GENERALIZED SAMPLING WITHOUT BANDLIMITING CONSTRAINTS

Michael Unser[1,2] *and* *Josiane Zerubia*[2]

[1] NIH, Bethesda, MD USA [2] INRIA, Sophia Antipolis, France

ABSTRACT

We investigate the problem of the reconstruction of a continuous-time function $f(x) \in \mathcal{H}$ from the responses of m linear shift-invariant systems sampled at $1/m$ the reconstruction rate, extending Papoulis' generalized sampling theory in two important respects. First, we allow for arbitrary (non-bandlimited) input signals (typ. $\mathcal{H} = L_2$). Second, we use a more general specification of the reconstruction subspace $V(\varphi)$, so that the output of the system can take the form of a bandlimited function, a spline, or a wavelet expansion. The system that we describe yields an approximation $\tilde{f} \in V(\varphi)$ that is consistent with the input $f(x)$ in the sense that it produces exactly the same measurements. We show that this solution can be computed by multivariate filtering. We also characterize the stability of the system (condition number). Finally, we illustrate the theory by presenting a new example of interlaced sampling using splines.

1. INTRODUCTION

In 1977, Papoulis introduced a powerful extension of Shannon's sampling theory, showing that a bandlimited signal could be reconstructed exactly from the samples of the responses of m linear shift-invariant systems, sampled at $1/m$th the Nyquist rate [4]. While the theory is elegant and covers many special cases described in the literature, we feel that the assumption of a bandlimited input function $f(x)$ is overly restrictive. Indeed, most real world signals are time or space limited which is in contradiction with the bandlimited hypothesis. Here, we propose a much less constrained formulation where the analog input signal can be almost arbitrary, typically $f(x) \in L_2$ where L_2 is the space of finite energy functions. This is possible only because we replace Papoulis and Shannon's principle of a perfect reconstruction by the weaker requirement of a *consistent* approximation. In other words, we want our reconstructed signal $\tilde{f}(x)$ to provide exactly the same measurements as $f(x)$ if it is re-injected into the system. To make the link with more recent signal representation theories such as splines and wavelets [3, 8], we also consider the more general reconstruction subspace $V(\varphi) = \text{span}\{\varphi(x-k)\}_{k\in Z}$ where the generating function $\varphi(x)$ is not necessarily $\text{sinc}(x)$.

In addition to the generalization of Papoulis's sampling theorem, we introduce a simple and practical reconstruction algorithm which takes the form of a multivariate matrix filter. We also derive a new stability coefficient that is useful for assessing the robustness of the proposed reconstruction method. Finally, we illustrate the theory with some new examples of interlaced sampling using splines.

2. FORMULATION AND ASSUMPTIONS

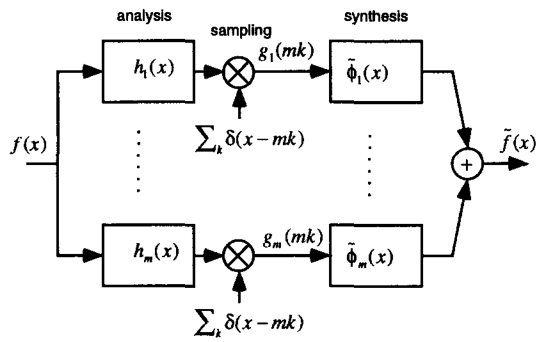

Figure 1: Generalized sampling procedure. The left part of the block diagram represents the measurement process which is performed by sampling the output of an m channel analysis filterbank. The right part describes the reconstruction process which involves the synthesis functions $\tilde{\phi}_i(x)$ in Theorem 1.

Our system is schematically represented in Fig. 1. We use a normalized sampling rate without any loss of generality. The continuous-time signal $f(x)$ is convolved with a bank of analysis filters $h_i(x), i = 1, \cdots m$, the responses of which are then sampled at $1/m$th the reconstruction rate to yield the measurement vector $\mathbf{g}_m(k) = (g_1(mk), g_2(mk), \cdots, g_m(mk))$. These discrete measurements are then combined to produce the continuous-time output $\tilde{f}(x)$. The system is essentially the same as the one considered by Papoulis except that the output $\tilde{f} \in V(\varphi)$ is only an approximation of the input $f \in \mathcal{H}$ where $\mathcal{H}$ is a class of functions considerably larger than $V(\varphi)$. To use an analogy, $\mathcal{H}$ is to $V(\varphi)$ what R is to Z.

For mathematical convenience, we describe the measurement process in terms of inner products

$$g_i(mk) = (h_i * f)(mk) = \langle f(x), \phi_i(x - mk)\rangle \quad (1)$$

where the equivalent analysis functions are $\phi_i(x) = h_i(-x)$. The reconstruction system works for almost any set of analysis filters $h_i(x)$, provided that some invertibility condition is met. We will now state our mathematical assumptions, emphasizing the main differences with Papoulis' initial formulation.

2.1. Extended class of input functions

In principle, we can consider almost any input function $f(x)$, except that we want to make sure that all measurement sequences are well-defined in the l_2-sense. Specifically, the class of admissible input functions $\mathcal{H}$ must be such is that there exists a finite constant C_ϕ such that

$$(a1) \quad \forall f \in \mathcal{H}, \ \ \|\mathbf{g}_m\|_{l_2} \le C_\phi \cdot \|f\|_{L_2}.$$

If the ϕ_i's are in L_2, then we can show that it is usually possible to consider any possible finite energy input function; i.e., $\mathcal{H} = L_2$. In the case where the ϕ_i's are Dirac delta functions (interlaced sampling), we must be slightly more conservative; for example, we can consider the class of C^1-continuous functions that decay like $O(|x|^{-r}), r = \frac{1}{2} + \epsilon, \epsilon > 0$.

2.2. Reconstruction subspaces

Our reconstructed signal $\tilde{f}$ belongs to the subspace of $\mathcal{H}$

$$V(\varphi) = \{\tilde{f}(x) = \sum_{k\in Z} c(k)\varphi(x-k) | c(k) \in l_2\} \quad (2)$$

where $\varphi(x)$ is a given *generating* function. This covers the bandlimited case with $\varphi(x) = \mathrm{sinc}(x)$, but also other more recent signal representation models such as splines and wavelets [3, 8]. We require that $V(\varphi)$ is a well-defined (closed) subspace of L_2. This is equivalent to the condition (cf. [1])

$$(a2) \quad 0 < A_\varphi \le \widehat{a}_\varphi(e^{j\omega}) \le B_\varphi < +\infty, \quad a.e.$$

where A_φ and B_φ are two positive constants (Riesz bounds); $\widehat{a}_\varphi(e^{j\omega})$ is the Fourier transform of the auto-correlation sequence $a_\varphi(k) = \langle \varphi(x-k), \varphi(x)\rangle$.

2.3. Consistent measurements

Because we have enlarged the class of admissible input functions to $\mathcal{H}$, we must give up Papoulis or Shannon's idea of an exact reconstruction. We will replace it with the notion of a consistent approximation of $f(x)$ in $V(\varphi)$, that is, a reconstruction $\tilde{f}(x) \in V(\varphi)$ that would produce the same set of measurements $\{g_i(mk), k \in Z\}_{i=1,\dots,m}$ if it was re-injected into the system. Specifically, we want to impose the consistency requirement for $k \in Z$ and $i = 1, \cdots, m$

$$\forall f \in \mathcal{H}, \langle \tilde{f}(x), \phi_i(x-mk)\rangle = \langle f(x), \phi_i(x - mk)\rangle. \quad (3)$$

This means that $f(x)$ and $\tilde{f}(x)$ are essentially equivalent to the end-user because they both look exactly the same through the measurement system which typically constitutes the only observation method available.

2.4. Invertibilty condition

Our solution involves a multivariate reconstruction filter Q, which is specified via a matrix inversion in the z-transform domain:

$$\widehat{\mathbf{Q}}(z) = \widehat{\mathbf{A}}_{\phi\varphi}^{-1}(z), \quad (4)$$

where $\widehat{\mathbf{A}}_{\phi\varphi}(z)$ is the z-transform of the input-output cross-correlation matrix sequence $\mathbf{A}_{\phi\varphi}(k)$ whose scalar entries are given by

$$[\mathbf{A}_{\phi\varphi}]_{i,j}(k) = (h_i * \varphi)(mk - j + 1). \quad (5)$$

We require this filter to be invertible in the sense that that there exists a constant M_Q such that

$$(a3) \quad M_Q^2 = \sup_{\omega\in[0,2\pi)} \lambda_{max}\left[\widehat{\mathbf{Q}}^T(e^{-j\omega}) \cdot \widehat{\mathbf{Q}}(e^{j\omega})\right] < +\infty,$$

where $\lambda_{max}[\cdot]$ denotes the largest eigenvalue. Note that this constant is also related to the minimum eigenvalue of $\widehat{\mathbf{A}}_{\phi\varphi}(e^{j\omega})$.

3. MAIN RESULTS

3.1. Generalized sampling theorem

Theorem 1 *Under assumptions* (a1), (a2), *and* (a3), *it is always possible to design a system that provides a consistent signal approximation in the sense of* (3) *for any input function* $f \in \mathcal{H}$. *The corresponding signal approximation admits the expansion*

$$\tilde{f}(x) = \sum_{i=1}^{m}\sum_{k\in Z} g_i(mk)\tilde{\phi}_i(x - mk) = \tilde{P}f(x), \quad (6)$$

and the underlying operator $\tilde{P}$ is a projector from $\mathcal{H}$ into $V(\varphi)$. The synthesis functions $\tilde{\phi}_i$ are given by

$$\tilde{\phi}_i(x) = \sum_{k\in Z} q_i(k)\varphi(x-k), \quad (i=1,\ldots,m) \qquad (7)$$

where the sequences $q_i(k)$ are determined as follows

$$\left[\begin{array}{ccc} \hat{q}_1(z) & \cdots & \hat{q}_m(z) \end{array}\right] = \left[\begin{array}{ccc} 1 & \cdots & z^{-m+1} \end{array}\right] \cdot \widehat{\mathbf{Q}}(z^m); \quad (8)$$

the filter matrix $\widehat{\mathbf{Q}}(z)$ is specified by (4).

For a complete proof and discussion, we refer to [6]. Here, we will examine some of the consequences of this result. First, because the operator $\tilde{P}$ is a projector, our result ensures a perfect reconstruction whenever the input signal is already included in the output space: $\forall f \in V(\varphi)$, $\tilde{P}f = f$. This corresponds to the more restrictive framework of conventional sampling theories. Second, this theorem extends Papoulis result in [4] which corresponds to the particular case $\mathcal{H} = V(\text{sinc}) = \mathrm{B}_\pi$ where B_π denotes the subspace of finite energy functions that are bandlimited to the frequency interval $\omega \in [-\pi,\pi]$. We also note that our theorem provides an explicit formula for the dual basis functions which is only implicit in Papoulis' paper. Finally, for $m=1$, we get back the generalized sampling theory for non-ideal acquisition devices proposed in [5].

3.2. Reconstruction algorithm

To derive the reconstruction algorithm, we use a vector representation of the reconstructed function

$$\tilde{f}(x) = \sum_{k\in Z} \mathbf{\Psi}^T(x-mk)\mathbf{c}_m(k) \qquad (9)$$

where the vector-sequence $\mathbf{c}_m(k) = (c(mk), c(mk+1), \cdots, c(mk+m-1))$ is the block (or polyphase) representation of the sequence of signal coefficients in (2), and where $\mathbf{\Psi}(x) = (\varphi(x), \varphi(x-1), \cdots, \varphi(x-m+1))$ is the corresponding vector generating function. Let us now re-inject $\tilde{f}$ into the system. By linearity, the consistency requirement (3) implies that

$$\mathbf{g}_m(k) \quad = \quad \sum_{k'\in Z} \langle \mathbf{\Phi}(x-mk), \mathbf{\Psi}^T(x-mk')\rangle \cdot \mathbf{c}_m(k'),$$

where we also use the vector representation $\mathbf{\Phi}(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_m(x))$ of the analysis functions in (3). Making the change of variable $l = k-k'$, we get

$$\mathbf{g}_m(k) = \sum_{l\in Z} \langle \mathbf{\Phi}(x-ml), \mathbf{\Psi}^T(x)\rangle \cdot \mathbf{c}_m(k-l),$$

a relation that can also be written in the form of a multivariate convolution

$$\mathbf{g}_m(k) = \sum_{l\in Z} \mathbf{A}_{\phi\varphi}(l)\mathbf{c}_m(k-l) = (\mathbf{A}_{\phi\varphi} * \mathbf{c}_m)(k), \quad (10)$$

where $\mathbf{A}_{\phi\varphi}(k) = \langle \mathbf{\Phi}(x-mk), \mathbf{\Psi}^T(x)\rangle$ is precisely the $m\times m$ matrix sequence defined by (5). Therefore, we can solve the system by inverse filtering

$$\mathbf{c}_m(k) = \sum_{l\in Z} \mathbf{Q}(l)\mathbf{g}_m(k-l) = (\mathbf{Q} * \mathbf{g}_m)(k), \qquad (11)$$

where $\widehat{\mathbf{Q}}(z)$ is defined by (4). Eq. (11) described a practical reconstruction algorithm that takes the form of a multivariate filter. This inverse filter is well-defined because of the invertibility condition (a3). Note that the above argument also proves the first part of Theorem 1, namely, the existence of a consistent signal approximation $\tilde{f}$ in $V(\varphi)$.

3.3. Stability analysis

Let $\widetilde{\mathbf{\Phi}}(x) = (\tilde{\phi}_1(x), \cdots, \tilde{\phi}_m(x))$ be the vector representation of the analysis functions in Theorem 1. In [7], we have established the following result:

Theorem 2 *The set $\{\widetilde{\mathbf{\Phi}}(x-mk)\}_{k\in Z}$ constitutes a Riesz basis of $V(\varphi)$. In other words, $\forall \tilde{f}(x) \in V(\varphi)$, there exists a sequence $\mathbf{g}(k) \in l_2^m$ and two strictly positive constants $A_{\tilde{\phi}}$ and $B_{\tilde{\phi}}$ such that*

$$(i) \qquad \tilde{f}(x) = \textstyle\sum_{k\in Z} \mathbf{g}^T(k)\widetilde{\mathbf{\Phi}}(x-mk)$$

$$(ii) \quad A_{\tilde{\phi}} \cdot \|\mathbf{g}\|_{l_2}^2 \le \|\tilde{f}(x)\|_{L_2}^2 \le B_{\tilde{\phi}} \cdot \|\mathbf{g}\|_{l_2}^2.$$

To show the relevance of this theorem to the stability issue, we consider a perturbation $\Delta\mathbf{g}$ on the measurements which results in a variation Δf on the output. By linearity, these perturbations also satisfy the norm inequality (*ii*). Combining those relations, we get

$$\frac{1}{\alpha_{\tilde{\phi}}}\left(\frac{\|\Delta\mathbf{g}\|_{l_2}}{\|\mathbf{g}\|_{l_2}}\right) \le \frac{\|\Delta\tilde{f}(x)\|_{L_2}}{\|\tilde{f}(x)\|_{L_2}} \le \alpha_{\tilde{\phi}}\left(\frac{\|\Delta\mathbf{g}\|_{l_2}}{\|\mathbf{g}\|_{l_2}}\right), \quad (12)$$

where

$$\alpha_{\tilde{\phi}} = \sqrt{\frac{B_{\tilde{\phi}}}{A_{\tilde{\phi}}}}. \qquad (13)$$

Thus, we may interpret $\alpha_{\tilde{\phi}}$ as the condition number of the system. To compute the frame bounds explicitly, we use Theorem 2.1 in [2], which yields

$$\left\{\begin{array}{l} A_{\tilde{\phi}} = \operatorname*{ess\,inf}_{\omega\in[0,2\pi)} \lambda_{min}\left[\widehat{\mathbf{A}}_{\tilde{\phi}}(e^{j\omega})\right] \\ B_{\tilde{\phi}} = \operatorname*{ess\,sup}_{\omega\in[0,2\pi)} \lambda_{max}\left[\widehat{\mathbf{A}}_{\tilde{\phi}}(e^{j\omega})\right], \end{array}\right. \qquad (14)$$

where the $m \times m$ matrix $\widehat{\mathbf{A}}_{\widetilde{\phi}}(z)$ is the z-transform of the autocorrelation matrix sequence $\mathbf{A}_{\widetilde{\phi}}(k) = \langle \widetilde{\mathbf{\Phi}}(x - mk), \widetilde{\mathbf{\Phi}}^T(x) \rangle$.

3.4. Examples

To illustrate the theory, we consider the case of interlaced sampling ($m = 2$) with a reconstruction in the space of cubic splines (i.e., φ is the cubic B-spline). The analysis functions are $\phi_1(x) = \delta(x)$, and $\phi_2(x) = \delta(x - \Delta t)$, where Δt is the sampling offset. The corresponding cubic spline reconstruction functions in Theorem 1 for $\Delta t = 1/2$ are shown in Fig. 2a. Note how they take the value one at the position of their respective sample, and how they vanish at all other sampling locations (circles). Fig. 2b displays the condition number of the system as a function of the offset parameter Δt. The algorithm has the most favorable behavior around $\Delta t = 1$ (uniform sampling) with a region of relative stability for $0.3 \leq \Delta t \leq 1.7$. Beyond that point, the conditioning of the system deteriorates rapidly. This is not surprising since the system is obviously underdetermined for the limiting cases $\Delta t = 0$ or $\Delta t = 2$ where the same samples are collected twice.

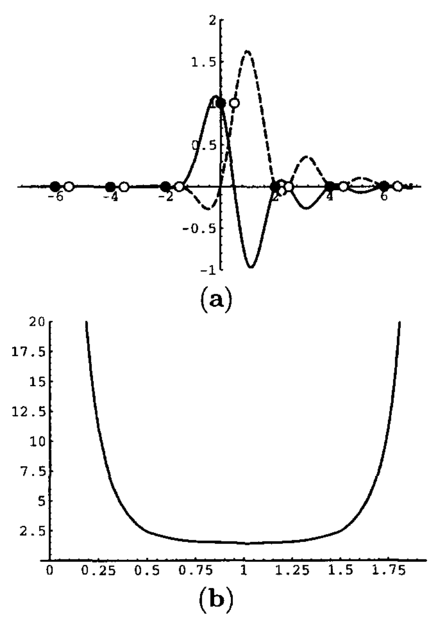

Figure 2: Interlaced sampling with cubic spline reconstruction. (a) Reconstruction functions $\widetilde{\phi}_1(x)$ and $\widetilde{\phi}_2(x)$ for $\Delta t = 1/2$; (b) Stability curve as a function of Δt.

4. REFERENCES

[1] A. Aldroubi and M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numerical Functional Analysis and Optimization*, vol. 15, no. 1-2, pp. 1-21, 1994.

[2] A. Aldroubi, "Oblique projections in atomic spaces," *Proc. Amer. Math. Soc.*, vol. 124, no. 7, pp. 2051-2060, 1996.

[3] S.G. Mallat, "A theory of multiresolution signal decomposition: the wavelet representation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. PAMI-11, no. 7, pp. 674-693, 1989.

[4] A. Papoulis, "Generalized sampling expansion," *IEEE Trans. Circuits and Systems*, vol. 24, no. 11, pp. 652-654, 1977.

[5] M. Unser and A. Aldroubi, "A general sampling theory for non-ideal acquisition devices," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 2915-2925, 1994.

[6] M. Unser and J. Zerubia, "A generalized sampling theory without bandlimiting constraints," INRIA Report, 1996.

[7] M. Unser, J. Zerubia, "Generalized sampling: stability and performance analysis," INRIA Report, 1996.

[8] M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding.* Englewood Cliffs, NJ: Prentice Hall, 1995.