly discussed in Section II-C below. In fact, such a formulation is not entirely appropriate because in most applications the exact position of the signal with respect to the origin is unknown *a priori*. One way to reflect this fact is to use a modified criterion corresponding to the average energy over all possible integer shifts of our input $x \in l_2$, which results in a vanishing of the second term in (6). A second approach, which is the one taken here, is to investigate the stochastic case (cf. Section III). These two approaches are essentially equivalent and all our results for stationary processes can be easily adapted to the "shift-invariant" deterministic case by simply replacing the power spectral density $S_x(f)$ by the square modulus of the Fourier transform of x.

C. The Least Squares Pyramid

The same formulation is also applicable for the design of a pyramid that minimizes the energy of the residual error. The basic pyramid structure introduced by Burt and Adelson is shown in Fig. 2 [9]. A full pyramid representation is obtained by applying this decomposition in a hierarchical fashion. The main difference with the wavelet transform is that this representation is overcomplete. However, it is simpler to implement and the redundancy can be used advantageously to reduce the effect of errors using quantization feedback [10], [14].

In general, the analysis and synthesis filters need not be identical and the system can be designed to minimize the approximation error [10]. The corresponding approximation space is specified by the synthesis filter u and is defined by

$$V_n = \left\{ \tilde{x}(k) = \sum_{l \in Z} c(l) u(k - 2l),\ c \in l_2 \right\}. \tag{7}$$

It can be shown that a necessary and sufficient condition for V_u to be a closed subspace of l_2 is

$$0 < m \le |U(f)|^2 + |U(f + \tfrac{1}{2})|^2 \le M < +\infty \qquad a.e., \tag{8}$$

where m and M are two constants and where $U(f)$ is the Fourier transform of $u(k)$. The optimal (biorthogonal) prefilter that provides the least squares approximation of the input signal is

$$\mathring{U}(f) := \frac{2U(f)}{|U(f)|^2 + |U(f + \frac{1}{2})|^2}. \tag{9}$$

By definition, the error in the least squares pyramid is orthogonal to $\tilde{x}$, which implies that

$$\|e\|^2 = \|x\|^2 - \|\tilde{x}\|^2. \tag{10}$$

An orthogonal basis of the same approximation space can be constructed by suitable linear combination of the generating sequence u. The orthogonalized synthesis filter is identical to the conjugate analysis filter and is given by

$$H(f) = \frac{\sqrt{2}\, U(f)}{\sqrt{|U(f)|^2 + |U(f + \frac{1}{2})|^2}}. \tag{11}$$

Clearly, it also satisfies condition (1), which suggests a close connection between this pyramid and the QMF bank in Fig. 1. In fact, there is a direct equivalence expressed by the set of identities

$$\begin{cases} \tilde{x} = x_{uu} = x_{hh} \\ e = x - \tilde{x} = x_{gg}. \end{cases} \tag{12}$$

It is illustrated graphically in Fig. 3. Hence, the conclusion that the problems of optimal design for the wavelet transform and least squares pyramid are equivalent. What can also be seen from those results is that the optimization error in the deterministic case can be made zero by simply choosing $u = x$ [or its orthogonalized version given by (11)], provided that the signal satisfies the admissibility condition (8). This is very similar to the specification of a deterministic matched filter.

III. Approximation of Stationary Processes

The previous formulas apply to the representation of deterministic signals in l_2. We now consider the case in which the sequence $\{x(k)\}_{k \in Z}$ is a realization of a wide sense stationary process with power spectral density (PSD) $S_x(f)$.

A. Link Between the Deterministic and Stochastic Formulations

The previous notion of energy (squared l_2-norm) needs now to be replaced by the average power of a signal, which is achieved through the following limit process

$$E\{x \cdot y\} = \lim_{N \to +\infty} \frac{1}{2N+1} E\left\{ \sum_{k=-N}^{+N} x(k)\, y(k) \right\}, \tag{13}$$

$$S_x(f) = \lim_{N \to +\infty} \frac{1}{2N+1} E\left\{ \left| \sum_{k=-N}^{+N} x(k) e^{-j2\pi k f} \right|^2 \right\} \tag{14}$$

where the expectation is taken over the ensemble of all realizations. This correspondence permits all previous norm equations to be rewritten in terms of the expectation operator $E\{\cdot\}$. In particular, the orthogonality of x_{hh} and x_{gg} now implies that the lower resolution approximation $\tilde{x}$ and the error signal are uncorrelated. Equivalently, we obtain the direct translation of the least squares condition (10)

$$E\{e^2\} = E\{x^2\} - E\{\tilde{x}^2\}. \tag{15}$$

The simplest way to evaluate $E\{\tilde{x}^2\}$ is to use the identities (3) and (12) and observe that downsampling in the stationary case divides the average power by the sampling factor. Therefore,

$$E\{\tilde{x}^2\} = E\{x_h^2\} = \frac{1}{2} \int_0^1 |H(f)|^2 S_x(f)\, df. \tag{16}$$

The main difference with the deterministic case (6) is that the 2nd cross-term has disappeared.

B. Optimal Filter Selection

Our main result can now be stated in the following theorem:

Theorem 1: For a stationary process with PSD $S_x(f)$, the residual error is minimized if the frequency response of the filter $H(f)$ is such that for $f \in [0, 1/2]$

$$|H(f)| = \begin{cases} \sqrt{2}, & \text{if } f \in \Omega_1 = \{f \in [0, \tfrac{1}{4}) \colon S_x(f) \geq S_x(f + \tfrac{1}{2})\} \\ & \quad \cup \{f \in (\tfrac{1}{4}, \tfrac{1}{2}] \colon S_x(f) > S_x(f + \tfrac{1}{2})\} \\ 0, & \text{if } f \in \Omega_2 = \{f \in [0, \tfrac{1}{4}) \colon S_x(f) < S_x(f + \tfrac{1}{2})\} \\ & \quad \cup \{f \in (\tfrac{1}{4}, \tfrac{1}{2}] \colon S_x(f) \leq S_x(f + \tfrac{1}{2})\}. \end{cases}$$

Ω_1 and Ω_2 are two sets of measure $1/4$ that partition the frequency interval $[0, 1/2]$, with the property

$$f \in \Omega_1 \Leftrightarrow \tfrac{1}{2} - f \in \Omega_2. \tag{17}$$

The optimal choice is therefore to select an ideal filter with a bandpass region Ω_1 and a stop band Ω_2. This solution has also the property that the coefficients in both channels are decorrelated for they do not share common spectral components. This simple selection process is illustrated in Fig. 4.

Theorem 1 implies that the half-band ideal lowpass filter is optimal whenever

$$\forall f \in [0, \tfrac{1}{4}],\ S_x(f) \geq S_x(f + \tfrac{1}{2}). \tag{18}$$

This constraint is not overly restrictive and should be applicable to most practical cases. Because of the symmetry and periodicity of the spectral density, (18) can be translated into a simple inequality constraint for the highpass portion of the spectrum with $f \in (1/4, 1/2)$, as illustrated in Fig. 4(a). In particular, condition (18) is satisfied when $S_x(f)$ is a nonincreasing function for $f \in (0, 1/2)$.

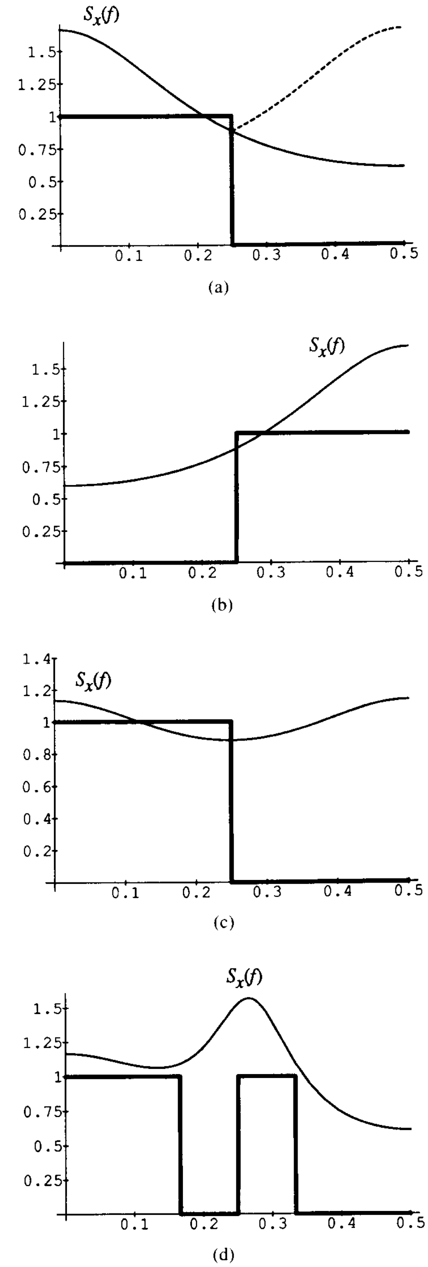

Fig. 4. Examples of optimal filter selection. (a) lowpass spectrum (Markov-1 with $\rho = 0.25$), (b) highpass spectrum (Markov-1 with $\rho = -0.25$), (c) spectrum with an axis of symmetry around $f = 1/4$, (d) composite spectrum.

Proof: Starting from (16), the range of integration is split into two parts and the average power is rewritten as

$$E\{\tilde{x}^2\} = \frac{1}{2} \int_0^{1/2} |H(f)|^2 S_x(f)\, df + \frac{1}{2} \int_0^{1/2} |H(f + \tfrac{1}{2})|^2 S_x(f + \tfrac{1}{2})\, df. \tag{19}$$

This expression is then further decomposed in terms of the integrals

$$E\{\tilde{x}^2\} = \frac{1}{2}\int_{\Omega_1} |H(f)|^2 S_x(f)\, df + \frac{1}{2}\int_{\Omega_2} |H(f)|^2 S_x(f)\, df.$$
$$+ \frac{1}{2}\int_{\Omega_1} |H(f+\tfrac{1}{2})|^2 S_x(f+\tfrac{1}{2})\, df$$
$$+ \frac{1}{2}\int_{\Omega_2} |H(f+\tfrac{1}{2})|^2 S_x(f+\tfrac{1}{2})\, df.$$

Using the definition of Ω_1 and Ω_2 in Theorem 1, we construct the upper bound

$$E\{\tilde{x}^2\} \le \frac{1}{2}\int_{\Omega_1} |H(f)|^2 S_x(f)\, df + \frac{1}{2}\int_{\Omega_2} |H(f)|^2 S_x(f+\tfrac{1}{2})\, df$$
$$+ \frac{1}{2}\int_{\Omega_1} |H(f+\tfrac{1}{2})|^2 S_x(f)\, df$$
$$+ \frac{1}{2}\int_{\Omega_2} |H(f+\tfrac{1}{2})|^2 S_x(f+\tfrac{1}{2})\, df$$

which, thanks to the QMF property (1), simplifies to

$$E\{\tilde{x}^2\} \le \int_{\Omega_1} S_x(f)\, df + \int_{\Omega_2} S_x(f+\tfrac{1}{2})\, df$$

and is independent of $H(f)$. Next, we use (17) and the symmetry properties of the spectral density to show that the two last integrals are in fact equivalent, which yields

$$E\{\tilde{x}^2\} \le 2\int_{\Omega_1} S_x(f)\, df. \tag{20}$$

It is then straightforward to verify that this bound is attained with the filter specified by Theorem 1. Finally, property (17) is used to show that this filter also satisfies the QMF condition (1). □

C. Comments

Since most signals encountered in practice tend to be predominantly lowpass, these results provide a good justification for the standard QMF design techniques which aim at obtaining a filter $H(f)$ with good lowpass characteristics [5].

Theorem 1 would also suggest that the use of an adaptive design may not offer any substantial advantages, at least for the class of stationary processes satisfying (18). In practice, this is only partially true because the total complexity of the system is usually limited. For filters with a fixed number of taps, the use of a signal-dependent approach will in general provide better performance because it has the capacity to adapt its limited resources to the spectral characteristics of the input signal. Some examples of this kind of behavior can be found in [7]. Another potential disadvantage of using ideal filters (or some close approximations) is that they may induce ringing artifacts. They also require more computations.

Because of the equivalence property (12), the same remarks also apply for the design of pyramid filters. An example that deserves some comment is the family of least squares spline pyramids [15]. We have previously observed experimentally that the performance of such pyramids increases with the order of the spline, although there was a tendency to saturation for $n \ge 3$. In light of our previous convergence results, Theorem 1 now provides us with a good theoretical explanation of this phenomenon: the least squares spline pyramid is asymptotically optimal because the corresponding analysis and interpolation filters both converge to the ideal lowpass filter as the order of the spline goes to infinity [16].

These observations are also consistent with the results of Antonini *et al.* who observed experimentally that wavelet transforms with higher regularity index tend to perform better for image coding [17]. For most classes of wavelets (e.g., splines, Daubechies' wavelets), the regularity increases with the complexity of the filters. A higher regularity index is usually also associated with a better separation of the different frequency bands.

Obviously, the criteria of optimality considered here (maximum energy compaction and decorreletedness) are only relevant for coding applications in which one uses a global quantization strategy. Our theoretical results do not exclude the possibility that better performance can be obtained with sub-optimal short kernel wavelets, provided that one uses some form of adaptive processing.

IV. The Continuous Case

The continuous counterpart of the previous optimization problem is the selection of a scaling function for the minimum error approximation of functions of the continuous-time variable t at a given scale. Here, we will first consider a less constrained version of this problem which is the approximation of a function in the space generated by the integer translates of a function $\varphi(x)$. Note that φ is not, at this stage, required to be a valid scaling function. We will then determine an optimal generating function that is such that the approximation (or sampling) error is minimized.

A. Least Squares Approximation in V

This section provides a brief review of the main results of the general sampling theory developed in [18]. In this formulation, the process of sampling is viewed as a procedure for obtaining the minimum L_2-norm approximation of a function $x(t)$ in the approximation space

$$V_\varphi = \left\{ \tilde{x}(t) = \sum_{k\in Z} c(k)\varphi(t-k),\ c \in l_2,\ t \in R \right\}. \tag{21}$$

The only restriction on the choice of φ is that V_φ is a closed subspace of L_2. A necessary and sufficient condition is

$$0 < m \le \sum_{k\in Z} |L(f-k)|^2 \le M < +\infty \qquad a.e. \tag{22}$$

where m and M are two constants and where $L(f)$ is the Fourier transform of $\varphi(t)$ [18]. This constraint is much weaker than the conditions required for a multiresolution analysis of L_2 [1]. An equivalent orthogonal generating function of V_φ is $\phi(t)$ whose Fourier transform is

$$F(f) = \frac{L(f)}{\sqrt{\sum_{k\in Z} |L(f-k)|^2}}. \tag{23}$$

The signal approximation $\tilde{x}$, which is the orthogonal projection of x on V_φ, is therefore given by

$$\tilde{x}(t) = \sum_{k\in Z} c_\phi(k)\phi(t-k), \qquad c_\phi(k) = \langle x(t), \phi(t-k)\rangle_{L_2}. \tag{24}$$

The corresponding signal processing block diagram is shown in Fig. 5.

This representation has certain properties that are essential to our analysis. First, it can be readily verified that the Fourier transform of ϕ satisfies the condition

$$\sum_{k\in Z} |F(f-k)|^2 = 1. \tag{25}$$

Second, it has the norm preserving property

$$\|\tilde{x}\|^2_{L_2} = \|c_\phi\|^2_{l_2}, \tag{26}$$

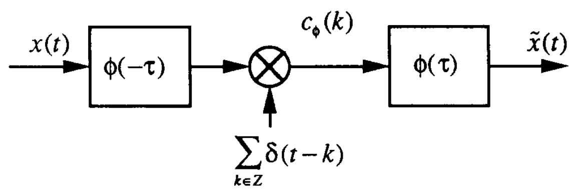

Fig. 5. Signal processing system for the minimum L_2-norm approximation of a signal in V_ϕ. Sampling is obtained by multiplication by a sequence of dirac functions.

simply because $\{\phi(t - k), k \in Z\}$ is an orthogonal basis of V_ϕ. Finally, $\tilde{x}$ is by definition orthogonal to the residual error $e = x - \tilde{x}$. Hence, it is also true that

$$\|e\|_{L_2}^2 = \|x\|_{L_2}^2 - \|\tilde{x}\|_{L_2}^2 = \|x\|_{L_2}^2 - \|c_\phi\|_{l_2}^2 \tag{27}$$

which, once again, indicates that minimizing $\|e\|^2$ is equivalent to maximizing $\|\tilde{x}\|^2$.

B. Optimal Filter Selection

Let us now assume that the continuous-time input of the system in Fig. 5 is a realization of a wide sense stationary process with PSD $S_x(f)$. To analyze this situation, the previous notion of the squared L_2-norm of a signal must be replaced by the average power $E\{x^2\}$. The link with the L_2 formulation is provided by the following limit process

$$E\{x^2\} = \lim_{T\to+\infty} \frac{1}{2T} E\left\{\int_{-T}^{T} |x(t)|^2\, dt\right\} \tag{28}$$

$$S_x(f) = \lim_{T\to+\infty} \frac{1}{2T} E\left\{\left|\int_{-T}^{T} x(t) e^{-j2\pi f} dt\right|^2\right\}. \tag{29}$$

This allows the translation of the norm preserving property (26) into the relation

$$E\{\tilde{x}^2\} = E\{c_\phi^2\}. \tag{30}$$

Likewise, (27) implies that condition (15) is satisfied as well. To state our main result, we first define $k_0(f)$ as the smallest positive integer for which the following inequality is true

$$\forall k \in Z, \quad S_x(f + k_0(f)) \ge S_x(f + k). \tag{31}$$

Theorem 2: For a continuous-time stationary random process with PSD $S_x(f)$, the sampling (or approximation) error is minimized if the Fourier transform of ϕ is such that for $f \in Z^+$

$$|F(f)| = \begin{cases} 1, & f \in \Omega_1 = \{\nu + k_0(\nu) : \nu \in [0, \frac{1}{2}]\} \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

Proof: As in the previous case, minimizing the approximation error is equivalent to maximizing the average power of the approximating signal $\tilde{x}$, or equivalently, the coefficient sequence c_ϕ. It can be readily shown that the spectral density of the sampled sequence c_ϕ is [19, p. 336]

$$S_{c_\phi}(f) = \sum_{k\in Z} |F(f + k)|^2 S_x(f + k)$$

The average power is then obtained by integration

$$E\{c_\phi^2\} = 2\int_0^{+1/2} \sum_{k\in Z} |F(f + k)^2 S_x(f + k)\, df. \tag{33}$$

By using (31) and (25), we construct the following upper bound

$$\begin{aligned} E\{c_\phi^2\} &\le 2\int_0^{+1/2} \sum_{k\in Z} |F(f + k)|^2 S_x(f + k_0(f))\, df \\ &= 2\int_0^{+1/2} S_x(f + k_0(f))\, df \end{aligned}$$

It is then easy to verify that this bound is achieved for any filter satisfying condition (32). □

C. Comments

Theorem 2 implies that the classical sampling function $\phi(t) = \text{sinc}\,(t)$ (ideal lowpass filter) is optimal for all stationary processes with PSD satisfying the constraint

$$\forall f \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ S_x(f) \ge S_x(f + k), \qquad k \in Z. \tag{34}$$

The corresponding approximation space V_{opt} is the space of functions bandlimited to the frequency interval $[-1/2, 1/2]$. It is precisely the class of functions considered in Shannon's sampling theorem.

The function sinc (t) also satisfies all the properties of a scaling function and hence generates a multiresolution analysis in the sense defined by Mallat [1]. The corresponding wavelet $\psi(t) = \cos(3\pi(t - 1)/2)\,\text{sinc}\,((t - 1)/2)$ is the modulated sinc (or ideal bandpass) function. It follows that this wavelet is "optimal" for the approximation of stationary signals satisfying condition (34) up to the resolution $\Delta = 1$. Note also that this wavelet is an analytical function and that it is in this sense "infinitely" regular.

Interestingly, there are a number of wavelet transform constructions that converge asymptotically to this limit. The better known example is the family of Battle–Lemarié spline wavelets which converge to an ideal bandpass filter as the order of the spline goes to infinity [20]. A procedure for constructing general families of wavelets with such asymptotic properties has been proposed recently [21]. There is also some evidence that the bandpass characteristics of Daubechies' family of compactly supported wavelets improve as a function of the index N [13]. Our results therefore suggest that there seems to be an advantage in using higher order wavelets for coding applications. Obviously, there is also a price to be paid because higher order always means greater complexity. In practice, one should seek a solution that provides a good compromise in terms of performance and complexity and this is precisely what filter design is all about.

Condition (35) is satisfied for signals with nonincreasing spectral density, although this latter constraint is a much stronger one. For such signals, this inequality still holds if the sampling rate is changed arbitrarily. The sinc-wavelet transform therefore provides an optimal approximation of such signal for all scales.

VI. Conclusion

This note first dealt with the problem of a signal decomposition into two subbands. This led to the derivation of optimal QMF filter banks for the representation of stationary processes. Such filters maximize the differences in the variance distribution across subbands and therefore result in the largest coding gain in the case of separate channel encoding. For the class of stationary processes with predominant lowpass characteristics [the exact condition is given by (34)], the optimal solution is the classical ideal half-band filter bank, which justifies the use of conventional QMF design techniques.

The extension of this formulation to the continuous case has also revealed a new aspect of the optimality of bandlimited signal representations. An implication is that the modulated-sinc wavelet transform is optimal for the representation at all scales of the whole class of stationary random processes with non-increasing spectral power density.

Although ideal filters are difficult to implement in practice, they can be approached as closely as desired by increasing the complexity in the algorithm. For instance, one can obtain closer and closer approximations of the ideal bandpass decomposition (sinc-wavelet) with most families of wavelet transforms (splines, Daubechies, etc.) by increasing the regularity index. These results provide a simple explanation for the performance improvement that is usually observed with higher order transforms (or filter banks). The optimal filter banks provided by this analysis also lead to very simple performance bounds that can be used as "gold standards" for assessing the efficiency of a given subband coder. Finally, these results can easily be adapted for the optimal representation of finite energy signals with a fixed (deterministic) shape and a uniform shift distribution (c.f. comment at the end of Section II-B).

Acknowledgment

While this paper was under review, the author became aware of the work of M. K. Tsatsanis and G. B. Giannakis [22]. These authors approached a similar problem from the different perspective of a principal component decomposition in the frequency domain using the polyphase matrix representation. They present an elegant derivation of the optimal solution for the discrete P-band case.

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Adapted Local Trigonometric Transforms and Speech Processing

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***Abstract*—We use an algorithm based on the adapted-window Malvar transform to decompose digitized speech signals into a local time-frequency representation. We present some applications and experimental results for a signal compression and automatic voiced–unvoiced segmentation. This decomposition provides a method of parameter simplification which appears to be useful for detecting fundamental frequencies, and characterizing formants.**

I. Introduction

In this note, we apply an algorithm, based on the local trigonometric orthonormal basis and the adapted local trigonometric transform, to decompose digitized speech signals into orthogonal elementary waveforms. This algorithm leads to a local time-frequency representation which is well adapted to analysis–synthesis, compression, and segmentation. We present some applications and experimental results for signal compression and automatic voiced–unvoiced segmentation. Furthermore, compression provides a simplified decomposition which appears to be useful for detecting fundamental frequencies and characterizing formants.

We begin with a clean, digitized speech signal. The signal is decomposed into a local trigonometric orthonormal basis which consists of cosines or sines multiplied by smooth cutoff functions. This basis is described by Coifman and Meyer [3] and by Malvar

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