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A GENERAL HILBERT SPACE FRAMEWORK FOR THE DISCRETIZATION OF CONTINUOUS SIGNAL PROCESSING OPERATORS

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ABSTRACT

We present a unifying framework for the design of discrete algorithms that implement continuous signal processing operators. The underlying continuous-time signals are represented as linear combinations of the integer-shifts of a generating function φ_i with $(i=1,2)$ (continuous/discrete representation). The corresponding input and output functions spaces are $V(\varphi_1)$ and $V(\varphi_2)$, respectively. The principle of the method is as follows: we start by interpolating the discrete input signal with a function $s_1 \in V(\varphi_1)$. We then apply a linear operator T to this function and compute the minimum error approximation of the result in the output space $V(\varphi_2)$. The corresponding algorithm can be expressed in terms of digital filters and a matrix multiplication. In this context, we emphasize the advantages of B-splines, and show how a judicious use of these basis functions can result in fast implementations of various types of operators. We present design examples of differential operators involving very short FIR filters. We also describe an efficient procedure for the geometric affine transformation of signals. The present formulation is general enough to include most earlier continuous/discrete signal processing techniques (e.g., standard bandlimited approach, spline or wavelet-based) as special cases.

Keywords: continuous signal processing, splines, bandlimited functions, wavelets, differentiation, operator design, fast algorithms, recursive filters, filter design.

1. INTRODUCTION

The standard method for designing continuous signal processing operators is to assume a bandlimited signal model and to derive the corresponding discrete operators accordingly. This approach is well suited for shift-invariant operators which transform bandlimited functions into other bandlimited functions; these signals can all be represented exactly by their sample values (Shannon's sampling theorem). The natural approach in this case is to evaluate the frequency response of the continuous operator and to perform a Fourier domain computation using the FFT.[12]

In practice, however, there are many applications in which more localized signal domain techniques may be preferable. For instance, it is customary to estimate derivatives using finite differences rather then by multiplication with $j\omega$ in the frequency domain.[7] Likewise, practitioners commonly use simple interpolation techniques (nearest neighbor, linear) to shift signals or to perform sampling-rate conversion (e.g., image zooming).[8, 10, 11] The reason for this preference is that local approaches are usually simpler to implement and substantially faster. In addition, they do not give rise to Gibbs oscillations.

The purpose of this paper is to introduce a Hilbert space framework that unifies these various approaches and provides some general design principles. The main reason for considering more general classes of functions than the traditional (bandlimited) ones is to open up new options for solving certain types of problems, and to derive algorithms that are computationally more efficient. For instance, polynomial splines are well suited for performing geometric transformations because the underlying basis functions are compactly supported[28, 31]; bandlimited representations, on the other hand, lack this essential property. Note that the general methodology developed here is also directly applicable in the context of the multiresolution theory of the wavelet transform, which uses the same type of continuous Hilbert space signal representations.[4, 9, 32]

1.1 Notations and operators

L_2 is the space of measurable, square-integrable, real-valued functions $s(x)$, $x \in R$. L_2 is a Hilbert space whose metric $\|\cdot\|$ (the L_2-norm) is derived from the inner product

$$\langle r,s\rangle = \langle r(x),s(x)\rangle = \int_{-\infty}^{+\infty} r(x)s(x)dx. \tag{1}$$

We use the "hat" symbol to denote the Fourier transform of the continuous-time function $s \in L_2$:

$$\hat{s}(\omega) = \int_{-\infty}^{+\infty} s(x)e^{-j\omega x}dx. \tag{2}$$

l_2 is the vector space of square-summable sequences (or discrete signals) $a(k)$, $k \in Z$. The convolution between two sequences a and b is denoted by $(b*a)(k)$. The sequence $b(k)$ can be viewed as a discrete convolution operator (or digital filter) that is applied to the signal $a \in l_2$. This filter is characterized by its transfer function (z-transform): $B(z) = \sum_{k\in Z} b(k)z^{-k}$; the corresponding Fourier transform is obtained by replacing z by $e^{j\omega}$. An important result concerning the stability and the reversibility of such operators is given by the following proposition.[2]

Proposition 1 : *A sequence $b(k)$ defines an invertible discrete convolution operator from l_2 into itself if and only if there exists two positive constants m and M such that $m \le |B(e^{j\omega})| \le M$, almost everywhere.*

This condition insures the existence and stability of the inverse filter, which we denote by

$$(b)^{-1}(k) \xleftrightarrow{z-transform} \frac{1}{B(z)}. \tag{3}$$

2. CONTINUOUS/DISCRETE SIGNAL REPRESENTATIONS

The general design approach that we propose is reminiscent of the finite element method developed for solving partial differential equations.[21] Although the problem is initially formulated in the continuous domain, we want to develop signal processing algorithms that are discrete in nature and can be effectively implemented on a computer. We are also interested in investigating non-bandlimited solutions, which implies a need for continuous-to-discrete and discrete-to-continuous conversion mechanisms that are more general than the traditional approach dictated by Shannon's sampling theory.[17] Our formulation, which is inspired from the theory of the wavelet transform, is to force the input and output of our system to lie in certain signal subspaces generated from the translates of a generating function φ. Thanks to this particular Hilbert space structure, signals that are functions of the continuous variable x can also be represented by sequences of numbers (discrete representation) that can be processed numerically.

In this section, we start by reviewing the corresponding signal representation and approximation mechanisms. We then concentrate on the special case of polynomial spline representations, and recall a number of relevant properties of the underlying basis functions (B-splines).

2.1 Signal subspaces

A general approach to specify continuous signal representations is to consider the class of functions generated from the integer translates of a single function $\varphi(x)$.[2, 20] The corresponding function space $V(\varphi) \subset L_2$ is defined as

$$V(\varphi) = \left\{ s(x) = \sum_{k \in Z} c(k) \varphi_k : c(k) \in l_2 \right\} \tag{4}$$

where $\varphi_k := \varphi(x-k)$. The only restriction on the choice of the *generating* function φ is that the set $\{\varphi(x-k)\}_{k \in Z}$ is a Riesz basis of $V(\varphi)$; this is equivalent to the condition

$$A \le A_\varphi(e^{j\omega}) = \sum_{k \in Z} \left|\hat{\varphi}(\omega + 2\pi k)\right|^2 \le B \quad a.e., \tag{5}$$

where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(x)$, and where A and B are two strictly positive constants.[2] Note that $A_\varphi(z)$ is also the z-transform of the autocorrelation sequence $a_\varphi(k) = \langle \varphi(x), \varphi(x+k) \rangle$. The admissibility condition (5) insures that each function $s(x)$ in $V(\varphi)$ is uniquely characterized by the sequence of its coefficients $c(k)$.

This formulation is quite general and covers many signal representation models that have been used in the literature. Examples of interest are the class of bandlimited functions (with $\varphi(x) = \text{sinc}(x)$), and polynomial spline representations which are considered in more detail in Section 2.4. Other special cases are the various subspaces associated with the wavelet transform and multiresolution analysis; this connection is further discussed elsewhere.[1]

This type of continuous/discrete model can also be extended for representing multi-dimensional signals in L_2^p, for example, images (p=2) or volumes (p=3). The simplest approach is to consider tensor-product representations using a separable multi-dimensional generating function

$$\varphi(x_1, \cdots, x_p) = \prod_{k=1}^{p} \varphi(x_k), \tag{6}$$

which can be constructed from any admissible 1D generating function $\varphi \in L_2$. The corresponding tensor-product representation will usually inherit the properties of the underlying 1D representation. In addition, many of the corresponding digital algorithms will be separable, which essentially means that the transformation can be implemented by successive 1D processing along the various dimensions of the data. Thus, it makes sense to first develop the one-dimensional theory even if one is interested in image and multidimensional signal processing applications.

2.2 Signal approximation in $V(\varphi)$

Given an arbitrary continuously varying function $s(x) \in L_2$, we can write its least squares approximation in our signal space $V(\varphi)$ (orthogonal projection) as

$$P_{V(\varphi)}\{s\} = \sum_{k \in Z} c(k) \varphi_k \tag{7}$$

where the coefficients are computed as the inner products

$$c(k) = \langle s, \mathring{\varphi}_k \rangle = \int_{-\infty}^{+\infty} s(x) \mathring{\varphi}(x-k) dx. \tag{8}$$

The corresponding analysis function $\mathring{\varphi} \in V(\varphi)$ is the semi-orthogonal *dual* of φ; it is given by[2]

$$\mathring{\varphi}(x) = \sum_{k \in Z} (a_\varphi)^{-1}(k) \varphi(x-k) \quad \xleftrightarrow{\text{Fourier}} \quad \frac{\hat{\varphi}(\omega)}{\sum_{k \in Z} \left|\hat{\varphi}(\omega + 2\pi k)\right|^2} \tag{9}$$

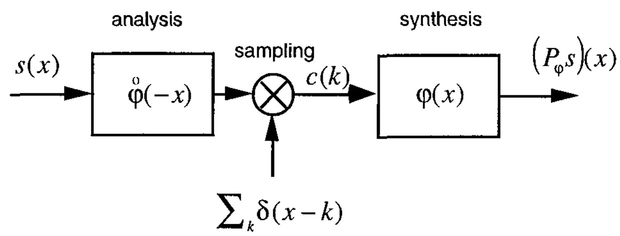

Fig. 1 : Equivalent signal processing system for the approximation of a function in $V(\varphi)$. The sampling operation is modeled by a multiplication with a sequence of Dirac impulses. The solid rectangular boxes represent convolution operators (analog filters).

where $(a_\varphi)^{-1}$ is the convolution inverse of the autocorrelation sequence $a_\varphi(k)$. Equation (7) can also be implemented through the block diagram in Fig. 1. If the generating function is orthonormal (i.e., $a_\varphi(k) = \delta[k]$), then it is its own dual, and the prefilter is just the time-reversed version of the post-filter. In particular, for $\varphi(x) = \mathrm{sinc}(x)$, we get the standard discretization procedure dictated by Shannon's sampling theory, which uses an ideal lowpass filter prior to sampling in order to suppress aliasing.

2.3 Approximation power

The quality of the approximation given by (7) will depend on the order of the representation which is an intrinsic property of the underlying function space. The order, which is an approximation theoretic concept, provides the rate of decay of the error when the sampling step h goes to zero. More precisely, we will say that the function space $V(\varphi)$ has an order of approximation N if there exists a constant C_φ such that for any function s that is N times differentiable: [6]

$$\left\| s - P_{V_h(\varphi)} s \right\| \le C_\varphi \cdot h^N \cdot \left\| s^{(N)} \right\|, \qquad (10)$$

where $P_{V_h(\varphi)}\{s\}$ denotes the orthogonal projection of s into the rescaled approximation space $V_h(\varphi) = \mathrm{span}\{\varphi(x/h - k)\}_{k \in Z}$ at sampling step h. The bound (10) relates the approximation error to the norm of the Nth derivative of s. The implication is that higher order representations usually result in a smaller approximation error (even if the sampling step is one as in our case). Smooth (or regular) basis functions are also very desirable because they typically correspond to a smaller constant C_φ.[23]

Interestingly, the order of approximation N only depends on the ability of the generating function to reproduce polynomials of degree n=N-1.[18, 20] This requirement can also be expressed in the frequency domain:

Strang-Fix conditions[18, 20] : A generating function φ has an Nth order of approximation if and only if $\hat{\varphi}^{(n)}(\omega) = 0$ for $\omega = 2\pi k,\ k \in Z, k \neq 0$, and n=0,...,N-1.

In other words, the condition for an Nth order of approximation is that the Fourier transform of φ has zeros of multiplicity N at all non-zero frequencies that are integer multiples of 2π. In the theory of the wavelet transform, the order of approximation corresponds to the number of vanishing moments of the bi-orthogonal (or orthogonal) analysis wavelet. [4, 19, 22]

2.4 Polynomial splines

The simplest generating function is the unit rectangular pulse which generates the family of piecewise constant functions. Corresponding algorithms turn out to be computationally very efficient, but the approximation power of the representation is poor (N=1). At the other extreme, we have the bandlimited model which has good approximation properties but which suffers from the serious drawback that its basis functions are not compactly supported. As a result, it is difficult to design sinc-based algorithms that are computationally realistic (unless the computations can be performed in the Fourier domain). In addition, the bandlimited model often gives rise to truncation artifacts that manifest themselves as Gibbs oscillations. Splines are of interest because they offer a good compromise solution between those two extremes.

Polynomial splines of degree n can be generated by taking φ to be Schoenberg's central B-spline of degree n,[13] which we denote by $\beta^n(x)$. These generating functions are constructed from the $(n+1)$-fold convolution of a rectangular pulse:

$$\beta^n(x) = \beta^0 * \beta^{n-1}(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \beta^{n-1}(x)dx, \tag{11}$$

where

$$\beta^0(x) = \begin{cases} 1 & -\frac{1}{2} \le x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}. \tag{12}$$

B-splines generating functions, and spline representations in general, turn out to be especially attractive for our purpose. Their main advantages are as follows:

(*i*) B-splines are compactly supported. Moreover, they are the shortest known functions with an order of approximation $N=n+1$. As we shall see, this short support property is a key consideration for computational efficiency.

(*ii*) B-splines are smooth and well behaved functions. Splines of degree n are $(n-1)$ continuously differentiable (cf. right hand side of (11)). As a result, splines have excellent approximation properties. Precise convergence rates and error bounds are also available.[16, 29]

(*iii*) Splines have a simple analytical form (piecewise polynomial) which greatly facilitates their manipulation.[5, 25]

(*iv*) Splines have multiresolution properties that make them very suitable for constructing wavelet functions and for performing multi-scale processing.[9, 24]

(*v*) The family of polynomial splines also provides design flexibility. By increasing the degree n, we can progressively switch from the simplest piecewise constant (n=0) and piecewise linear (n=1) representations to the other extreme, which corresponds to a bandlimited signal model ($n \to +\infty$).[3, 14, 15]

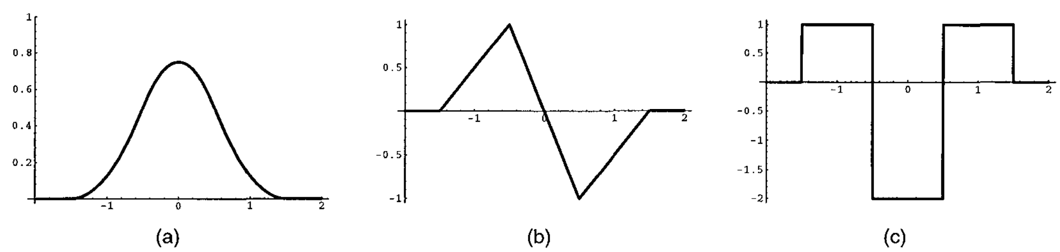

Fig. 2 : Example of a B-spline function and its derivatives. (a) Quadratic B-spline (n=2), (b) first derivative (piecewise linear), (c) second derivative (piecewise constant).

We briefly illustrate property (*iii*) by giving the formula for the exact differentiation of a B-spline of degree n :

$$\frac{d\beta^n(x)}{dx} = \beta^{n-1}(x+\tfrac{1}{2}) - \beta^{n-1}(x-\tfrac{1}{2}). \tag{13}$$

This equation can be applied recursively to obtain higher order derivatives. This process is illustrated with the quadratic B-spline in Fig. 2. The main point is that the pth derivative of a spline yields another spline with a corresponding reduction of the degree $n_2 = n - p$. One slight complication is that the position of the basis functions is shifted by half a sampling step for odd order derivatives. As we shall see in Section 4.1, we can avoid this nuisance by approximating the result of the differentiation in the function space of our choice.

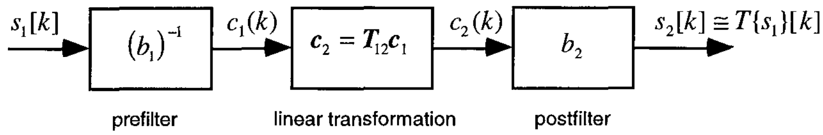

Fig. 3 : Discrete implementation of a continuous signal processing operator.

3. GENERAL DESIGN PROCEDURE

We are interested in the problem of the discretization of continuous signal processing operators. We will therefore consider an operator T from L_2 (or an appropriate subspace of L_2) into L_2, that acts on a signal $s(x)$ of the continuous variable $x \in R$, and that transforms it into another signal $T\{s\}(x)$ that is continuously defined as well. A typical example is the differentiation operator: $T\{s\} = ds(x)/dx$. Our only requirement is that the operator T is *linear*; i.e.,

$$\forall s_1, s_2 \in L_2, \forall \alpha, \beta \in R, \quad T\{\alpha s_1 + \beta s_2\} = \alpha T\{s_1\} + \beta T\{s_2\}. \tag{14}$$

Our approach uses a representation of the input signal in the continuous/discrete Hilbert space $V(\varphi_1)$, and provides a representation of the output in $V(\varphi_2)$. The principle of the method is as follows. We start by selecting a model of our input signal $s_1 \in V(\varphi_1)$ to which we apply the continuous transformation operator T. We then project the transformed signal $T\{s_1\}$ onto the output space $V(\varphi_2)$. The resulting output $s_2 \cong T\{s_1\}$ will be the least squares approximation of $T\{s_1\}$ in $V(\varphi_2)$. This procedure leads to a discrete algorithm that is summarized by the block diagram in Fig. 1. The various components of this system are described below.

3.1 Signal interpolation

In practice, the input signal is specified by its discrete sample values at the integers: $s(k) \in l_2$. The first step is therefore to fit this data sequence with a continuous function $s_1 \in V(\varphi_1)$, which has the following parametric representation

$$s_1(x) = \sum_{k \in Z} c_1(k) \varphi_1(x-k). \tag{15}$$

If we require the model to provide an exact interpolation, we obtain the following condition:

$$s(k) = s_1(x)\big|_{x=k} = (b_1 * c_1)(k), \tag{16}$$

where b_1 is the discretized version of the generating function

$$b_1(k) := \varphi_1(x)\big|_{x=k}. \tag{17}$$

Hence, if the discrete convolution operator b_1 is invertible (cf. Proposition 1), then we can solve (16) by simple inverse digital filtering

$$c_1(k) = \left((b_1)^{-1} * s\right)(k), \tag{18}$$

which is the interpolation part of the algorithm. If the generating function φ_1 is interpolating (i.e., if it is one at the origin and vanishes at all other integers), then $(b_1)^{-1}$ is the identity operator and the expansion coefficients in (15) are the sample values of the signal. A typical example where no prefiltering is necessary is $\varphi_1(x) = \text{sinc}(x)$. We should note, however, that the cost of prefiltering in the block diagram in Fig. 3 is usually negligible, and that we are much more interested in selecting the shortest possible generating function within the given subspace $V(\varphi_1)$.

3.2 Transformation and approximation

The next step is to apply the transformation T to $s_1(x)$, which, by linearity, yields $T\{s_1\}(x)=\sum_{k\in Z} c_1(k)T\{\varphi_{1,k}\}$. The transformed model $T\{s_1\}$ is then approximated in the least square sense by another continuous function $s_2 \in V(\varphi_2)$ of the form

$$s_2(x)=\sum_{l\in Z} c_2(l)\varphi_{2,l} \tag{19}$$

where

$$c_2(l)=\langle T\{s_1\},\mathring{\varphi}_{2,k}\rangle. \tag{20}$$

Substituting the expression of $T\{s_1\}$ in (20) and using the linearity of the inner product, we show that the output signal coefficients can be obtained from the following (infinite-dimensional) matrix equation

$$\boldsymbol{c}_2=\boldsymbol{T}_{12}\boldsymbol{c}_1 \tag{21}$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are the coefficients vectors. $\boldsymbol{T}_{12}=[T_{12}(k,l)]$ is the projection matrix of the operator T from $V(\varphi_1)$ into $V(\varphi_2)$, with entry

$$(k,l)\in Z^2,\quad T_{12}(k,l)=\langle T\{\varphi_{1,k}\},\mathring{\varphi}_{2,l}\rangle. \tag{22}$$

3.3 Model resampling

At this point, the transformation of our input signal is entirely specified through (18) and (21). In many applications, it is appropriate to provide a more direct representation of the result in terms of the sample values of the signal $s_2 \cong T\{s_1\}$. These digital signal values are obtained from the c_2's by discrete convolution (post-filter)

$$s_2[k]:=s_2(x)\big|_{x=k}=(b_2 * c_2)(k) \tag{23}$$

where the convolution kernel b_2 is simply the re-sampled version of the generating function φ_2 :

$$b_2(k):=\varphi_2(x)\big|_{x=k}. \tag{24}$$

3.4 Implementation considerations

The efficiency of the algorithm will primarily depend on the sparseness of the transformation matrix $\boldsymbol{T}_{12}$. For these reasons, we would like the functions that appear in the inner product (22) to be as short as possible. Since in most cases of interest, the essential support of $T\{\varphi_{1,k}\}$ will be proportional to the support of the argument $\varphi_{1,k}$, it is therefore desirable to select $\varphi_1(x)$ and $\mathring{\varphi}_2(x)$ as short as possible. However, computational efficiency has also to be counterbalanced by approximation theoretic considerations; in other words, we should select the order of the model so as to maintain the approximation error within an acceptable error bound (cf. Section 2.3).

Since the shortest known function that has an order of approximation N is the B-spline of degree $n=N-1$, an attractive design choice is

$$\begin{cases}\varphi_1(x)=\beta^{n_1}(x)\\ \mathring{\varphi}_2(x)=\beta^{n_2}(x)\end{cases} \tag{25}$$

where the input and output degree parameters n_1 and n_2 can be adjusted according to the requirements of the application. Note that with this particular choice for $\mathring{\varphi}_2$, we are implicitly expressing the result of the transformation in the dual basis of the B-splines.[27]

For this particular choice of basis functions, the prefilter is an all-pole system that can be implemented efficiently using a fast recursive algorithm.[26] The corresponding post-filter is

$$B_2(z) = \frac{B_1^{n_2}(z)}{B_1^{2n_2+1}(z)} \tag{26}$$

where $B_1^n(z)$ denotes the z-transform of a discrete B-spline of degree n.[25] The numerator of (26) is a simple FIR filter; the denominator is a symmetric all-pole system that can be implemented using the recursive technique mentioned previously.[26]

4. SPECIALIZED VERSIONS OF THE ALGORITHM

The expensive step in this sequence of operations is the evaluation of (21), which also requires the precomputation of the transformation matrix $\boldsymbol{T}_{12}$. We will now look at special classes of operators T (e.g., shift-invariant, or affine) that result in very simple implementations.

4.1 Shift-invariant operators

Let S_Δ be the shift operator by Δ; i.e., $S_\Delta\{s\} = s(x-\Delta)$. An operator T is said to be shift-invariant if it commutes with the shift operator : $T\{S_\Delta s\} = S_\Delta T\{s\}$. In this case, (21) reduces to a discrete convolution equation (digital filter solution)

$$c_2(k) = (t_{12} * c_1)(k) \tag{27}$$

where

$$t_{12}(k) = T\{\overset{\circ}{\varphi}{}_2^T * \varphi_1\}(x)\big|_{x=k}. \tag{28}$$

• *Example 1 (bandlimited model)* : Let $\varphi_1 = \varphi_2 = \overset{\circ}{\varphi}_2 = \mathrm{sinc}(x)$ and T be the convolution operator with frequency response $\hat{T}(\omega)$. Then t_{12} is the digital filter with the frequency response $\hat{T}(\omega)$ for $\omega \in [-\pi, \pi]$. The pre- and post-filters are the identity and we get the conventional bandlimited solution. This approach is the most appropriate for an implementation in the Fourier domain. Its main drawback is that the processing is non-local; i.e., the digital filter t_{12} is typically not compactly supported.

• *Example 2 (polynomial spline model)* : Let us now consider the B-spline solution proposed in (25). The corresponding cross-correlation function is $(\overset{\circ}{\varphi}{}_2^T * \varphi_1)(x) = (\beta^{n_1} * \beta^{n_2})(x) = \beta^{n_1+n_2+1}(x)$. Thus, the filter t_{12} can be obtained by applying the operator T to a B-spline of degree $n_1 + n_2 + 1$. This approach is well suited for designing differential operators. As an illustrative example, we consider the simple case n_1=1 and $n_2 = 0$, which has the advantage that no pre- nor post-filtering is required (i.e., $\beta^1(x)$ is an interpolation kernel, and $\beta^0(x)$ is an orthogonal generating function). Using the formulas for the differentiation of a quadratic B-spline (cf. graph in Fig. 2), it is not difficult to derive the corresponding FIR differentiation filters:

$$T_{12}^{(1)}(z) = \tfrac{1}{2}(z - z^{-1}) \quad \text{(first derivative)}$$

$$T_{12}^{(2)}(z) = z - 2 + z^{-1} \quad \text{(second derivative)}.$$

These are the 1st and 2nd order central difference operators that are commonly used in practice. Better quality approximations can also be obtained by using higher order basis functions. Note that we can even obtain an "exact" formula if $T\{\varphi_1\} \in V(\varphi_2)$, which is obviously not the case for the example above. For instance, the 2nd derivative of a spline of degree n_1 is a spline of degree $n_2 = n_1 - 2$. Such "exact" digital filtering solutions have been considered previously.[25]

4.2 Pseudo-shift invariant operators (or affine-type transformations)

A pseudo-shift invariant operator is an operator that pseudo-commutes with the shift operator; i.e., $T\{S_\Delta s\} = S_{f(\Delta)}T\{s\}$ where $f(\Delta)$ an invertible function of Δ. An interesting member of this family is the affine operator: $T\{s\} := s(ax+b)$. Since $T\{S_\Delta s\} = s((x-\Delta)/a+b)$, the corresponding pseudo-commuting function is $f(\Delta) = \Delta / a$. For this general class of operators, we rewrite (22) as

$$\begin{aligned} T_{12}(k,l) &= \langle T\{\varphi_1(x-k)\}, \overset{o}{\varphi}_2(x-l)\rangle = \langle S_{f(k)}T\{\varphi_1(x)\}, \overset{o}{\varphi}_2(x-l)\rangle \\ &= \langle T\{\varphi_1(x)\}, \overset{o}{\varphi}_2(x-l+f(k))\rangle = \tau_{12}(l-f(k)) \end{aligned}$$

where the auxiliary function $\tau_{12}(x)$ is defined as

$$\tau_{12}(x) = \overset{o}{\varphi}{}_2^{T} * T\{\varphi_1\}(x). \tag{29}$$

Hence, the matrix equation (21) reduces to the much simpler linear expansion formula

$$c_2(l) = \sum_{k\in Z} c_1(k)\tau_{12}(x-f(k))\Big|_{x=l}. \tag{30}$$

Therefore, we can obtain a relatively efficient implementation by pre-computing the kernel $\tau_{12}(x)$ and eventually storing it in a lookup table.

In particular, this approach provides a simple general mechanism for scaling signals by factors that are not restricted to powers of two, as is usually the case in the context of the wavelet transform. As an illustrative example, we consider the simplest piecewise constant model $\varphi_1 = \varphi_2 = \overset{o}{\varphi}_2 = \beta^0(x)$ and consider the dilation/contraction operator $T_\alpha\{s\} := s(x/\alpha)$ where α is any non-zero scaling factor. The corresponding expansion kernel is

$$\tau_{12}(x;\alpha) = \int_{-\infty}^{+\infty} \beta^0(x-y)\beta^0(y/\alpha)dy = \begin{cases} h & 0 \le |x| < x_0 \\ h - h\frac{(|x|-x_0)}{x_1-x_0} & x_0 \le |x| < x_1 \\ 0, & x_1 \le |x| \end{cases} \tag{31}$$

where the height h and knot constants x_0 and x_1 are defined as follows

$$\begin{cases} h = \min\{1, |\alpha|\} \\ x_0 = \big|1-|\alpha|\big|/2 \\ x_1 = (1+|\alpha|)/2 \end{cases} . \tag{32}$$

This function is piecewise linear. An example for $\alpha = 3/2$ is shown in Fig. 4. Note that the corresponding affine transformation formula (30) is similar to a linear interpolation except that it uses the modified sampling kernel $\tau_{12}(x;\alpha)$ that depends explicitly on the scaling factor α. The advantage of the least squares scheme is that it provides an implicit protection against aliasing errors (i.e., it includes the additional prefiltering step in Fig. 1).

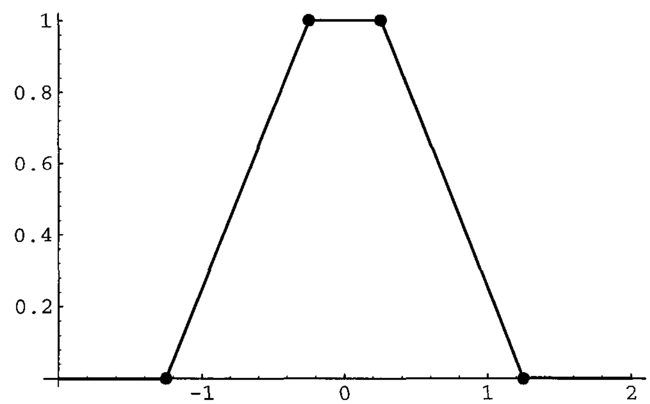

Fig. 4 : Example of trapezoidal sampling function for the implementation of an affine transformation with α=3/2 for the piecewise constant model.

Similar least squares signal rescaling algorithms can also be derived using higher order spline models.[28] What makes this type of approach feasible is the fact that φ_1 and $\mathring{\varphi}_2$ can be selected so that the function τ_{12} is compactly supported (cf. (29)). For a given polynomial spline model, the present formulation with $n_1 = n_2$ yields an algorithm that is consistently superior to the conventional interpolation approach that uses a direct signal re-sampling.[28] The method can also be extended for geometrical affine transformations in higher dimensions.[30]

Finally, we note that the compact support condition for $\tau_{12}(x)$ cannot be achieved in the traditional bandlimited framework; for this reason, there is no practical sinc-based algorithm for this particular problem unless α is an integer factor.

5. CONCLUSION

We have presented a general approach for the design of continuous signal processing operators. Although the problem is entirely formulated in the continuous domain, the solution is provided by a digital algorithm that can implemented numerically. In this sense, we have a formulation that is analogous to the finite element approach for the discretization of partial differential equations. The approach has a sound mathematical foundation that takes its roots in the theory of the wavelet transform. The input and output representation subspaces are well-defined Hilbert spaces with a convenient integer-shift-invariant structure. The design methodology uses a rather standard least squares criterion. As a result, the solution (orthogonal projection) is optimal in a well defined sense.

The present formulation is general enough to provide a unifying framework for most of the work that has been done so far in the area of continuous/discrete signal processing. Special cases include the traditional bandlimited approach, most instances of spline processing, and operator design in the wavelet domain. Presently, we have only looked at a very small subset of operators and identified some special cases that resulted in efficient implementations. The full potential of the method still remains to be explored.

We have shown, at least qualitatively, that we can improve the quality of the approximation by increasing the order of the input and output representation spaces. However, there are still a number of related performance issues that need to be addressed more quantitatively. In particular, it is not yet entirely clear how the order of the input space influences the quality of the approximation of the operator, although we would tend to select $n_1 \geq n_2$. It would therefore be of great interest to derive error bounds that show the explicit dependency on both of these variables.

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