APPLIED AND COMPUTATIONAL HARMONIC ANALYSIS **3,** 201–214 (1996)
ARTICLE NO. 0016

Quasi-Orthogonality and Quasi-Projections

MICHAEL UNSER[1]

Biomedical Engineering and Instrumentation Program, Building 13, Room 3N17, National Center for Research Resources, National Institutes of Health, Bethesda, Maryland 20892-5766

Communicated by Charles K. Chui

Received July 11, 1995; revised November 8, 1995

Our main concern in this paper is the design of simplified filtering procedures for the quasi-optimal approximation of functions in subspaces of L_2 generated from the translates of a function $\varphi(x)$. Examples of signal representations that fall into this framework are Schoenberg's polynomial splines of degree n, and the various multiresolution spaces associated with the wavelet transform. After a brief review of the relation between the order of approximation of the representation and the concept of quasi-interpolation (Strang–Fix conditions), we investigate the implication of these conditions on the various basis functions and their duals (vanishing moment and quasi-interpolation properties). We then introduce the notion of quasi-duality and show how to construct quasi-orthogonal and quasi-dual basis functions that are much shorter than their exact counterparts. We also consider the corresponding quasi-orthogonal projection operator at sampling step h and derive asymptotic error formulas and bounds that are essentially the same as those associated with the exact least-squares solution. Finally, we use the idea of a perfect reproduction of polynomials of degree n to construct short kernel quasi-deconvolution filters that provide a well-behaved approximation of an oblique projection operator.

1. INTRODUCTION

The standard formula for the interpolation of a function on a uniform grid is

$$s(x) = \sum_{k \in Z} s(k)\varphi(x-k), \tag{1}$$

where φ is an interpolation kernel and where the expansion coefficients $s(k)$ are the integer samples of the function (or signal) $s(x) \in V(\varphi) = \text{span}\{\varphi(x-k)\}_{k\in Z}$. A well-known example is the cardinal series representation of band-limited functions with $\varphi(x) = \text{sinc}(x)$ [32, 22, 14]. *Quasi-interpolation* [24, 12, 10, 7] refers to a weaker form of interpolation that holds for polynomials of degree n

$$\forall p_n \in \pi^n, \quad p_n(x) = \sum_{k\in Z} p_n(k)\varphi(x-k), \tag{2}$$

where π^n denotes the subspace of polynomials of degree n. This important concept was introduced by Strang and Fix [24] as a theoretical tool for deriving error bounds for the approximation of functions using a general representation model of the form

$$s_h(x) = \sum_{k\in Z} c_h(k)\varphi\left(\frac{x}{h}-k\right), \tag{3}$$

where h is the sampling step and where the basis functions are uniformly spaced and rescaled accordingly. Specifically, these authors were able to prove that the simple quasi-interpolant condition (2) could be translated into the following remarkable error bound for the approximation of a smooth L_2-function $s(x)$:

$$\min_{c_h} \|s - s_h\| \leqslant C \cdot h^{n+1} \cdot \|s^{(n+1)}\|; \tag{4}$$

where $s^{(L)}$ denotes the Lth derivative of s and where $C = C(\varphi)$ is a constant that does not depend on s. During the past few years, this theory has been extended to the more general class of functional-based operators (cf. [11, 5, 16])

$$Q_\lambda s(x) = \sum_{k\in Z} \lambda[s(x+k)]\varphi(x-k), \tag{5}$$

such that $\forall p_n \in \pi^n, Q_\lambda p_n = p_n$, where λ: $L_2 \to R$ is a suitable linear functional (e.g., the inner product with some distribution or analysis function $\tilde{\varphi}$). Particular examples of signal representations that fall into this framework are polynomials splines [20, 19, 21, 28], and, more recently, the various types of wavelet basis and scaling functions [8, 18, 31]. In the latter case, the approximation order also corresponds to the number of vanishing moments of the wavelet function [23, 26, 25, 6].

In addition to their theoretical relevance, quasi-interpolants are useful for designing approximate interpolation formulas that are much easier to implement than their exact

[1]Fax: (301) 496-6608. E-mail: unser@helix.nih.gov.

1063-5203/96 $18.00

counterparts without any appreciable loss in performance [7]. In particular, quasi-interpolation schemes preserve the $O(h^{n+1})$ behavior of the error which is the best achievable (cf. (4)). Their main advantage is that it is possible to construct quasi-interpolation kernels that have a much smaller support than the true interpolator for a particular function space. For example, in the case of cubic splines, the shortest quasi-interpolant corresponds to a 3 tap FIR filter, while the exact interpolator (cardinal or fundamental spline) has an infinite support [4].

The purpose of this paper is to further explore the concept of quasi-interpolation for convolution-based signal approximations schemes that are more involved—and usually more accurate—than a simple interpolation (cf. Fig. 1). In particular, we will consider orthogonal and oblique projection operators and derive the quasi-interpolation properties of the underlying basis functions. We will also introduce the notions of quasi-orthogonality and quasi-projections and see how these can result in simplified implementation formulas. The main idea that we want to promote is that we can gain in flexibility by being less stringent in enforcing certain projection constraints. The basic requirement is a perfect reproduction for polynomials of degree n which, as in the quasi-interpolation case, keeps the error within the same bound as the intrinsic approximation error due to the choice of the representation space itself (cf. (4)). This property is consistent with the generalized notion of quasi-interpolation investigated by de Boor and others [11, 5]. However, our design is more constrained because of a stricter definition of quasi-duality which results in a smaller approximation error (higher degree of concordance with the corresponding least squares solution).

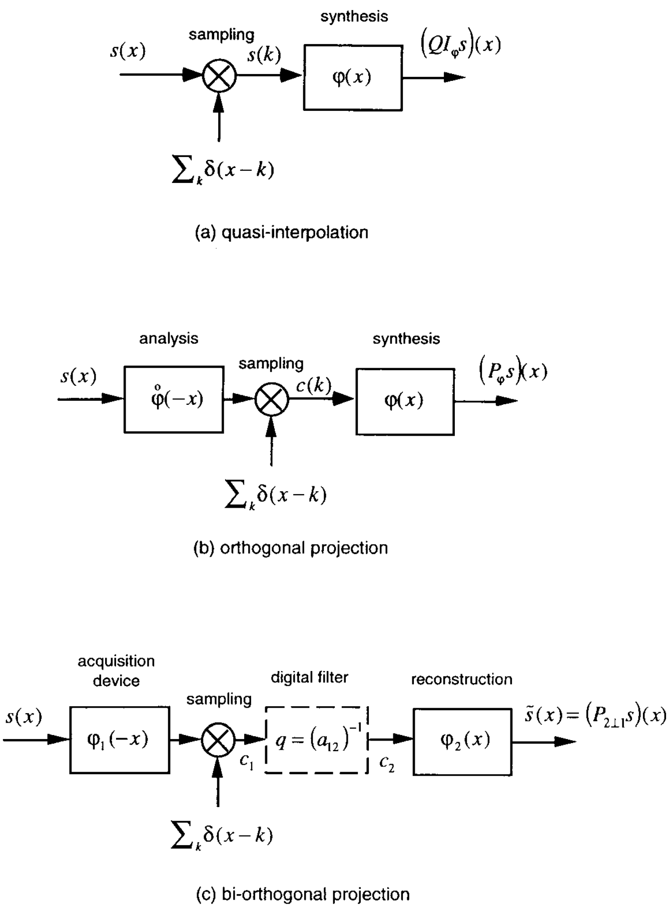

FIG. 1. Three alternative solutions for the approximation of L_2-signals in $V(\varphi) = V(\varphi_2)$: (a) (quasi-)interpolation, (b) least-squares approximation (orthogonal projection), (c) biorthogonal projection (generalized deconvolution). The sampling operation is modeled by a multiplication with the sequence of Dirac impulses $\sum_{k\in Z}\delta(x-k)$. The solid rectangular boxes represent continuous convolution operators (analog filters).

The presentation is organized as follows. In Section 2, we start by introducing the relevant signal representations and review the two projection schemes for constructing signal approximations in those subspaces. In Section 3, we consider signal representations with an order of approximation L and investigate the implication of this property on the underlying basis functions and their duals. In Section 4, we introduce the notions of quasi-orthogonality and quasi-dual basis functions and derive a corresponding error bound which is essentially the same as for the orthogonal case. In Section 5, we consider the approximation of an oblique (or bi-orthogonal) projector and show how this technique can be useful for designing deconvolution filters. Finally, in Section 6, we conclude with some general comments on the proposed methods and provide some future direction of research.

2. PRELIMINARIES: SIGNAL REPRESENTATION AND APPROXIMATION

2.1. Signal Subspaces

A general approach to specify continuous signal representations is to consider the class of functions generated from the integer translates of a single function $\varphi(x)$ [3, 24]. The corresponding function space $V(\varphi) \subset L_2$ is defined as

$$V(\varphi) = \left\{ s(x) = \sum_{k\in Z} c(k)\varphi(x-k) | c \in l_2 \right\}, \tag{6}$$

where l_2 is the vector space of square-summable sequences. The only restriction on the choice of the *generating* function φ is that the set $\{\varphi(x-k)\}_{k\in Z}$ is a Riesz basis of $V(\varphi)$; this is equivalent to the condition

$$A \leqslant \hat{a}_\varphi(\omega) = \sum_{k\in Z} |\hat{\varphi}(\omega + 2\pi k)|^2 \leqslant B \quad \text{a. e.,} \tag{7}$$

where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(x)$, and where A and B are two strictly positive constants [3]. This constraint ensures that each function $s(x)$ in $V(\varphi)$ is uniquely characterized by the sequence of its coefficients $c(k)$.

This formulation is quite general and covers many signal representation models that have been used in the literature. Examples of interest are the class of bandlimited functions (with $\varphi(x) = \operatorname{sinc}(x)$), and polynomial spline representations which can be generated by taking φ to be the B-spline of degree n. Other special cases are the various subspaces associated with the wavelet transform and multiresolution analysis; this connection is further discussed in [2].

2.2. Orthogonal Projection

Given an arbitrary function $s \in L_2$, we can determine its minimum L_2-error approximation in $V(\varphi)$. This corresponds to the orthogonal projection of s onto $V(\varphi)$, which is given by

$$(P_\varphi s)(x) = \sum_{k\in Z} \langle s(x), \mathring{\varphi}(x-k)\rangle \varphi(x-k), \tag{8}$$

where $\mathring{\varphi} \in V(\varphi)$ is the dual of φ. This function is defined as

$$\mathring{\varphi}(x) = \sum_{k\in Z} (a_\varphi)^{-1}(k)\varphi(x-k), \tag{9}$$

where $(a_\varphi)^{-1}$ represents the convolution inverse of the autocorrelation sequence $a_\varphi(k) := \langle \varphi(x), \varphi(x-k)\rangle$. Note that this latter sequence is invertible because of the admissibility condition (7). Equation (8) can be implemented through the block diagram in Fig. 1b. The main difference with the (quasi-) interpolation approximation scheme in Fig. 1a is that the signal is prefiltered prior to sampling. If the generating function is orthonormal (i.e., $a_\varphi(k) = \delta[k]$), then it is its own dual, and the prefilter is just the time-reversed version of the postfilter. In particular, for $\varphi(x) = \operatorname{sinc}(x)$, we get the standard discretization procedure dictated by Shannon's sampling theory, which uses an ideal lowpass filter prior to sampling in order to suppress aliasing.

2.3. Oblique Projection

Under the least-squares constraint, the dual analysis function $\mathring{\varphi}$ is uniquely specified (cf. (9)). In practice, this is often too constraining a condition. For instance, this makes it impossible to choose analysis and synthesis filters that are both compactly supported, unless φ is also orthogonal to start with. In some other cases, the analysis function may correspond to the impulse response of an acquisition device which is typically given a priori. In order to deal with those situations and gain in flexibility, we can consider the more general sampling procedure in Fig. 1c, where the analysis and synthesis functions φ_1 and φ_2 are (almost) arbitrary, except that they satisfy the admissibility condition (7). The system includes an additional digital correction filter q that can be specified so that the input signal $s(x)$ and its approximation $\tilde{s}(x)$ are consistent in the sense that they yield the same measurements:

$$c_1(k) := \langle \varphi_1(x-k), s(x)\rangle = \langle \varphi_1(x-k), \tilde{s}(x)\rangle. \tag{10}$$

Under those constraints, the system is uniquely determined and the desired approximation $\tilde{s}(x)$ corresponds to the projection of s onto $V(\varphi_2)$ perpendicular to the analysis space $V(\varphi_1)$ [27]. It is given by

$$\tilde{s}(x) = P_{2\perp 1}s(x) = \sum_{k\in Z} ((a_{12})^{-1} * c_1)(k)\varphi_2(x-k), \tag{11}$$

where $q = (a_{12})^{-1}$ is the convolution inverse of the cross-correlation sequence $a_{12}(k) := \langle \varphi_1(x-k), \varphi_2(x)\rangle$.

A necessary and sufficient condition for this solution to be well defined is that the angle θ_{12} between the subspaces $V(\varphi_1)$ and $V(\varphi_2)$ is less than 90° [1]. This quantity is given by (cf. [27])

$$\cos(\theta_{12}) = \operatorname*{ess\,inf}_{\omega\in[0,\pi]} |\hat{r}_{12}(\omega)|, \tag{12}$$

where $|\hat{r}_{12}(\omega)|$ is the *spectral coherence* function defined by

$$|\hat{r}_{12}(\omega)| = \frac{\sum_{k\in Z} \overline{\hat{\varphi}_1(\omega+2\pi k)}\hat{\varphi}_2(\omega+2\pi k)}{\sqrt{\sum_{k\in Z} |\hat{\varphi}_1(\omega+2\pi k)|^2}\sqrt{\sum_{k\in Z} \hat{\varphi}_2(\omega+2\pi k)|^2}}. \tag{13}$$

Note that $|\hat{r}_{12}(\omega)| = 1$ (or $\cos(\theta_{12}) = 1$) if and only if $\varphi_1 \in V(\varphi_2)$ (or $V(\varphi_1) = V(\varphi_2)$), in which case the approximation given by (11) is equivalent to the least-squares solution (8) with $\varphi = \varphi_2$. The geometrical interpretation is also useful in providing us with an indication on how good our approximation is compared to the least-squares solution. Specifically, we have the error bound (cf. [27], Theorem 3)

$$\forall s \in L_2, \quad \|s - P_2 s\| \leqslant \|s - P_{2\perp 1}s\| \leqslant \frac{1}{\cos\theta_{12}} \|s - P_2 s\|, \tag{14}$$

where $P_2 s$ denotes the orthogonal projection of s onto $V(\varphi_2)$. This is a way of indicating that the orthogonal and biorthogonal projections are essentially equivalent, at least as far as the magnitude of the approximation error is concerned.

2.4. Strang–Fix Conditions and Approximation Order

It is often of interest to choose a sampling step h that is different from the unit step that has been used so far to keep the presentation simple. The representation model is then given by (3) and the approximation formulas described in the two previous sections can easily be adapted by appropriate scaling and normalization. As h gets smaller, the approximation error $\|s - s_h\|$ generally decreases and eventually becomes negligible as h goes to zero. As mentioned in the Introduction, the general behavior of this error as a function of h depends on the ability of the representation to reproduce polynomials up to a certain degree n. This result is expressed by the Strang–Fix conditions [24], which relate the approximation power of the representation to the spectral characteristics of the generating function φ. These authors considered the case where φ has compact support, but their result has been extended for noncompact φ with polynomial decay [17, 15, 13].

Strang–Fix Conditions. Let φ be a generating function with appropriate decay (compact support [24], or polynomial decay [15, 13]). The following statements are equivalent:

(i) The function space $V(\varphi)$ reproduces all polynomials of degree $n = L-1$, which is equivalent to saying that there exists a function $\varphi_{QI} \in V(\varphi)$ (not necessarily unique) that is a quasi-interpolant of order L.

(ii) There exists a function $\varphi_{QI} \in V(\varphi)$ (the same as in condition (i)) such that

$$\hat{\varphi}_{QI}(2\pi k) = \delta[k] \leftrightarrow \sum_{k\in Z} \varphi_{QI}(x-k) = 1 \qquad (15)$$

$$\hat{\varphi}^{(m)}_{QI}(2\pi k) = 0 \quad (m = 1, \ldots, L-1) \leftrightarrow \sum_{k\in Z} (x-k)^m \varphi_{QI}(x-k) = 0, \qquad (16)$$

where $\hat{\varphi}^{(m)}_{QI}(\omega)$ denotes the mth derivative of the Fourier transform of φ_{QI}.

(iii) $\hat{\varphi}(\omega)$, the Fourier transform of φ, is nonvanishing at the origin and has zeros of at least multiplicity L at all nonzero frequencies that are integer multiples of 2π.

(iv) The approximation error at step size h is bounded as

$$\forall s \in W_2^L, \qquad \inf_{s_h(x\cdot h)\in V(\varphi)} \|s - s_h\| \leqslant C \cdot h^L \cdot \|s^{(L)}\|, \quad (17)$$

where W_2^L is Sobolev's space of order L, i.e., the space of smooth functions whose L first derivatives are defined in the L_2-sense (bounded energy).

The maximum value of L for which any of these conditions is satisfied defines the order of approximation of the representation. With this definition the order is one larger than the degree n. For example, polynomial splines of degree n have an order of approximation $L = n + 1$.

Interestingly, the present stability condition (7) ensures that the least-squares approximation is controlled in the sense specified by Strang and Fix because $\|c_h\|^2/h \leqslant A^{-1} \cdot \|P_h s\|^2 \leqslant A^{-1} \cdot \|s\|^2$, where A is the lower frame bound. This last condition is required to support the implication from (iv) to (i).

In practice, the simplest test for determining the order of approximation L of a certain representation space $V(\varphi)$ is to check for property (iii).

3. QUASI-INTERPOLATION PROPERTIES

In this section, we derive a number of properties of the basis functions of $V(\varphi)$ that are directly related to the order of approximation of the representation. Although the proofs are elementary, we believe that most of these results have not been described before.

To be on the safe side when we integrate polynomials against φ or $\mathring{\varphi}$, we will assume that both functions have exponential decay. Note that the stronger condition "φ has compact support" implies that $\mathring{\varphi}$ has exponential decay (cf. (9)). In principle, only one function can be compactly supported unless φ is orthogonal as well.

PROPOSITION 1. *Let φ and its dual $\mathring{\varphi}$ be two generating functions with exponential decay. If φ is a quasi-interpolant of degree n then the same is true for $\mathring{\varphi}$.*

Proof. $\mathring{\varphi}$ also generates $V(\varphi)$ so that its Fourier transform has the required vanishing properties at all nonzero frequencies that are integer multiple of 2π (cf. condition (iii)). We therefore only need to consider the behavior of its Fourier transform at the origin (cf. condition (ii)). For this purpose, we consider the following equivalent form of the quasi-interpolant property of φ (interpolation of all monomials up to degree n):

$$\sum_{k\in Z} k^m \varphi(x-k) = x^m \quad (m = 0, \ldots, n). \qquad (18)$$

We then multiply each side by $\mathring{\varphi}(x)$ and integrate, which yields

$$\begin{aligned}\int_{-\infty}^{+\infty} x^m \, \mathring{\varphi}(x)\, dx &= \sum_{k\in Z} k^m \langle \varphi(x-k), \mathring{\varphi}(x)\rangle \\ &= \sum_{k\in Z} k^m \delta[k] = \begin{cases} 1, & \text{if } m = 0 \\ 0, & m = 1, \ldots, n \end{cases}\end{aligned} \qquad (19)$$

simply because $\langle \varphi(x-k), \mathring{\varphi}(x-l)\rangle = \delta[k-l]$ (biorthogonality property). Note that we need the decay conditions on φ and $\mathring{\varphi}$ in order to be able to permute the sum and the integral in the first line of (19). The left-hand side of (19) represents the mth moment of $\mathring{\varphi}$; it also corresponds to the value of the mth derivative of the Fourier transform of this function at the origin. Hence, we have established the desired result and also shown that the integral and the n first moments of φ and $\mathring{\varphi}$ are identical (n vanishing moments):

$$\int_{-\infty}^{+\infty} \mathring{\varphi}(x)\, dx = \int_{-\infty}^{+\infty} \varphi(x)\, dx = 1 \qquad (20)$$

$$\int_{-\infty}^{+\infty} x^m \, \mathring{\varphi}(x)\, dx = \int_{-\infty}^{+\infty} x^m \varphi(x)\, dx = 0 \quad (m = 1, \ldots, n). \quad ■ \qquad (21)$$

PROPOSITION 2. *If φ is a quasi-interpolant of degree n then*

$$\forall p_n \in \pi^n, \quad \int_{-\infty}^{+\infty} p_n(x)\varphi(x-k)\,dx = p_n(k). \tag{22}$$

In other words, quasi-interpolants have the ability to sample polynomials. For this class of functions, their effect is the same as that of the Dirac delta function. An implication is that a quasi-interpolant can be used as its own dual for reproducing polynomials

$$\forall p_n \in \pi^n, \quad p_n(x) = \sum_{k\in Z} \langle p_n(x), \varphi(x-k)\rangle \varphi(x-k), \tag{23}$$

where the term $\langle p_n(x), \varphi(x-k)\rangle$ should be interpreted as an integral rather than an inner product in the conventional L_2-sense. By extension, we also have that

$$\forall p_n \in \pi^n, \quad p_n(x) = \sum_{k\in Z} \langle p_n(x), \varphi_1(x-k)\rangle \varphi_2(x-k), \tag{24}$$

where φ_1 and φ_2 are any quasi-interpolants of degree n. In particular, this is true for $\varphi_1 = \mathring{\varphi}$ and $\varphi_2 = \varphi$, which is the case of our interest. Other related polynomial reproduction properties are discussed in [11] and [5].

Proof. Because of the moment conditions (20) and (21), we have that

$$\forall p_n \in \pi^n, \quad \int_{-\infty}^{+\infty} p_n(x)\varphi(x)\,dx = p_n(0).$$

We can apply this formula to $p_n(x-y)$ which is also a polynomial in x; the result then follows by a simple change of variable in the integral. ■

PROPOSITION 3. *If φ is the Lth order generating function with exponential decay, then $\eta = \mathring{\varphi}^T * \varphi$ is a symmetrical (quasi-) interpolant of order $2L$.*

Proof. The Fourier transform of η is given by

$$\hat{\eta}(\omega) = \frac{|\hat{\varphi}(\omega)|^2}{\sum_{k\in Z} |\hat{\varphi}(\omega + 2\pi k)|^2}. \tag{25}$$

The decay condition implies that $\hat{\varphi}(\omega) \in C^\infty$, while the Lth order approximation property specifies that $\hat{\varphi}(\omega)$ has zeros of multiplicity L at all nonzero frequencies that are integers multiples of 2π. The denominator of (25) is the Fourier transform of the sampled autocorrelation sequence $a_\varphi(k)$ which is exponentially decaying as well. Thus, $\hat{a}_\varphi(\omega) \in C^\infty$. Since $\hat{a}_\varphi(\omega)$ is also nonvanishing (stability condition), this implies that $\hat{\eta}(\omega)$ has zeros of multiplicity $2L$ at the critical frequencies. Thus, $\eta(x)$ has an $(2L)$th order of approximation and reproduces polynomials of degree $2L-1$. In addition, we can show that $\eta(x)|_{x=k} = \delta[k]$ by considering the 2π-periodized version of (25), which is one everywhere. Hence, η is the (unique) interpolating function associated with the higher order space $V(\eta) = V(\varphi * \varphi^T)$, which also implies that it is a quasi-interpolant. ■

Proposition 3 is closely related to Theorem 2.1 in [5], which deals with the more general situation where the analysis function is not necessarily the dual of φ. The essential difference is that the corresponding cross-correlation function is only a quasi-interpolator of order L (instead of $2L$ for the present case).

Let φ_1 and φ_2 be two equivalent generating functions of $V(\varphi)$. Then, it is always possible to find a sequence p_{12} such that

$$\varphi_2(x) = \sum_{k\in Z} p_{12}(k)\varphi_1(x-k). \tag{26}$$

In addition, we know that the sequence p_{12} defines a reversible digital filter in the sense that there exist two strictly positive constants m and M such that $m \leqslant |\hat{p}_{12}(\omega)|^2 \leqslant M$ (cf. [3, Proposition 1]).

Interestingly, the function $\eta(x)$ in Proposition 3 is invariant to such changes of basis. Specifically, if φ_2 is an equivalent generating function of $V(\varphi_1)$, then

$$\begin{aligned}
\hat{\eta}_2(\omega) &= \frac{|\hat{\varphi}_2(\omega)|^2}{\sum_{k\in Z} |\hat{\varphi}_2(\omega + 2\pi k)|^2} \\
&= \frac{|\hat{p}_{12}(\omega)|^2 |\hat{\varphi}_1(\omega)|^2}{|\hat{p}_{12}(\omega)|^2 \sum_{k\in Z} |\hat{\varphi}_1(\omega + 2\pi k)|^2} \\
&= \frac{|\hat{\varphi}_1(\omega)|^2}{\sum_{k\in Z} |\hat{\varphi}_1(\omega + 2\pi k)|^2} = \hat{\eta}_1(\omega),
\end{aligned}$$

since $\hat{p}_{12}(\omega)$ is 2π-periodic and nonvanishing.

PROPOSITION 4. *Let φ_1 and φ_2 be two exponentially decaying functions related to each other through* (26) *and consider the three following statements:* (i) *φ_1 is a quasi-interpolant of degree n*, (ii) *φ_2 is a quasi-interpolant of degree n, and* (iii) $\hat{p}_{12}^{(m)}(0) = \delta[m]$, $m = 0, \ldots, n$. *Then, we have the following implications:* (i) *and* (ii) ⇒ (iii); (i) *and* (iii) ⇒ (ii); (ii) *and* (iii) ⇒ (i).

Proof. We start by rewriting (26) in the Fourier transform domain as

$$\hat{\varphi}_2(\omega) = \hat{p}_{12}(\omega) \cdot \hat{\varphi}_1(\omega).$$

An implicit assumption is that $\hat{\varphi}_1(0) \neq 0$, which implies that $\hat{p}_{12}(\omega) = \hat{\varphi}_2(\omega)/\hat{\varphi}_1(\omega)$ has the same smoothness at $\omega = 2\pi k$ as $\hat{\varphi}_1$ and $\hat{\varphi}_2$ around the origin. Thus, we only need to consider the behavior of the Fourier transforms of φ_1 and φ_2 at $\omega = 0$. Taking the mth derivative, we get

$$\hat{\varphi}_1^{(m)}(0) = \sum_{k=0}^{m} \binom{m}{k} \hat{p}_{12}^{(k)}(0) \cdot \hat{\varphi}_2^{(m-k)}(0).$$

Starting with $m = 0$, we immediately conclude that

$$\hat{\varphi}_1(0) = 1 \quad \text{and} \quad \hat{\varphi}_2(0) = 1 \Rightarrow \hat{p}_{12}(0) = \hat{p}_{12}(2\pi k) = 1.$$

We then work our way up from $m = 1$ to n and use the previously established relations to show that

$$\hat{\varphi}_1^{(m)}(0) = 0 \quad \text{and} \quad \hat{\varphi}_2^{(m)}(0) = 0 \Rightarrow \hat{p}_{12}^{(m)}(0) = \hat{p}_{12}^{(m)}(2\pi k) = 0,$$

which ultimately proves the first implication: (i) and (ii) $\Rightarrow$ (iii). The same procedure can also be used to prove the two other implications. ■

By simple permutation, we see that the conditions in Proposition 4 also apply for the inverse sequence $\hat{p}_{21}(\omega) = 1/\hat{p}_{12}(\omega)$. For illustration purposes, let us consider the case of the interpolator which can always be written in the form (26) with

$$\hat{p}_{12}(\omega) = \frac{1}{\sum_{k\in Z} \hat{\varphi}_1(\omega + 2\pi k)},$$

where φ_1 is a quasi-interpolant. In this case, it is simpler to work with the inverse sequence and it is straightforward to show that

$$\hat{p}_{21}^{(m)}(0) = \sum_{k\in Z} \hat{\varphi}_1^{(m)}(0 + 2\pi k) = \delta[m],$$

which proves that the interpolator is indeed a quasi-interpolant (which should not come as a surprise). A less obvious result is the following proposition.

PROPOSITION 5. *If φ is an Lth order generating function with exponential decay, then there exists an orthogonal function $\phi \in V(\varphi)$ that is a quasi-interpolant of degree $n = L - 1$.*

Proof. Let us select a generating function φ that is a quasi-interpolant. Since the corresponding dual is a quasi-interpolant as well, we know from Proposition 4 that the function

$$\hat{q}(\omega) = \frac{1}{\sum_{k\in Z} |\hat{\varphi}(\omega + 2\pi k)|^2}$$

has the required flatness properties at the origin. One particular choice of orthogonal generating function is $\phi(x) = \sum_{k\in Z} p(k)\varphi(x - k)$, where the sequence p is symmetric and has the real-valued Fourier transform

$$\hat{p}(\omega) = \frac{1}{\sqrt{\sum_{k\in Z} |\hat{\varphi}(\omega + 2\pi k)|^2}}.$$

Clearly, $\hat{q}(\omega) = \hat{p}(\omega) \cdot \hat{p}(\omega)$ and we can express the mth derivative of this function as

$$\hat{q}^{(m)}(\omega) = \sum_{k=0}^{m} \binom{m}{k} \hat{p}^{(k)}(\omega) \cdot \hat{p}^{(m-k)}(\omega).$$

By setting $\omega = 0$ and working our way up from $m = 0$ to n as before, we show that

$$\hat{q}^{(m)}(0) = \delta[m] \Rightarrow \hat{p}^{(m)}(0) = \delta[m] \quad (m = 0, \ldots, n).$$

The desired result then directly follows from Proposition 4. ■

Note that many other orthogonal generation functions can be obtained by considering complex square roots of $\hat{q}(\omega)$, which differ from $\hat{p}(\omega)$ above by a phase factor only. Such orthogonal functions will in general not be quasi-interpolants. For example, the compactly supported Daubechies scaling functions are not quasi-interpolants, although the underlying spaces have the required approximation properties [9]. On the other hand, the orthogonal Battle–Lemarié spline functions, which are symmetrical, are quasi-interpolants of degree n, where n is the degree of the splines.

PROPOSITION 6. *For real-valued quasi-interpolants of degree n, the moments of the functions φ and $\mathring{\varphi}$ up to order $2n + 1$ are identical up to an alternating change of sign*

$$m_\varphi^m = \int_{-\infty}^{+\infty} x^m \varphi(x)\, dx = (-1)^{m-1} \int_{-\infty}^{+\infty} x^m\, \mathring{\varphi}(x)\, dx,$$

$$m = 1, \ldots, 2n + 1 \qquad (27)$$

(*the first n ones being zero as a result of Proposition* 1).

The implication of Proposition 6 in the symmetric case is that the Fourier transforms of φ and $\mathring{\varphi}$ have an exact opposite behavior near the origin. This is simply because the odd order moments of a symmetric functions are all zero.

Another implication is that the even order moments of ϕ (the orthogonal generating function in Proposition 5) are zero up to order $2n + 1$. This also means that the modulus of $\hat{\phi}(\omega)$ (which is the same for all orthogonalized versions of φ) will exhibit a higher degree of flatness at the origin. Specifically, we have that $|\hat{\phi}(\omega)| = 1 + O(\omega^{2n+2})$ as $\omega \to 0$, since all odd derivatives of a symmetric function are zero at the origin. Thus, in the symmetric case, the prefiltering by ϕ introduces relatively little distortion, at least for signals that are sufficiently smooth (i.e., predominantly lowpass). This may provide a justification for the quadrature formula $\langle s(x), \phi(x - k)\rangle \cong s(k)$ which is frequently used for the finer scale initialization of the wavelet transform [9, 18].

Proof. The mth moment corresponds to the mth derivative of the Fourier transform at the origin. Let us therefore consider the mth derivative of the function $\hat{\eta}(\omega)$ defined in Proposition 3, which we choose to expand as

$$\hat{\eta}^{(m)}(\omega) = \sum_{k=0}^{m} \binom{m}{k} \hat{\mathring{\varphi}}^{(k)}(\omega) \bar{\hat{\varphi}}^{(m-k)}(\omega). \qquad (28)$$

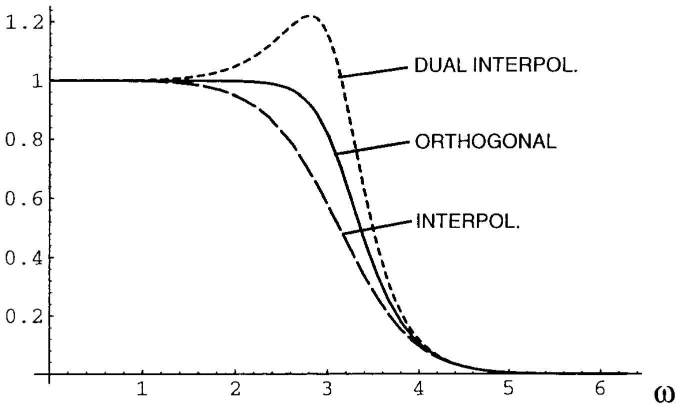

FIG. 2. Fourier transforms of various quasi-interpolating cubic spline functions ($L = 4$). Shown are the Orthogonal spline (solid line), interpolating (or fundamental) spline (dashed line), and dual of the interpolating spline (dotted line).

If φ is real-valued, then $\bar{\hat{\varphi}}(\omega) = \hat{\varphi}(-\omega)$, which implies that $\bar{\hat{\varphi}}^{(n-k)}(\omega) = (-1)^{n-k}\hat{\varphi}^{(n-k)}(-\omega)$. We already know that the first n derivatives of $\hat{\mathring{\varphi}}$ and $\hat{\varphi}$ are zero at the origin. Thus, for $m = n+1$ and $\omega = 0$, there are only two nonzero terms in (28), and we get that

$$\hat{\eta}^{(n+1)}(0) = \hat{\mathring{\varphi}}(0)(-1)^{n+1}\hat{\varphi}^{(n+1)}(0) + \hat{\mathring{\varphi}}^{(n+1)}(0)\hat{\varphi}(0) = 0.$$

Since $\hat{\mathring{\varphi}}(0) = \hat{\varphi}(0) = 1$ (cf. (20)), we conclude that

$$\hat{\mathring{\varphi}}^{(n+1)}(0) = (-1)^n\hat{\varphi}^{(n+1)}(0),$$

which is equivalent to (27) for $m = n + 1$. The same reasoning also applies for the other values of $m \leqslant 2n + 1$. ■

Proposition 6 provides us with a better understanding of the qualitative behavior of some of the filters previously described in the context of polynomial spline sampling theory [28]. Examples of such functions for the cubic spline case are shown in Fig. 2. The lower function represents the transfer function of a cubic spline interpolator. Its Taylor development at the origin is $\hat{\varphi}(\omega) = 1 - \omega^4/720 - \omega^6/3024 - 7\omega^8/259200 + O(\omega^9)$. The corresponding prefilter has exactly the opposite behavior at the origin, as exemplified by its Taylor series: $\hat{\mathring{\varphi}}(\omega) = 1 + \omega^4/720 + \omega^6/3024 + 17\omega^8/604800 + O(\omega^9)$. The degree of flatness of these functions at the origin is characteristic of a fourth-order quasi-interpolant. Also represented is the orthogonal Battle–Lemarié cubic spline function, which lies in between the two previous functions. In fact, it corresponds precisely to their geometric mean. This function is substantially flatter at the origin and has seven vanishing moments as predicted by the theory. Its behavior near the origin (Taylor series) is $\hat{\phi}(\omega) = 1 - \omega^8/2419200 + O(\omega^9)$ as $\omega \to 0$.

We end this section by briefly showing how the quasi-interpolant properties established so far can simplify the pointwise analysis of the approximation error. For this purpose, we first define the orthogonal projection operator at step size h

$$P_h s(x) = \sum_{k\in Z} \frac{1}{h}\left\langle s(x),\ \mathring{\varphi}\left(\frac{x}{h} - k\right)\right\rangle \varphi\left(\frac{x}{h} - k\right)$$
$$= \left\langle s(y), \frac{1}{h}K_\varphi\left(\frac{x}{h}, \frac{y}{h}\right)\right\rangle, \quad (29)$$

where

$$K_\varphi(x, y) = \sum_{k\in Z} \varphi(x-k)\ \mathring{\varphi}(y-k) \quad (30)$$

is the reproducing kernel of the Hilbert space $V(\varphi)$; the factor $1/h$ provides the proper inner-product normalization. Assuming that $s(x) \in C^{2L}$, we use a standard Taylor series argument to derive the asymptotic error formula (cf. [6, 25])

$$\begin{aligned} s(x) &- (P_h s)(x) \\ &= -\sum_{m=0}^{2L-1} \frac{h^m}{m!} e_m\left(\frac{x}{h}\right) s^{(m)}(x) + O(h^{2L}) \\ &= -\sum_{m=L}^{2L-1} \frac{h^m}{m!} e_m\left(\frac{x}{h}\right) s^{(m)}(x) + O(h^{2L}) \quad \text{as } h \to 0, \end{aligned} \quad (31)$$

where the error functions $e_m(x)$ are defined as

$$e_m(x) = \int_{-\infty}^{+\infty} (y-x)^m K_\varphi(x, y)\, dy. \quad (32)$$

Because of the polynomial reproduction properties, the L first error terms are zero, which explains the pointwise $O(h^L)$ behavior of the error. Thanks to our previous results, we can now derive two relatively simple formulas for the error terms up to order $2L - 1$.

PROPOSITION 7. *If φ is a quasi-interpolant of order L as in Proposition 1 then*

$$e_m(x) = m_{\mathring{\varphi}}^m + (-1)^m \sum_{k\in Z}(x-k)^m \varphi(x-k) \quad (33)$$

$$= (-j)^m \sum_{\substack{k\in Z \\ k\neq 0}} \hat{\varphi}^{(m)}(2\pi k) e^{j2\pi kx} \quad \text{for } m = 0, \ldots, 2L-1 \quad (34)$$

(*the first L ones being zero as a result of the Strang–Fix conditions*).

Proof. We expand $e_m(x)$ as

$$\begin{aligned} e_m(x) &= \int_{-\infty}^{+\infty} \sum_{k\in Z}[(y-k)-(x-k)]^m \varphi(x-k)\,\mathring{\varphi}(y-k)dy \\ &= \sum_{k\in Z}\sum_{l=0}^{m} \binom{m}{l} (-1)^l (x-k)^l \varphi(x-k) \\ &\quad \times \int_{-\infty}^{+\infty} (y-k)^{m-l}\,\mathring{\varphi}(y-k)\cdot dy \\ &= \sum_{l=0}^{m} \binom{m}{l} \left((-1)^l \sum_{k\in Z}(x-k)^l \varphi(x-k)\right) \\ &\quad \times \int_{-\infty}^{+\infty} x^{m-l}\,\mathring{\varphi}(x)\,dx, \end{aligned}$$

where we use Lebesgue's dominated convergence theorem to justify the various permutations of the infinite sum.[2] Since φ and $\mathring{\varphi}$ are both quasi-interpolants of order L (cf. Proposition 1), for most terms in the binomial expansion, there is typically at least one factor that is zero (cf. the Strang–Fix condition (ii) and the moment properties (20) and (21)). In fact for $m < 2L$, the only nonvanishing terms are the first and last ones:

$$e_m(x) = \int_{-\infty}^{+\infty} x^m\,\mathring{\varphi}(x)\,dx + (-1)^m \sum_{k\in Z}(x-k)^m \varphi(x-k) \quad \text{for } m = 0,\dots,2L-1.$$

The Fourier series representation (34) is obtained by observing that the rightmost term corresponds to the periodization of the function $g(x) = (-x)^m\varphi(x)$ whose Fourier transform is $\hat{g}(\omega) = (-j)^m\hat{\varphi}^{(m)}(\omega)$. The DC component ($k = 0$) is taken care of by using the moment relation in Proposition 6. ■

4. QUASI-ORTHOGONALITY

In certain cases, it may be of interest to compute an approximate form of the orthogonal projection of a function $s \in L_2$ using an approximate representation of the dual $\mathring{\varphi}$ in (8). Alternatively, one may also wish to use a single analysis/synthesis function that is approximately orthogonal but satisfies some other properties that are desirable for the application at hand (i.e., compact support). The idea is therefore to relax the orthogonality constraint, but in a way that essentially preserves the approximation power of the method.

Our definition of the notion of quasi-duality is primarily motivated by Proposition 3, which gives a direct link between the concepts of duality and interpolation. A more rigorous justification will be provided by the error analysis in Section 4.2.

Definition 1. Two generating functions $\varphi_1 \in V(\varphi)$ and $\varphi_2 \in V(\varphi)$ are *quasi-duals* of order L if $(\varphi_1^T * \varphi_2)(x)$ is a quasi-interpolant of order $2L$.

Definition 2. A generating function is *quasi-orthogonal* or order L if it is its own quasi-dual.

In particular, we note that a sufficient condition for a real-valued function to be quasi-orthogonal of order L is that its moment of order zero to be one and its higher order moments for $m = 1,\dots,2L-1$ all be zero. These constraints are consistent with Proposition 6.

4.1. Construction of Quasi-Dual Basis Functions

The design of such functions is relatively straightforward and primarily amounts to constructing a discrete sequence $q(k)$ subject to the moment constraints

$$m_q^n = \sum_{k\in Z} k^n q(k) = C_n \quad (n = 0,\dots,2L-1), \tag{35}$$

where $C_0,\dots,C_{2L-1}$ are some specified constants. This yields a linear system of equations in terms of the unknown coefficients $q(k)$. In particular, we can find the sequence q of minimal length ($2L$) by solving a Vandermonde system of equations, which is guaranteed to have a unique solution. Alternatively, we may also consider longer sequences. Since these are no longer uniquely defined, this gives us the possibility of introducing additional problem-related constraints.

To be more specific, let us consider the design of a quasi-orthogonal function $\phi_{QO} \in V(\varphi)$, which we choose to represent as

$$\phi_{QO}(x) = \sum_{k\in Z} q(k)\varphi(x-k). \tag{36}$$

Taking the mth derivative of the Fourier transform of ϕ_{QO} at the origin, we get

$$\hat{\phi}_{QO}^{(m)}(0) = \sum_{i=0}^{m} \binom{m}{i} \hat{\varphi}^{(i)}(0)\hat{q}^{(m-i)}(0), \tag{37}$$

a relation that can also be rewritten in terms of the moments of these functions

$$m_\phi^m = \sum_{i=0}^{m} \binom{m}{i} m_\varphi^i m_q^{m-i}. \tag{38}$$

[2] Specifically, for a fixed x, we consider the above expression as the limit of the integral of the convergent series of functions $f_N(y) = (y-x)^m \sum_{k=-N}^{+N} \varphi(x-k)\,\mathring{\varphi}(y-k)$. Using the decay conditions on φ and $\mathring{\varphi}$, we derive the corresponding upper bound $|f_N(y)| \leqslant C\cdot|y-x|^m\cdot e^{-(\gamma/2)|y-x|}$, which is integrable. This proves that the sum of the integrals converges to the integral of the limit. Likewise, we can show that $\sum_{k=-N}^{+N}(x-k)^l\varphi(x-k) \leqslant C_l < \infty$, which justifies the permutation of two sums.

We use the simplest quasi-orthogonality specification which is to set all higher order moments to zero. This leads to the following triangular system of equation in term of the unknowns $m_q^0, \ldots, m_q^{2L-1}$:

$$\begin{bmatrix} m_\varphi^0 & \cdots & & & 0 \\ m_\varphi^1 & m_\varphi^0 & & & \\ m_\varphi^2 & 2m_\varphi^1 & m_\varphi^0 & & \vdots \\ m_\varphi^3 & 3m_\varphi^2 & 3m_\varphi^1 & m_\varphi^0 & \\ & & & & \ddots \end{bmatrix} \cdot \begin{bmatrix} m_q^0 \\ m_q^1 \\ \vdots \\ \vdots \\ m_q^{2L-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \quad (39)$$

Since $m_\varphi^0 \neq 0$ (cf. condition (iii)), the system has a unique solution that can be determined by straightforward Gaussian elimination. Hence, we can get the required constants in (35) and proceed to determine an admissible sequence $q(k)$. As an example, we have considered the case where φ is a B-spline of degree n. The corresponding coefficients of the quasi-orthogonal splines for $n = 0, \ldots, 3$ are given in Table 1. The frequency responses of the orthogonal cubic splines and its shortest quasi-orthogonal approximation are shown in Fig. 3. The quasi-orthogonal solution provides a reasonably good approximation that is obviously better for lower frequencies. In any case, it is a better substitute for $\phi(x)$ than the standard quasi-interpolation solution, which has only half as many vanishing moments at the origin.

Finally, we note that we can apply the same design procedure for constructing quasi-duals of φ, except that we need to consider the moments of the autocorrelation function $\varphi * \varphi^T$ instead of those of φ.

4.2. Error Analysis

In order to deal with the approximation error issue, we define the quasi-projection operator that computes an approximation of a signal $s \in L_2$ at the sampling step h

$$QP_h s(x) = \sum_{k\in Z} \frac{1}{h} \left\langle s(x), \tilde{\varphi}\left(\frac{x}{h} - k\right)\right\rangle \varphi\left(\frac{x}{h} - k\right), \quad (40)$$

where $\tilde{\varphi}$ is an Lth order quasi-dual of φ (both with exponential decay).

By construction, the quasi-dual function $\tilde{\varphi}$ will have the same moments as $\mathring{\varphi}$ up to order $2L-1$. Assuming that φ is a quasi-interpolant of order L, we can use exactly the same manipulations as in Proposition 7 to derive the corresponding error functions which turn out to be the same as before for $m = 0, \ldots, 2L-1$. The implication of this analysis is that the quasi-projection and least-squares approximations are in agreement up to order $2L$:

$$QP_h s(x) - P_h s(x) = O(h^{2L}) \quad \text{as } h \to 0. \quad (41)$$

This is better than most quasi-interpolators which are only within $O(h^L)$ of the optimal solution. In particular, (41) implies that the quasi-projection is asymptotically optimal.

Another indication that the global behavior of the quasi-projector is very similar to that of the orthogonal projector is provided the following L_2-error bound.

TABLE 1
B-Spline Coefficients of the Shortest Quasi-Orthogonal Splines for $n = 0, \ldots, 3$

n	B-spline coefficients
0	$(+1)$
1	$(-\frac{1}{12}, +\frac{7}{6}, -\frac{1}{12})$
2	$(+\frac{37}{1920}, -\frac{97}{480}, +\frac{437}{320}, -\frac{97}{480}, +\frac{37}{1920})$
3	$(-\frac{41}{7560}, +\frac{311}{5040}, -\frac{919}{2520}, +\frac{12223}{7560}, -\frac{919}{2520}, +\frac{311}{5040}, -\frac{41}{7560})$

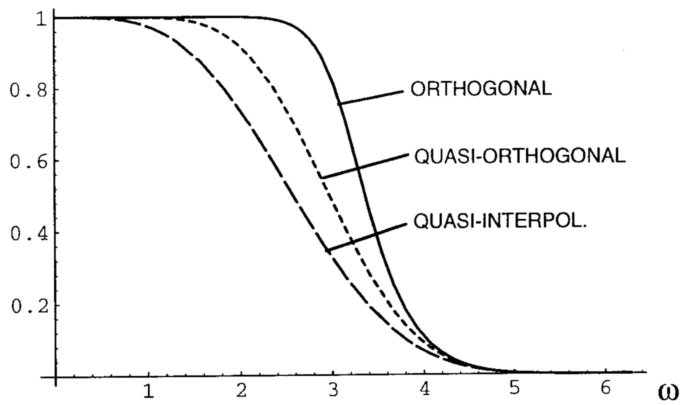

FIG. 3. Fourier transforms of the orthogonal (solid line), quasi-orthogonal (dotted line), and shortest quasi-interpolant (dashed line) cubic spline functions.

THEOREM 1. *If $V(\varphi)$ has an Lth order of approximation, then the approximation error is such that*

$$\forall s \in W_2^{2L},$$
$$\|s - QP_h s\|_{L_2} \leqslant C_1 \cdot h^{2L} \cdot \|s^{(2L)}\|_{L_2} + C_2 \cdot h^L \cdot \|s^{(L)}\|_{L_2}, \quad (42)$$

where $C_1 = C_1(\varphi, \tilde{\varphi})$ and $C_2 = C_2(\varphi, \tilde{\varphi})$ are two constants that do not depend on s.

The proof of this result is given in the Appendix. The important point here is that this result is essentially the same as the one derived in [29] for the exact least-squares solution, except for the values of the constants C_1 and C_2. This bound provides a finer characterization of the error as the one associated with the Strang–Fix conditions, but it is still consistent with this result. The advantage of the present form is that the constant C_2 in front of h^L can be significantly smaller than the constant C in the more standard quasi-interpolation bound (4). The reason for this is that the second term in (42) accounts for a portion of the error only: the out-of-band contribution e_2 (cf. (A-4) and (A-8)). The fact that we can construct an $O(h^{2L})$ bound for the remaining part of the error (cf. (A-3) and (A-6)) is really what

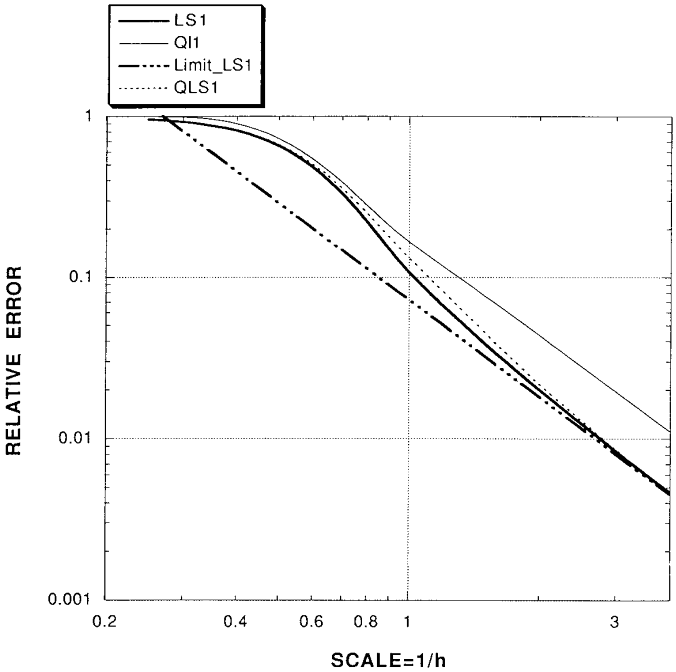

FIG. 4. Relative L_2-error as a function of the sampling step h for the approximation of the function $f(x) = -x \cdot \exp(-x^2/2)$ using linear splines. The three approximation methods are interpolation (QI1), quasi-orthogonal projection (QLS1), and orthogonal projection (least-squares solution) (LS1). The dashed line at the bottom represents the optimal asymptote predicted by the theory.

distinguishes the present approach from the more standard interpolation and quasi-interpolation schemes which yield only an $O(h^L)$ term. Consequently, we end up with a sharper result, at least for smaller values of h.

To illustrate these properties, we have considered the problem of approximating a wavelet function (first derivative of a Gaussian) using linear splines ($L = 2$) with a step size h. The corresponding error curves, which were all computed numerically, are given in Fig. 4. The graph clearly illustrates the fact that the quasi-orthogonal approximation (dotted line) is superior to an interpolation (QI1) in the sense that it is much closer to the minimum error solution (LS1). In particular, the LS1 and QLS1 curves are in perfect agreement at the two ends of the scale. All curves display the characteristic asymptotic $O(h^L)$ decay, but the interpolation approach is biased toward higher values. The QLS1 approach, on the other hand, is asymptotically optimal.

5. QUASI-PROJECTION AND DECONVOLUTION

We now go back to the block diagram in Fig. 1c and consider alternatives to the biorthogonal projection operator described in Section 2.3. As before, we assume that the analysis and synthesis functions φ_1 and φ_2 are given *a priori*. This particular setting typically corresponds to a deconvolution problem where the analysis function represents the impulse response of the acquisition device.

In order to gain in flexibility, we relax our initial measurement consistency constraint (10) which led to a solution that is a projector, but did not leave any further degrees of freedom. Instead, we simply require that this condition be enforced for polynomials of degree $n = L - 1$, where L is the approximation order of the representation space. Specifically, the constraint is

$$\forall p_n \in \pi^n, \quad p_n(x) = \sum_{k \in Z} (q * c_{p_n})(k) \varphi_2(x - k), \tag{43}$$

where the coefficients $c_{p_n}(k)$ are the corresponding measurements

$$c_{p_n}(k) = \int_{-\infty}^{+\infty} p_n(x) \varphi_1(x - k)\, dx. \tag{44}$$

In particular, we can consider the test functions

$$p_{2m}(x) = \sum_{l \in Z} l^m \varphi_2(x - l) \quad (m = 0, \ldots, n), \tag{45}$$

which are linearly independent and constitute a basis of π^n; these are indeed polynomials because $V(\varphi_2)$ has an order of approximation $L = n + 1$. We can determine their measurements values explicitly

$$c_{p_{2m}}(k) = \sum_{l \in Z} l^m \langle \varphi_1(x - k), \varphi_2(x - l) \rangle = \sum_{l \in Z} l^m a_{12}(k - l),$$

where a_{12} is the sampled cross correlation between φ_1 and φ_2 (cf. Section 2.3). By substituting these values in (43) and using the fact the set constitutes a basis of π^n, we express our initial requirement as a *quasi-identity* condition on the auxiliary sequence p

$$\sum_{k \in Z} k^m p(l - k) = l^m \quad (m = 0, \ldots, n), \tag{46}$$

where p is defined as

$$p(k) = (q * a_{12})(k). \tag{47}$$

The term quasi-identity refers to the fact that the sequence $p(k)$ is equivalent to the identity convolution operator for all polynomial sequences of degree n. Our problem therefore reduces to the design of a digital filter q that is a quasi-inverse of a_{12}. For this purpose, we make use of the following result.

PROPOSITION 8. *An equivalent form of the quasi-identity condition of degree n* (46) *is*

$$m_p^m = \sum_{k \in Z} k^m p(k) = \begin{cases} 1, & m = 0 \\ 0, & m = 1, \ldots, n. \end{cases} \tag{48}$$

Proof. By making the appropriate change of variable, we rewrite and expand the left-hand side of (46) as

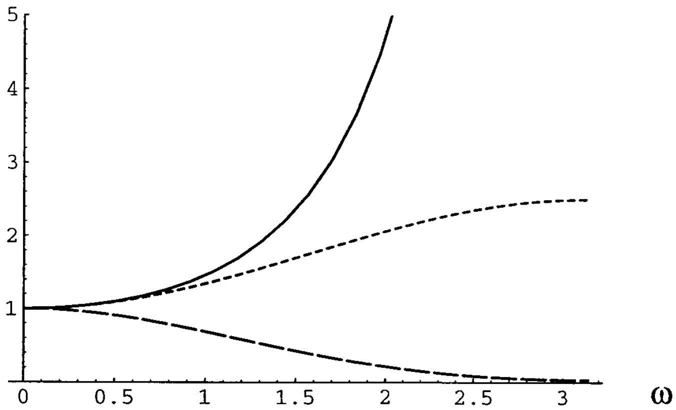

FIG. 5. Frequency responses of the biorthogonal (solid line) and quasi-projection (dotted line) filters for the deconvolution of a Gaussian blur (dashed line) using a cubic B-spline signal representation.

$$\sum_{k\in Z} k^m p(l-k) = \sum_{k\in Z}(l-k)^m p(k) = \sum_{k\in Z}\sum_{i=0}^{m}\binom{m}{i} l^m(-1)^{m-i} m_p^{m-i}.$$

We then work our way up for $m = 0$ to n to show that the two sets of equations are equivalent. For the direct part of the implication, we also use the fact that the identity (46) must be true for all l's. ■

Therefore, we can design the filter q by using essentially the same moment-based approach as before. Specifically, we start by evaluating the moments $m^0_{a_{12}}, \ldots, m^n_{a_{12}}$ of the cross-correlation sequence a_{12}. We then determine the corresponding moments of q by solving an $L \times L$ triangular system of equation similar to (39). The only requirement for obtaining a well-defined solution is that $m^0_{a_{12}} = \hat{a}_{12}(0) \neq 0$, which is a relatively weak constraint. Finally, we compute the filter coefficients $q(k)$ in the same way as before, except that the number of constraints is smaller (L).

One possible solution for specifying longer filters would to perform an optimization under constraints by minimizing the quadratic deviation between p and the identity, as suggested in [30] for the first-order case. Another criterion that potentially could be optimized is the mean square estimation error assuming a particular signal plus noise measurement model (constrained Wiener solution).

As a result of this general design strategy, the frequency response of the quasi-projection filter q has the same behavior at the origin (Taylor series up to order n) as the exact biorthogonal solution $(a_{12})^{-1}$. This can be shown through the simple argument

$$\hat{q}(\omega)\cdot\hat{a}_{12}(\omega) = 1 + O(\omega^L) \Rightarrow \hat{q}(\omega) = \frac{1}{\hat{a}_{12}(\omega)} + O(\omega^L) \quad \text{as } \omega \to 0.$$

Consequently, quasi-deconvolution filters should always perform a good signal recovery in the low frequency range.

For illustration purposes, let us consider the example of a Gaussian blur ($\varphi_1(x) = (2\pi\sigma)^{-1/2}\exp(-x^2/(2\sigma^2))$, which $\sigma^2 = \frac{1}{2}$) with a reconstruction in the space cubic splines ($L = 4$) using the cubic B-spline as generating function. The corresponding shortest fourth-order quasi-projector is a 3-tap symmetric FIR filter. Its frequency response is shown in Fig. 5 next to the inverse filter solution $1/\hat{a}_{12}(\omega)$. As expected, both solutions are in near perfect agreement at the origin. In terms of stability, the quasi-projection filter is better behaved in the sense that it provides a less drastic correction in the higher frequency range.

In order to reconcile the present approach with the quasi-orthogonal projection method in Section 4, it is of interest to represent the combined effect of the measurement device and correction filter q in term of an equivalent analysis function

$$\tilde{\varphi}_1(x) = \sum_{k\in Z} q^T(k)\varphi_1(x-k). \tag{49}$$

This function is also useful in establishing the connection with the generalized quasi-interpolation theory of de Boor [11]; it corresponds to the measure μ used by this author. If q is designed according to our specification, then $\tilde{\varphi}_1$ turns out to be a quasi-dual of φ_1, but in a weaker sense as in the definition given in Section 4:

PROPOSITION 9. *If the function spaces $V(\varphi_1)$ and $V(\varphi_2)$ have an order of approximation L_1 and L_2, respectively, and if the sequence q is an $(L_1 + L_2)$th order quasi-inverse of $a_{12}(k) = \langle\varphi_1(x), \varphi_2(x-k)\rangle$, then the cross-correlation function $(\tilde{\varphi}_1^T * \varphi_2)$ is a quasi-interpolant of order $(L_1 + L_2)$.*

Proof. We start by stating the following lemma, which is a rather direct consequence of condition (iii) in Section 3. ■

LEMMA 1. *Let φ be a generating function that has an Lth order of approximation, and let $b_\varphi(k) = \varphi(x)|_{x=k}$ be its sampled representation. Then, $\hat{b}_\varphi(\omega)$ and $\hat{\varphi}(\omega)$ have the same Lth-order Taylor series at the origin.*

Observing that $\varphi_{12} = \varphi_1^T * \varphi_2$ is a valid generating function with an $(L_1 + L_2)$th order of approximation, we use the above lemma to show that $\hat{\varphi}_{12}(\omega)$ and $\hat{a}_{12}(\omega)$ have the same Taylor series of order $(L_1 + L_2)$ at the origin. The argument is then essentially

$$\hat{\tilde{\varphi}}_{12}(\omega) = \hat{q}(\omega)\cdot\hat{\varphi}_{12}(\omega) = \hat{q}(\omega)\cdot\hat{a}_{12}(\omega) + O(\omega^{L_1+L_2}) = 1 + O(\omega^{L_1+L_2}).$$

So far, we have considered only the case $L_1 = 0$, which is the most likely to occur in practice. However, if φ_1 has a higher order of approximation, it may be of interest to design a filter q using more vanishing moment constraints to get a solution that is as close as possible to the quasi-orthogonal one in Section 4. Such a strategy will ensure that both solutions are equivalent if $\varphi_1 \in V(\varphi_2)$. Moreover, we can use the same argument as in Section 4.2 to show that the quasi-projection solution can be designed to be within $O(h^{L_1+L_2})$ of the optimal one (orthogonal projection). This is possible because the Taylor series of the oblique and orthogonal projection errors are identical up to order $L_1 + L_2 - 1$ (cf. [25, Theorem 2]).

6. CONCLUSION

Our goal in this paper has been twofold. First, we provided a detailed characterization of the quasi-interpolant properties of various types of basis functions associated with an Lth-order representation space. Second, we used the key idea of a perfect reproduction of polynomials of degree n to design approximate solutions to a variety of problems. These included the construction of quasi-orthogonal and quasi-dual basis functions which can be used to implement quasi-projection operators. Although the design recipes all turn out to be relatively straightforward (i.e., matching the Lth-order Taylor series of the Fourier transform of a sequence—or a function—to a given prototype), this type of approach has a more profound justification which takes its roots in approximation theory. Specifically, we were able to show that it provides the same type of error behavior as the exact solution (orthogonal projection).

We also applied these ideas to the construction of quasi-projection deconvolution filters. Although our method performs only an approximate deconvolution, it has potentially much to offer and may turn out to be a good practical alternative to the exact inverse filter (or biorthogonal) solution. The arguments that can be listed in its favor are the following:

- Quasi-projection filters can be designed to be much shorter than their exact counterparts; they usually result in faster algorithms. In this respect, the quasi-projection formalism can offer an elegant solution to the filter truncation problem.
- The exact inverse filter may have potential instabilities; these tend be substantially attenuated for their approximate counterparts, especially when the filters are short. The inverse filter solution is also more sensitive to an incorrect estimation of the point-spread function of the acquisition device.
- The exact inverse filter solution is rigorously valid only in an idealized noise-free situation. In practice, it has a tendency to overemphasize noise which may result in an unnecessary degradation of the signal-to-noise ratio. The quasi-projection formalism offers the prospect of an optimization that could take into account measurement noise. This is a possibility that remains to be investigated.

APPENDIX: QUASI-ORTHOGONAL ERROR ESTIMATE

The present derivation uses essentially the same steps as the proof for the orthogonal case, which is reported in [29]. The main difference is in the way the constants are estimated. We start by computing the Fourier transform of (40)

$$(QP_h s)\,\hat{}\,(\omega) = \hat{\varphi}(h\omega) \sum_{k\in Z} \overline{\hat{\tilde{\varphi}}(h\omega) + 2\pi k)} \cdot \hat{s}\left(\omega + \frac{2\pi}{h}k\right) \tag{A-1}$$

and choose to decompose the approximation error in the Fourier domain as

$$\hat{s}(\omega) - (QP_h s)\,\hat{}\,(\omega) = \hat{e}_1(\omega) + \hat{e}_2(\omega), \tag{A-2}$$

where

$$\hat{e}_1(\omega) = [1 - \overline{\hat{\tilde{\varphi}}(h\omega)}\hat{\varphi}(h\omega)]\hat{s}(\omega) \tag{A-3}$$

$$\hat{e}_2(\omega) = \sum_{\substack{k\in Z\\ k\neq 0}} \hat{\varphi}(h\omega) \cdot \overline{\hat{\tilde{\varphi}}(h\omega + 2\pi k)} \cdot \hat{s}\left(\omega + \frac{2\pi}{h}k\right). \tag{A-4}$$

Using Parseval's relation, we evaluate the first error term

$$\|e_1\| = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{s}(\omega)|^2[1 - \hat{\eta}_Q(h\omega)]^2 d\omega\right)^{1/2},$$

where $\hat{\eta}_Q(\omega) = \overline{\hat{\tilde{\varphi}}}(\omega) \cdot \hat{\varphi}(\omega)$. Recalling that η_Q is a quasi-interpolator of order $2L$, we use a standard Taylor series argument to show that

$$|1 - \hat{\eta}_Q(h\omega)| \leqslant C_1 \cdot (h\omega)^{2L},$$

where

$$C_1 = \frac{1}{(2L)!} \sup_{\xi} |\hat{\eta}_Q^{(2L)}(\xi)|. \tag{A-5}$$

This yields the first part of the error bound

$$\|e_1\| \leqslant C_1 \cdot h^{2L} \cdot \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\omega|^{4L} |\hat{s}(\omega)|^2 d\omega\right)^{1/2}. \tag{A-6}$$

To estimate the second term, we use a different technique:

$$\|e_2\| = \sup_{f\in L_2, \|f\|=1} \langle e_2, f\rangle = \sup_{f\in L_2, \|f\|=1} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\hat{f}(\omega)}$$
$$\times \sum_{k\neq 0} \hat{\varphi}(hw) \cdot \overline{\hat{\tilde{\varphi}}(h\omega + 2\pi k)}\hat{s}\left(\omega + \frac{2\pi}{h}k\right) d\omega.$$

We then apply the Cauchy–Schwarz inequality over the sum and the integral:

$$\|e_2\| \leqslant \sup_f \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{k\neq 0} |\hat{f}(\omega)|^2 |\hat{\tilde{\varphi}}(h\omega + 2\pi k)|^2 d\omega \right]^{1/2} \times \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{k\neq 0} |\hat{\varphi}(h\omega)|^2 |\hat{s}\left(\omega + \frac{2\pi}{h}k\right)|^2 d\omega \right]^{1/2}.$$

The first factor is bounded by

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{k\neq 0} |\hat{f}(\omega)|^2 |\hat{\tilde{\varphi}}(h\omega + 2\pi k)|^2 d\omega \leqslant \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \sum_{k\in Z} |\hat{\tilde{\varphi}}(h\omega + 2\pi k)|^2 d\omega \leqslant B_{\tilde{\varphi}} \cdot \|f\|^2,$$

where $B_{\tilde{\varphi}} \cong 1/A_{\varphi}$ is the upper frame bound of the quasi-dual function $\tilde{\varphi}$ (c.f. (7)). Using this inequality and making the change of variable $\xi = \omega + 2\pi k/h$, we get

$$\|e_2\| \leqslant \sqrt{B}_{\tilde{\varphi}} \cdot \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{s}(\xi)|^2 \sum_{k\neq 0} |\hat{\varphi}(h\xi - 2\pi k)|^2 d\xi \right]^{1/2}. \tag{A-7}$$

Since $|\varphi(\omega)|^2$ has zeros of order $2L$ at $\omega = 2\pi k, k \in Z/\{0\}$, we can use the estimate

$$\sum_{k\neq 0} |\hat{\varphi}(h\omega + 2\pi k)|^2 \leqslant \frac{(h\omega)^{2L}}{(2L)!} \sup_{\xi} \left| \sum_{k\neq 0} (|\hat{\varphi}|^2)^{(2L)}(\xi + 2\pi k) \right|,$$

which is derived in [29]. Combining this resut with (A-7), we find that

$$\|e_2\| \leqslant C_2 \cdot h^L \cdot \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\omega|^{2L} |\hat{s}(\omega)|^2 d\omega \right)^{1/2}, \tag{A-8}$$

where

$$C_2 = \sqrt{(B_{\tilde{\varphi}}/(2L)!) \sup_{\xi} \left| \sum_{k\neq 0} (|\hat{\varphi}^2|)^{(2L)}(\xi + 2\pi k) \right|}. \tag{A-9}$$

The decay conditions on φ and $\tilde{\varphi}$ guarantee that the constants C_1 and C_2 are finite. Finally, putting things together, we end up with

$$\|s - QP_h s\|_{L_2} \leqslant \|e_1\| + \|e_2\| \leqslant C_1 \cdot h^{2L} \cdot \|s^{(2L)}\|_{L_2} + C_2 \cdot h^L \cdot \|s^{(L)}\|_{L_2}.$$

ACKNOWLEDGMENTS

The author thanks Akram Aldroubi for helpful comments and suggestions, and Ingrid Daubechies for a very stimulating interaction and her crucial contribution to the proof of Theorem 1.

REFERENCES

1. A. Aldroubi, Oblique projections in atomic spaces, *Proc. Amer. Math. Soc.,* to appear.
2. A. Aldroubi and M. Unser, Families of multiresolution and wavelet spaces with optimal properties, *Numer. Func. Anal. Optim.* **14**, No. 5–6 (1993), 417–446.
3. A. Aldroubi and M. Unser, Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory, *Numer. Funct. Anal. Optim.* **15**, No. 1 & 2 (1994), 1–21.
4. A. Aldroubi, M. Unser, and M. Eden, Cardinal spline filters: Stability and convergence to the ideal sinc interpolator, *Signal Process.* **28**, No. 2 (1992), 127–138.
5. H. G. Burchard and J. Lei, Coordinate order of approximation by functional-based approximation operators, *J. Approx. Theory* **82**, No. 2 (1995), 240–256.
6. S. Cambanis and E. Masry, Wavelet approximation of deterministic and random signals: Convergence properties and rates, *IEEE Trans. Inform. Theory* **40**, No. 4 (1994), 1013–1029.
7. C. K. Chui and H. Diamond, A characterization of multivariate quasi-interpolation formulas and applications, *Numer. Math.* **57** (1990), 105–121.
8. I. Daubechies, Orthogonal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.
9. I. Daubechies, "Ten Lectures on Wavelets," SIAM, Philadelphia, 1992.
10. C. de Boor, The polynomials in the linear span of integer translates of a compactly supported function, *Const. Approx.* **3** (1987), 199–208.
11. C. de Boor, Quasi-interpolation and approximation power of multivariate splines, *in* "Computation of Curves and Surfaces" (W. Dahmen *et al.,* Eds.), pp. 313–345, Kluwer, Dordrecht, 1990.
12. C. de Boor and G. Fix, Spline approximation by quasi-interpolants, *J. Approx. Theory* **8** (1973), 19–45.
13. E. J. Halton and W. A. Light, On local and controlled approximation order, *J. Approx. Theory* **72** (1993), 268–277.
14. A. J. Jerri, The Shannon sampling theorem—its various extensions and applications: A tutorial review, *Proc. IEEE* **65,** No. 11 (1977), 1565–1596.
15. R.-Q. Jia and J. Lei, Approximation by multi-integer translates of functions having global support, *J. Approx. Theory* **72** (1993), 2–23.
16. J. Lei, L_p-approximation by certain projection operators, *J. Math. Anal. Appl.* **185** (1994), 1–14.
17. W. A. Light, Recent developments in the Strang–Fix theory for approximation orders, *in* "Curves and Surfaces" (P. J. Laurent, A. L. Méhauté, and L. L. Schumaker, Eds.), pp. 285–292, Academic Press, Boston, 1991.
18. S. G. Mallat, A theory of multiresolution signal decomposition: the wavelet representation, *IEEE Trans. Pattern Anal. Machine Intell.* **11,** No. 7 (1989), 674–693.
19. I. J. Schoenberg, Contribution to the problem of approximation of equidistant data by analytic functions, *Quart. Appl. Math.* **4** (1946), 45–99, 112–141.
20. I. J. Schoenberg, Cardinal interpolation and spline functions, *J. Approx. Theory* **2** (1969), 167–206.
21. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.
22. C. E. Shannon, Communication in the presence of noise, *Proc. IRE* **37** (1949), 10–21.

23. G. Strang, Wavelets and dilation equations: a brief introduction, *SIAM Rev.* **31** (1989), 614–627.

24. G. Strang and G. Fix, A Fourier analysis of the finite element variational method, *in* "Constructive Aspect of Functional Analysis," pp. 796–830, Edizioni Cremonese, Rome, 1971.

25. W. Sweldens and R. Piessens, Asymptotic error expansions for wavelet approximations of smooth functions, II, *Numer. Math.* **68,** No. 3 (1994), 377–401.

26. W. Sweldens and R. Piessens, Quadrature formulae and asymptotic error expansions for wavelet approximations of smooth functions, *SIAM J. Numer. Anal.* **31,** No. 4 (1994), 1240–1264.

27. M. Unser and A. Aldroubi, A general sampling theory for non-ideal acquisition devices, *IEEE Trans. Signal Process.* **42,** No. 11 (1994), 2915–2925.

28. M. Unser, A. Aldroubi, and M. Eden, Polynomial spline signal approximations: Filter design and asymptotic equivalence with Shannon's sampling theorem, *IEEE Trans. Inform. Theory* **38,** No. 1 (1992), 95–103.

29. M. Unser and I. Daubechies, "On the Approximation Power of Convolution-Based Least Squares Versus Interpolation," NCRR Report 53/95, National Institutes of Health, 1995.

30. M. Unser and M. Eden, FIR approximations of inverse filters and perfect reconstruction filter banks, *Signal Process.* **36,** No. 2 (1994), 163–174.

31. M. Vetterli and J. Kovacevic, "Wavelets and Subband Coding," Prentice–Hall, Englewood Cliffs, NJ, 1995.

32. E. T. Whittaker, The Fourier theory of the cardinal functions, *Proc. Math Soc. Edinburgh* **1** (1929), 169–176.