# On the Approximation Power of Convolution-Based Least Squares Versus Interpolation

Michael Unser, *Senior Member, IEEE*, and Ingrid Daubechies, *Member, IEEE*

***Abstract*— There are many signal processing tasks for which convolution-based continuous signal representations such as splines and wavelets provide an interesting and practical alternative to the more traditional sinc-based methods. The coefficients of the corresponding signal approximations are typically obtained by direct sampling (interpolation or quasi-interpolation) or by using least squares techniques that apply a prefilter prior to sampling. Here, we compare the performance of these approaches and provide quantitative error estimates that can be used for the appropriate selection of the sampling step $h$. Specifically, we review several results in approximation theory with a special emphasis on the Strang–Fix conditions, which relate the general $O(h^L)$ behavior of the error to the ability of the representation to reproduce polynomials of degree $n = L - 1$. We use this theory to derive pointwise error estimates for the various algorithms and to obtain the asymptotic limit of the $L_2$-error as $h$ tends to zero. We also propose a new improved $L_2$-error bound for the least squares case. In the process, we provide all the relevant bound constants for polynomial splines. Some of our results suggest the existence of an intermediate range of sampling steps where the least squares method is roughly equivalent to an interpolator with twice the order. We present experimental examples that illustrate the theory and confirm the adequacy of our various bound and limit determinations.**

## I. Introduction

INTERPOLATION is one of the basic operations in signal processing. It is used extensively in picture processing to rotate and rescale images or to correct for spatial distortions. While the signal processing theory traditionally emphasizes the sinc-interpolation for bandlimited functions [1], this method is rarely used in practice because of the slow decay of sinc$(x)$. Instead, practitioners usually rely on short kernel methods such as bilinear interpolation [2], cubic convolution [3]–[5], or polynomial spline interpolation [6], [7], which are much more efficient to implement, especially in higher dimensions. These methods are all convolution-based in the sense that they use an interpolation model of the form

$$s_h(x) = \sum_{k \in Z} c_h(k) \varphi\left(\frac{x}{h} - k\right) \tag{1}$$

where $h$ is the sampling step and $\varphi(x)$ the basic interpolation kernel. The expansion coefficients in (1) typically correspond to the samples of the input function $s(x)$ taken on a uniform grid: $c_h(k) = s(hk)$ (cf. Fig. 1). More recently, researchers have proposed a systematic formulation of this class of representations using essentially the same Hilbert space framework as developed in the context of the wavelet transform [8]–[10]. This led to the design of "second generation" methods for the continuous representation of signals based on the principle of a minimum error approximation [11]–[13]. The corresponding least squares solution can be obtained through a simple modification of the basic interpolation procedure, which consists of applying an appropriate prefilter to $s(x)$ prior to sampling (cf. Fig. 2). This form of preprocessing is akin to the use of an anti-aliasing lowpass filter in conventional sampling theory, except that the optimal prefilter is not necessarily ideal. In fact, one may even take a reverse perspective and choose the representation such that the prefilter is particularly well behaved. A good example of this is the use of B-spline filters for computing polynomial spline approximations [12]. This approach has been used advantageously to design practical least squares methods for image resizing [14], and, more recently, affine transformations [15]. For a given signal model, these methods generally outperform the standard interpolation procedures.

From all the examples above, it appears that such nonbandlimited signal representations have a lot to offer for signal processing—both in terms of computational efficiency and simplicity of implementation. However, two basic questions remain. First, how should one select the sampling step $h$? Is there any analog of Shannon's sampling theorem that tells us that for bandlimited signals, we must choose a sampling frequency that is above the Nyquist rate? Second, what is the quality of the approximation? Are least squares approaches really superior to the simpler interpolation schemes that are commonly used in practice?

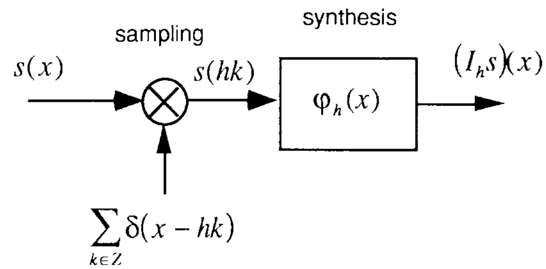


Fig. 1. Block diagram representation of a convolution-based interpolator. Sampling is achieved by multiplication with the sequence of Dirac impulses $\Sigma_{k \in Z} \delta(x - hk)$. The impulse response of the reconstruction filter is $\varphi_h(x) = \varphi(x/h)$.

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M. Unser is with the Biomedical Engineering and Instrumentation Program, National Center for Research Resources, National Institutes of Health, Bethesda, MD 20892-5766 USA.

I. Daubechies is with the Program in Applied and Computational Mathematics and Department of Mathematics, Princeton University, Princeton, NJ 08540 USA.

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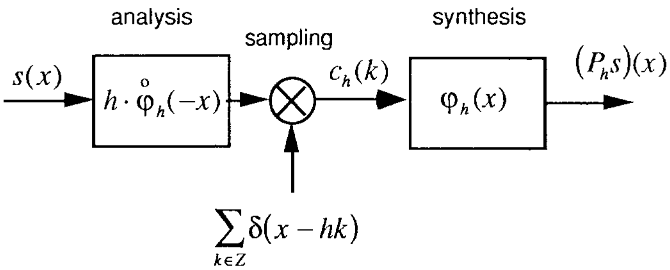


Fig. 2. Convolution-based least squares signal approximation. The impulse response of the optimal prefilter is $h \cdot \mathring{\varphi}(-x/h)$, where $\mathring{\varphi}$ is the dual of $\varphi$. The signal approximation $P_h s$ corresponds to the orthogonal projection of $s$ onto the signal subspace $V_h =$ span $\{\varphi(x/h - k)\}_{k \in Z}$.

What is effectively required to answer these questions is a detailed characterization of the error as a function of $h$. This turns out to be a key issue in approximation theory, and there are many fundamental results available in this area of mathematical research. However, most of this theory has not yet been brought to the attention of the signal processing community. Of special relevance is the general error bound provided by Strang and Fix in the early 1970's [16]. Specifically, let the function $s_h$ in (1) represent an $L$th-order[1] approximation of the (finite energy) function $s(x) \in L_2$ at the sampling step $h$. Then, we have the error bound

$$\min_{c_h} \|s - s_h\| \leq C \cdot h^L \cdot \left[\frac{1}{\pi} \int_0^{+\infty} |\omega^L \hat{s}(\omega)|^2 \, d\omega\right]^{1/2} \tag{2}$$

where $C$ is a constant that does not depend on $s$, and where $\hat{s}(\omega)$ denotes the Fourier transform of $s(x)$. The right-most term in (2) represents the energy of the $L$th derivative of $s$. Since we are interested in making the connection with Shannon's sampling theory, we can also interpret it as a measure of the bandwidth of the input signal $s$. However, there are two fundamental differences with the classical result for bandlimited signals. First, there is no special assumption (such as bandlimitedness) on the class of admissible input signals. The only requirement is that the function $s$ and its $L$th derivative are square integrable, which is a very weak smoothness constraint, at least when compared with bandlimitedness. Second, the $O(h^L)$ form of the bound suggests that the representation will never be exact. However, the error can be made arbitrarily small if the signal is sampled at a sufficient rate.

Although the bound (2) is of considerable theoretical interest, it needs to be made more specific and quantitative to be of direct use for signal processing. In particular, we need a better handle on the value of the constant $C$, which depends on the choice of the representation model (e.g., splines or wavelets) and on the algorithm used. In addition, the error bound (2) does not really distinguish between the traditional interpolation or quasiinterpolation schemes and the least squares approximation methods (orthogonal projection), which have been emphasized more recently. In fact, it is well known to approximation theorists that these methods are qualitatively equivalent, i.e., they all achieve the optimal $O(h^L)$ rate predicted by the theory [17], [18]. From a practical point of view, however, there are many reasons to expect that the least squares approaches should be superior—the question is: by how much?

Our aim with this paper is to address these important points and present the theoretical results that are relevant to the issue. The presentation will be partly tutorial with pointers to the relevant literature but will also include some new results that are specific to least squares approximation. In order to make it as self-contained and understandable as possible for a signal processing audience, we will present the derivation of all key results. As a byproduct, we will also characterize all the relevant constants explicitly. This will allow us to provide general guidelines and formulas that can be of direct use to the practitioner.

The paper is organized as follows. In Section II, we start with a characterization of the relevant function spaces and briefly review the three main methods for obtaining signal approximations:

1) interpolation;
2) quasi-interpolation;
3) least squares approximation.

We then discuss the Strang–Fix conditions, which provide a remarkable connection between the ability of the representation to reproduce polynomials of degree $n = L - 1$ and its approximation power as expressed by (2). In Section III, we investigate the pointwise behavior of the error for the various approximation methods using the Taylor series as our main tool. In Section IV, we essentially rederive the basic $L_2$-bound (2) for (quasi-)interpolators and provide a refined estimate for the least squares case, which strongly suggests that this latter method is indeed superior. We also present experimental error curves that support our speculation that there is an intermediate region where a least squares approximation of order $L$ is roughly equivalent to a (quasi-)interpolation of order $2L$. Finally, in Section V, we consider the asymptotic case and compute the limiting form of the various errors for $h$ sufficiently small.

### A. Notations

$L_2$ is the space of measurable, square-integrable, real-valued functions or signals $s(x), x \in R$. It is a Hilbert space whose metric $\|\cdot\|_2$ (the $L_2$-norm) is derived from the inner product

$$\langle s(x), r(x)\rangle := \int_{-\infty}^{+\infty} s(x) r(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{\hat{s}(\omega)} \hat{r}(\omega) \, d\omega. \tag{3}$$

where the right-hand side equality is Parseval's relation, and where $\hat{s}(\omega)$ and $\hat{r}(\omega)$ denote the Fourier transforms of $s$ and $r$, respectively. The $L_\infty$ or sup-norm is defined as

$$\|s\|_\infty = \lim_{p \to +\infty} \left[\int_{-\infty}^{+\infty} |s(x)|^p \, dx\right]^{1/p} = \sup_{x \in R} |s(x)|. \tag{4}$$

The class of smoothness of a signal will be specified by its appartenance to the Sobolev space $W_2^L$ (resp., $W_\infty^L$), which is the space of functions whose $L$ first derivatives are defined in the $L_2$ (resp. $L_\infty$) sense.

[1] An approximation procedure has an $L$th order of approximation if it can reproduce all polynomials of degree $n = l - 1$; a precise definition will be given in Section II-D.

## II. Preliminary Notions

### A. Convolution-Based Signal Representations

A general approach to specify continuous signal representations is to consider the class of functions generated from the integer translates of a single function $\varphi(x) \in L_2$ [13], [16]. We can adjust the resolution by varying the sampling step (or step size) $h$ and rescaling $\varphi$ accordingly. The corresponding function space $V_h(\varphi) \subset L_2$ is defined as

$$V_h(\varphi) = \left\{ s_h(x) = \sum_{k\in Z} c_h(k)\varphi\left(\frac{x}{h} - k\right) | c_h \in l_2 \right\} \quad (5)$$

where $l_2$ is the vector space of square-summable sequences. The only restriction on the choice of the *generating* function $\varphi$ is that the set $\{\varphi(x/h - k\}_{k\in Z}$ is a Riesz basis of $V_h(\varphi)$; this is equivalent to the condition

$$0 < A \le \hat{a}_\varphi(\omega) = \sum_{k\in Z} |\hat{\varphi}(\omega + 2\pi k)|^2 \le B < +\infty \quad \text{a.e.} \quad (6)$$

where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(x)$, and where the constants $A$ and $B$ are the so-called Riesz bounds [13]. This constraint ensures that each function $s_h(x)$ in $V_h(\varphi)$ is uniquely characterized by the sequence of its coefficients $c_h(k)$.

This formulation is quite general and includes all the signal interpolation models that were mentioned in the introduction, as well as many others. A special case that is also covered is Deslauriers and Dubuc's dyadic interpolation scheme in which the generating function is defined indirectly through a refinement equation [19]. Other examples of this nature are the various subspaces associated with the wavelet transform and multiresolution analysis; this connection is further discussed in [20].

### B. Interpolation and Quasi-Interpolation

The simplest way to represent a continuous signal $s(x) \in L_2$ in $V(\varphi)$ is to use its samples as the coefficients of the representation in (5). The corresponding "interpolation" operator, which is schematically represented by the block diagram in Fig. 1, is defined as

$$(I_h s)(x) = \sum_{k\in Z} s(hk)\varphi\left(\frac{x}{h} - k\right). \quad (7)$$

The operator $I_h$ is bounded, provided that the input signal is sufficiently smooth; for example, $s \in W_2^1$. Note that with this definition, the samples of the signal $x(x)$ and of its "interpolation" $(I_h s)(x)$ are not necessarily identical. To get a true interpolation (i.e., $\forall k \in Z, s(x)|_{x=hk} = (I_h s)(x)|_{x=hk}$), we need to select a generating function $\varphi_{\text{int}} \in V_1(\varphi)$ that satisfies the interpolation property

$$\varphi_{\text{int}}(x)|_{x=k} = \delta[k] \quad (8)$$

where $\delta[k]$ denotes the discrete unit impulse at the origin. For a given subspace $V_h(\varphi) \subset L_2$, the interpolation function $\varphi_{\text{int}}(x/h) \in V_h(\varphi)$ is generally unique [13]. A typical example is the sinc function, which is the interpolation kernel for bandlimited functions. An interpolator that is commonly used in image processing is Keys's short cubic convolution kernel (cf. [5])

$$\varphi_a^3(x) = \begin{cases} (a+2)|x|^3 - (a+3)|x|^2 + 1, & 0 \le |x| < 1 \\ a(|x|^3 - 5|x|^2 + 8|x| - 4), & 1 \le |x| < 2 \\ 0, & 2 \le |x|. \end{cases} \quad (9)$$

which is parameterized by $a$. Interpolation kernels may also be constructed by taking the autocorrelation of an orthogonal scaling function [21], [22].

Interestingly, it is possible to relax the interpolation condition without any noticeable loss in performance. This leads to the concept of a quasi-interpolation, which is a standard notion in approximation theory [23]–[26] but has not yet been exploited in signal processing. By definition, a quasi-interpolant of order $L = n + 1$ is a function $\varphi_{\text{QI}}$ that interpolates all polynomial $p_n(x)$ of degree $n$

$$\forall p_n(x) \in \pi^n, \quad \sum_{k\in Z} p_n(k)\varphi_{\text{QI}}(x-k) = p_n(x) \quad (10)$$

where $\pi^n$ denotes the space of polynomials of degree $n$. By rewriting this condition for the monomials $x^m, m = 0, \cdots, L-1$, it is not difficult to show that an equivalent formulation of this condition in the frequency domain is

$$\hat{\varphi}_{\text{QI}}(\omega)|_{\omega=2\pi k} = \delta[k] \quad (11)$$

$$\hat{\varphi}_{\text{QI}}^{(m)}(\omega)|_{\omega=2\pi k} = 0, \quad (m = 1, \cdots, L-1) \quad (12)$$

where $\hat{\varphi}_{\text{QI}}^{(m)}$ denotes the $m$th derivative of the Fourier transform of $\varphi_{\text{QI}}$. In other words, the transfer function of a quasi-interpolant of order $L$ has zeros of multiplicity $L$ at all nonzero frequencies that are integer multiples of $2\pi$ and is flat at the origin, i.e., $\hat{\varphi}_{\text{QI}}(\xi) = 1 + O(\xi^L)$ as $\xi \to 0$. Whether or not it is possible to construct quasi-interpolants within a certain subspace $V_h(\varphi)$ depends on its order of approximation; this is a notion that will be clarified in Section II-D (Strang–Fix conditions). Note that there are many possible quasi-interpolants within a particular subspace, a special case being the interpolator in (8). It is often desirable to select the one with the shortest possible support.

*Example:* Take $\varphi$ to be the centered B-spline of degree $n$: $\varphi_n = \varphi_0 * \varphi_{n-1}$ with $\varphi_0(x) = 1, -\frac{1}{2} \le x < \frac{1}{2}$ and 0 otherwise. Then, $\varphi_n$ is piecewise polynomial of degree $n$ and is $(n-1)$ times continuously differentiable. This function generates the standard space of polynomial splines of degree $n$ [27], [28]. The Fourier transform of $\varphi_n$ is

$$\hat{\varphi}_n(\omega) = \left(\frac{\sin\omega/2}{\omega/2}\right)^{n+1} = \text{sinc}^{n+1}(\omega/(2\pi)) \quad (13)$$

which is a function that has zeros of multiplicity $L = n + 1$ for $\omega = 2\pi k, k \ne 0$. If $n > 1$, then $\varphi_n$ is not interpolating and is quasi-interpolating only up to order 2 (because $\hat{\varphi}_n(\xi) = 1 + (n+1)\xi^2/32 + O(\xi^4)$ as $\xi \to 0$). Higher order quasi-interpolants can be constructed by suitable linear combination of B-splines. For instance, one can easily check that the cubic spline kernel

$$\varphi_{\text{QI}}(x) = (-\varphi_3(x+1) + 8\varphi_3(x) - \varphi_3(x-1))/6 \quad (14)$$

is a quasi-interpolant of order 4; in fact, it is the shortest one within the family of cubic splines. For comparison, the cubic interpolating function $\varphi_a^3(x)$ in (9) is a quasi-interpolant of order 3 only for the optimal choice $a = 1/2$ and even less otherwise. One can also construct an interpolating (or cardinal) spline by defining

$$\hat{\varphi}_{\mathrm{int},n}(2\pi f) = \frac{\hat{\varphi}_n(2\pi f)}{\sum_{k\in Z}\hat{\varphi}_n(2\pi f + 2\pi k)} = \frac{\mathrm{sinc}^{n+1}(f)}{\sum_{k\in Z}\mathrm{sinc}^{n+1}(f+k)}. \tag{15}$$

This function is also a quasi-interpolant of order $n+1$, but it is not compactly supported for $n > 1$. However, $\varphi_{\mathrm{int},n}(x)$ decays exponentially fast and can be implemented recursively [29].

### C. Convolution-Based Least Squares

A more sophisticated approach for obtaining a representation of the signal $s(x) \in L_2$ in $V_h(\varphi)$ is to determine its minimum $L_2$-norm approximation (orthogonal projection). This least squares approximation is given by (cf. [13])

$$(P_h s)(x) = \sum_{k\in Z} c(k)\varphi\left(\frac{x}{h} - k\right)$$
$$c(k) = \frac{1}{h}\left\langle s(x), \mathring{\varphi}\left(\frac{x}{h} - k\right)\right\rangle \tag{16}$$

where $\mathring{\varphi} \in V_1(\varphi)$ is the dual of $\varphi$ and where the factor $h^{-1}$ is an inner product normalization. The dual (or biorthogonal) function $\mathring{\varphi}$ is defined by

$$\mathring{\varphi}(x) = \sum_{k\in Z}(a_\varphi)^{-1}(k)\varphi(x-k) \overset{\mathrm{Fourier}}{\longleftrightarrow}$$
$$\hat{\mathring{\varphi}}(\omega) = \frac{\hat{\varphi}(\omega)}{\sum_{k\in Z}|\hat{\varphi}(\omega + 2\pi k)|^2} = \frac{\hat{\varphi}(\omega)}{\hat{a}_\varphi(\omega)} \tag{17}$$

where $(a_\varphi)^{-1}$ represents the convolution inverse of the sampled autocorrelation sequence $a_\varphi(k) := \varphi^T * \varphi(x)|_{x=k}$. The approximation procedure described by (16) can be interpreted in terms of the block diagram in Fig. 2. The signal is prefiltered with $h^{-1}\mathring{\varphi}(-x/h)$, sampled, and then reconstructed by convolution with the rescaled generating function $\varphi(x/h)$. The only difference with the (quasi-)interpolation procedure in Fig. 1 is the presence of the prefiltering module, which has a role similar to the antialiasing filter required in conventional sampling theory. In fact, if $\varphi(x) = \mathrm{sinc}(x)$, then the optimal prefilter is precisely Shannon's ideal lowpass filter with the appropriate cutoff at the Nyquist frequency.

The orthogonal projection operator on $V_h(\varphi)$ can also be expressed in the more compact form

$$(P_h s)(x) = \int_{-\infty}^{+\infty} s(y)\frac{1}{h}K\left(\frac{x}{h}, \frac{y}{h}\right)dy \tag{18}$$

where $K(x, y)$ is the reproducing kernel associated with the basic approximation space $V_1(\varphi)$:

$$K(x,y) = \sum_{k\in Z}\varphi(x-k)\mathring{\varphi}(y-k). \tag{19}$$

### D. Strang-Fix Conditions

As $h$ gets smaller, the approximation error $\|s - s_h\|$ generally decreases and eventually becomes negligible as $h$ goes to zero. The general behavior of this error as a function of $h$ depends on the ability of the representation to reproduce polynomials up to a certain degree $n$. This result is expressed by the Strang–Fix conditions [16], which relate the approximation power of the representation to the spectral characteristics of the generating function. Strang and Fix initially assumed that $\varphi$ is compactly supported, but their result has also be extended for noncompact $\varphi$ with sufficient polynomial decay at infinity [30], [31].

*1) Strang–Fix Conditions:* Let $\varphi$ be a valid generating function with appropriate decay. The following statements are equivalent:

*i)* The function spaces $V_h(\varphi)$ reproduce polynomials of degree $n = L - 1$, which is equivalent to say that there exists a function $\varphi_{\mathrm{QI}} \in V_1(\varphi)$ (not necessarily unique) that is a quasi-interpolant of order $L$.

*ii)* There exists a function $\varphi_{\mathrm{QI}} \in V_1(\varphi)$ (the same as in condition $(i)$) such that

$$\forall x \in R, \quad \sum_{k\in Z}\varphi_{\mathrm{QI}}(x-k) = 1 \tag{20}$$
$$\forall x \in R, \quad \sum_{k\in Z}(x-k)^m\varphi_{\mathrm{QI}}(x-k) = 0 \quad (m = 1, \cdots, L-1). \tag{21}$$

*iii)* $\hat{\varphi}(\omega)$, which is the Fourier transform of $\varphi$, is nonvanishing at the origin and has zeros of at least multiplicity $L$ at all nonzero frequencies that are integer multiples of $2\pi$.

*iv)* There exists a constant $C$ such that approximation error at step size $h$ is bounded as

$$\forall s \in W_2^L, \quad \inf_{s_h\in V_h(\varphi)}\|s - s_h\|_2 \le C\cdot h^L\cdot\|s^{(L)}\|_2. \tag{22}$$

*2) Remarks and Comments:*

1) The maximum value of $L$ for which any of these conditions is satisfied defines the order of approximation of the representation. With this definition, the order is one larger than the degree $n$. For example, polynomial splines of degree $n$ have an order of approximation $L = n + 1$.
2) The more standard way of expressing Condition *ii*) is through (11) and (12). In fact, these two sets of equations form a discrete Fourier transform pair since we are dealing with periodized signals.
3) Condition *iii*) provides the simplest practical test for determining the order of approximation $L$ of a certain representation space.
4) The whole strength of this result is the equivalence between a simple quasi-interpolation property and Condition *iv*). Note that with our definition of $V_h(\varphi)$, the approximation is controlled in the sense specified by Strang and Fix in [16] because $\|c_h\|^2/h \le A^{-1}\cdot\|P_h s\|^2 \le A^{-1}\cdot\|s\|^2$, where $A$ is the lower Riesz bound

in (6). This last condition is required for the proof of the implication from *iv*) to *iii*).

There is a rich approximation theory literature on the Strang–Fix conditions and their various multivariate extensions, including results for noncompactly supported $\varphi$ [30], [32], [33]; see also the surveys [17], [31], [34] and the references therein. Some of the most general results to date are provided by de Boor *et al.* [35].

In the remainder of the paper, we will have a closer look at the way in which the quasi-interpolation properties affect the approximation error. In particular, we will compare the performance of the (quasi-)interpolation and least squares signal approximation methods described previously.

## III. Pointwise Error Analysis

The simplest way to investigate the behavior of the error as a function of $h$ is to look at what happens to the signal locally. The basic tool for this analysis is the Taylor series expansion. Specifically, if our signal $s$ is $(n+1)$ times continuously differentiable (i.e., $s \in W_\infty^{n+1}$), we can write

$$s(y) = s(x) + (y-s)s^{(1)}(x) + \frac{(y-x)^2}{2!}s^{(2)}(x) + \cdots + \frac{(y-x)^n}{n!}s^{(n)}(x) + R_{n+1}(y) \quad (23)$$

where the remainder $R_{n+1}(y)$ is

$$R_{n+1}(y) = \frac{(y-x)^{n+1}}{n!} \cdot \int_0^1 (1-\tau)^n s^{(n+1)}(\tau y + (1-\tau)x))\, d\tau. \quad (24)$$

### A. Interpolation Error

Let us first consider the (quasi-)interpolation error

$$s(x) - (I_h s)(x) = s(x) - \sum_{k\in Z} s(hk)\varphi\left(\frac{x}{h} - k\right). \quad (25)$$

Replacing $s(hk)$ by its Taylor series development (23) with $y = hk$ and using the quasi-interpolation properties of $\varphi$, we get

$$s(x) - (I_h s)(x) = \frac{-h^L}{(L-1)!}\sum_{k\in Z}\left(k - \frac{x}{h}\right)^L \varphi\left(\frac{x}{h} - k\right) \cdot \left[\int_0^1 (1-\tau)^{L-1} s^{(L)}(\tau hk + (1-\tau)x))\, d\tau\right]. \quad (26)$$

The only remaining terms are those associated with the remainders of the Taylor series because $\varphi$ is designed to perfectly interpolate all expansion terms up to degree $n = L-1$. An immediate consequence is the following uniform estimate of the error (cf. [33]):

*Proposition 3.1:* If $\varphi$ is a quasi-interpolant of order $L$ with sufficient decay, then

$$\forall s \in W_\infty^L, \quad \|s - (I_h s)\|_\infty \le C_{\varphi,L} \cdot h^L \cdot \|s^{(L)}\|_\infty \quad (27)$$

where

$$C_{\varphi,L} = \frac{1}{L!}\sup_{x\in[0,1]}\sum_{k\in Z}|x-k|^L|\varphi(x-k)|.$$

*Proof:* Starting from (26), we get the following estimates:

$$|s(x) - (I_h s)(x)| \le \frac{h^L}{(L-1)!}\sum_{k\in Z}\left|\frac{x}{k} - k\right|^L \left|\varphi\left(\frac{x}{h} - k\right)\right| \cdot \int_0^1 (1-\tau)^{L-1}\sup_{x\in R}|s^{(L)}(x)|\, d\tau$$

$$|s(x) - (I_h s)(x)| \le \frac{h^L}{L!}\|s^{(L)}\|_\infty \sum_{k\in Z}\left|\frac{x}{k} - k\right|^L \left|\varphi\left(\frac{x}{h} - k\right)\right|.$$

□

In order to be applicable, this estimate requires that $\varphi$ decays sufficiently quickly. Specifically, for $C_{\varphi,L}$ to be finite, we need some polynomial decay at infinity

$$|\varphi(x)| \le K \cdot (1+|x|)^{-M}, \qquad M \ge L. \quad (28)$$

This is a relatively mild condition that is satisfied for any positive $M$ if $\varphi$ decays exponentially fast or if it has compact support. Interestingly enough, the converse statement of Proposition 3.1 is also true, but the proof requires considerably more work (cf. [33, Theorem 3.1]).

If $\varphi(x)$ decays like $O(x^{-(L+1)})$, we can improve our estimate by considering one more term in the Taylor series expansion. Using the same technique as before, we show that

$$\left|s(x) - (I_h s)(x) + \frac{h^L}{L!}E_L\left(\frac{x}{h}\right)s^{(L)}(x)\right| \le C_{\varphi,L+1} \cdot h^{L+1} \cdot \|s^{(L+1)}\|_\infty$$

where

$$E_L(x) = (-1)^L \sum_{k\in Z}(x-k)^L \varphi(x-k). \quad (29)$$

This leads to the pointwise estimate for $s \in W_\infty^{L+1}$

$$s(x) - (I_h s)(x) = -\frac{h^L}{L!}E_L\left(\frac{x}{h}\right)s^{(L)}(x) + O(h^{L+1}). \quad (30)$$

### B. Least Squares Error

To simplify the analysis of the error in the least squares case, we will use the reproducing kernel formalism [cf. (18)]. For this purpose, we first need to introduce what is yet another way of expressing the conditions for a $L$th-order approximation. We shall assume that $\varphi$ satisfies the decay condition (28).

*Proposition 3.2:* An equivalent form of conditions (20) and (21) is

$$e_0(x) = \int_{-\infty}^{+\infty} K(x,y)\, dy = 1 \quad (31)$$

$$e_m(x) = \int_{-\infty}^{+\infty} (y-x)^m K(x,y)\, dy = 0, \quad m = 1, \cdots, L-1 \quad (32)$$

where $K(x,y)$ is defined by (19).

*Proof:* Let $\mathring{\varphi}$ be the dual function for $\varphi$; then, $K$ is given by (19), which converges since $\varphi$ satisfies (28). Moreover, (28) implies that $\mathring{\varphi} \in L^1$. It then follows that $\int \mathring{\varphi}(x)\, dx \neq 0$ since otherwise, $\int K(x,y)\, dy = 0$, in contradition with (31). We can therefore always select a (possibly different) admissible $\varphi_Q \in V_1(\varphi)$ such that the moments of its dual $\mathring{\varphi}_Q$ are vanishing for $m = 1, \cdots, L-1$ and $\int \mathring{\varphi}_Q(x)\, dx = 1$ ($L$ linear constraints). Since the reproducing kernel is independent of the choice of a particular basis, we can then expand $e_m(x)$ as

$$\begin{aligned} e_m(x) &= \int_{-\infty}^{+\infty} \sum_{k\in Z} [(y-k)-(x-k)]^m \\ &\qquad \varphi_Q(x-k)\mathring{\varphi}_Q(y-k)\cdot dy \\ &= \sum_{k\in Z}\sum_{l=0}^{m} \binom{m}{l} (-1)^l (x-k)^l \varphi_Q(x-k) \\ &\qquad \cdot \int_{-\infty}^{+\infty} (y-k)^{m-l} \mathring{\varphi}_Q(y-k)\cdot dy \\ &= \sum_{l=0}^{m}\sum_{k\in Z} \binom{m}{l} (-1)^l (x-k)^l \varphi_Q(x-k) \\ &\qquad \cdot \int_{-\infty}^{+\infty} y^{m-l} \mathring{\varphi}_Q(y)\cdot dy. \end{aligned}$$

In order to justify the permutation of various sums and integrals, we use the decay conditions on $\varphi$; the argument uses Lebesgue's dominated convergence theorem. Because of our assumption on the moments of $\mathring{\varphi}_Q$, all terms are zero except the last one:

$$e_m(x) = (-1)^m \sum_{k\in Z} (x-k)^m \varphi_Q(x-k) \text{ for } m = 0, \cdots, L-1. \tag{33}$$

Thus, (31) and (32) imply that $\varphi_Q$ is a quasi-interpolant. Conversely, if $\varphi_Q$ is a quasi-interpolant of order $L$, then the same is true for its dual (cf. [36, Proposition 1]), and the moment conditions necessary for obtaining (33) are automatically satisfied. □

Using (18) and (31), we now write the approximation error as

$$s(x) - (P_h s)(x) = \int_{-\infty}^{+\infty} (s(x) - s(y)) \frac{1}{h} K\left(\frac{x}{h}, \frac{x}{h}\right) dy \tag{34}$$

where we use the rescaled version of the reproducing kernel. Substituting the Taylor series (23) in (34) and using the fact that $K(x,y)$ "kills" all difference monomials $(y-x)^m$ up to degree $n = L-1$, we end up with the contribution associated with the remainder only

$$s(x) - (P_h s)(x) = -\int_{-\infty}^{+\infty} R_L(y) \frac{1}{h} K\left(\frac{x}{h}, \frac{y}{h}\right) dy. \tag{35}$$

This leads to the following standard error bound (cf. [18], [37]–[39]).

*Proposition 3.3:* If $\varphi$ is such that the conditions in Proposition 3.2 are satistified, then

$$\forall s \in W_\infty^L, \quad \|s - (P_h s)\|_\infty \le C_{K,L} \cdot h^L \cdot \|s^{(L)}\|_\infty \tag{36}$$

where

$$C_{K,L} = \frac{1}{L!} \sup_x \left[\int_{-\infty}^{+\infty} |x-y|^L |K(x,y)|\, dy\right].$$

*Proof:* The remainder in (35) can also be written in the standard form

$$R_L(y) = \frac{(y-x)^L}{L!} s^{(L)}(\xi)$$

where $\xi$ is some value between $x$ and $y$. This leads to the estimate

$$\begin{aligned} |s(x) - (P_h s)(x)| &\le \frac{h^L}{L!} \cdot \|s^{(L)}\|_\infty \\ &\quad \cdot \int_{-\infty}^{+\infty} \left|\frac{y}{h} - \frac{x}{h}\right|^L \frac{1}{h} \left|K\left(\frac{x}{h}, \frac{y}{h}\right)\right| dy. \end{aligned}$$

We then make the change of variable $y' = y/h$ in the integral and take the supremum. □

Here, we also have to exert some caution and make sure that the bound in Proposition 3.3 is well defined. In practice, $\varphi$ and its dual $\mathring{\varphi}$ have at least exponential decay. In such a case, it can be shown that $|K(x,y)| \le C \cdot e^{-\gamma|x-y|}$ so that the constant $C_{K,L}$ is definitely finite. In fact, Proposition 3.3 is also valid if $\varphi$ has sufficient polynomial decay at infinity; this result can be obtained as a particular case of the general $L_p$ bound in [18, Theorem 2.2].

Considering one more term in the Taylor series expansion, we can use the same argument as before to show that for $s \in W_\infty^{L+1}$

$$s(x) - (P_h s)(x) = -\frac{h^L}{L!} e_L\left(\frac{x}{h}\right) s^{(L)}(x) + O(h^{L+1}) \tag{37}$$

where the auxiliary function $e_L(x)$ is defined as

$$e_L(x) = \int_{-\infty}^{+\infty} (y-x)^L K(x,y)\, dy. \tag{38}$$

Note that this last function can be shown to be equivalent to the monowavelets introduced in [38]. The estimate for the remainder term in (37) is essentially the same as in Proposition 3.3, except that it uses $(L+1)$ instead of $L$.

Similarly, we can obtain higher order asymptotic error expansions. Such estimates were recently described in the context of the wavelet transform; for more details, refer to [37]–[39].

### C. Comments

The present analysis indicates that the local behavior of the error is qualitatively the same in the (quasi-)interpolation and least squares cases. All estimates emphasize the characteristic $O(h^L)$ decay. The only difference is in the magnitude of the constants involved, which tend to be smaller in the least squares case.

The functions $E_L(x)$ and $e_L(x)$ that appear in the pointwise estimates (30) and (37) are both periodic with periodicity one. This suggests that the error (for $h$ sufficiently small) has an oscillatory behavior with an amplitude that is proportional to the $L$th derivative of the signal.

## IV. $L_2$ Error Bounds

For quantification purposes, it is usually more informative to investigate the behavior of the $L_2$ error. The most appropriate tool for this type of analysis is the Fourier transform. In order to get ready for this task, we first prove a useful lemma.

*Lemma 4.1:* If $F$ is $M$ times continuously differentiable and $F^{(m)}(2\pi k) = 0$ for all $k \in Z, k \neq 0$ and $m = 0, \cdots, M-1$, and where $F^{(M)}$ decays fast enough so that

$$\sum_k |F^{(M)}(\xi + 2\pi k)| \leq C \leq +\infty$$

then

$$\left|\sum_{k\neq 0} F(\omega + 2\pi k)\right| \leq \frac{|\omega|^M}{M!} \sup_{|\xi|\leq\pi} \left|\sum_{k\neq 0} F^{(M)}(\xi + 2\pi k)\right| \quad \text{for all } |\omega| \leq \pi.$$

*Proof:* We replace $F(\omega + 2\pi k)$ by its Taylor series expansion of order $M$ around $x = 2\pi k$ and perform the summation over all nonzero integers $k$. Because all derivatives up to order $M-1$ are zero, we end up with the summation of the remainders only [cf. (24)]

$$\sum_{k\neq 0} F(\omega + 2\pi k) = \frac{\omega^M}{(M-1)!} \sum_{k\neq 0} \int_0^1 F^{(M)}(2\pi k + \tau\omega)(1-\tau)^{M-1}\, d\tau.$$

Taking the absolute value and permuting the sum and the integral, we obtain

$$\left|\sum_{k\neq 0} F(\omega + 2\pi k)\right| \leq \frac{\omega^M}{(M-1)!} \sup_{|\xi|\leq\pi} \left|\sum_{k\neq 0} F^{(M)}(2\pi k + \xi)\right| \cdot \int_0^1 (1-\tau)^{M-1}\, d\tau$$

which yields the desired result since the value of the integral is precisely $1/M$. □

### A. Quasi-interpolants

Let us consider the block diagram in Fig. 1, and write down the sampling equation in the Fourier domain

$$(I_h s)^\wedge(\omega) = \sum_{k\in Z} \hat{s}\left(\omega + \frac{2\pi}{h}k\right)\hat{\varphi}(h\omega) \tag{39}$$

where $\hat{s}(\omega)$ denotes the Fourier transform of the input signal $s$. This leads to the following Fourier domain representation of the error:

$$\begin{aligned}\hat{s}(\omega) - (I_h s)^\wedge(\omega) &= \{\hat{s}(\omega)(1 - \hat{\varphi}(h\omega))\} + \left\{\sum_{k\neq 0} \hat{s}\left(\omega + \frac{2\pi}{h}k\right)\hat{\varphi}(h\omega)\right\} \\ &= \hat{E}_1(\omega) + \hat{E}_2(\omega)\end{aligned} \tag{40}$$

where the error is decomposed in its in-band and out-of-band components.

*Proposition 4.2:* If $\varphi$ is a quasi-interpolant of order $L$ with sufficient decay, then

$$\forall s \in W_2^L, \quad \|s - I_h s\|_2 \leq C_{\varphi,L} \cdot h^L \cdot \|s^{(L)}\|_2. \tag{41}$$

A stronger $L_p$ version of this result can be found in [30, Th. 4.1]. Here, we present our own proof for the $L_2$ case, mainly because some of the intermediate inequalities will be required to discuss the similarities and differences with the least squares case.

*Proof:* All computations are performed in the Fourier domain using Parseval's relation. We start with the first error term

$$\|E_1\|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{s}(\omega)|^2 |1 - \hat{\varphi}(h\omega)|^2\, d\omega \tag{42}$$

and consider the $L$th-order Taylor series expansion of $(1 - \hat{\varphi}(h\omega))$ around the origin. Since $\hat{\varphi}(0) = 1$ and $\hat{\varphi}^{(m)}(0) = 0$ for $m = 1, \cdots, L-1$, we have the following estimate:

$$|1 - \hat{\varphi}(h\omega)| \leq \frac{(h\omega)^L}{L!} \sup_\xi |\varphi^{(L)}(\xi)|$$

which, together with (42), implies that

$$\|E_1\| \leq \left(\frac{1}{L!}\sup_\xi |\varphi^{(L)}(\xi)|\right) \cdot h^L \cdot \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^{2L}|\hat{s}(\omega)|^2\, d\omega\right)^{1/2} = C_1 \cdot h^L \cdot \|s^{(L)}\|. \tag{43}$$

The second error term is

$$\|E_2\|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\sum_{k\neq 0} \hat{s}\left(\omega + \frac{2\pi}{h}k\right)\right|^2 |\hat{\varphi}(h\omega|^2\, d\omega.$$

We multiply $\hat{s}(\omega + 2\pi k/h)$ by $(1/k) \cdot k = 1$ and use the Schwarz inequality on the sum to get

$$\|E_2\|^2 \leq \underbrace{\left(\sum_{k\neq 0} \frac{1}{k^2}\right)}_{\alpha} \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{k\neq 0} k^2 \left|\hat{s}\left(\omega + \frac{2\pi}{h}k\right)\right|^2 \cdot |\hat{\varphi}(h\omega|^2\, d\omega.$$

Next, we make the change of variable $\xi = \omega + 2\pi k/h$, which yields

$$\begin{aligned}\|E_2\|^2 &\leq \alpha \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{s}(\xi|^2 \sum_{k\neq 0} k^2 |\hat{\varphi}(h\xi - 2\pi k|^2 \, d\xi \\ &\leq \left(\alpha \cdot \sup_{\xi} \frac{\sum_{k\neq 0} k^2|\hat{\varphi}(\xi + 2\pi k)|^2}{|\xi|^{2L}}\right) \cdot h^{2L} \\ &\quad \cdot \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^{2L}|\hat{s}(\omega)|^2 \, d\omega\right) = C_2 \cdot h^{2L} \cdot \|s^{(L)}\|^2. \end{aligned} \tag{44}$$

We now need to show that $C_2$ is finite and to estimate it. To do this, we use

$$k^2 \leq \frac{1}{2\pi^2}(|\xi + 2\pi k|^2 + |\xi|^2)$$

because

$$x^2 \leq 2[(x+y)^2 + y^2] = x^2 + (x+2y)^2.$$

This implies that

$$\begin{aligned}C_2 \leq \frac{\alpha}{2\pi^2}\Bigg[&\sup_{\xi} \frac{\sum_{k\neq 0}|\hat{\varphi}(\xi + 2\pi k)|^2}{|\xi|^{2L-2}} \\ &+ \sup_{\xi} \frac{\sum_{k\neq 0}|\xi + 2\pi k|^2 |\hat{\varphi}(\xi + 2\pi k)|^2}{|\xi|^{2L}}\Bigg].\end{aligned}$$

For each of these terms, we can use Lemma 4.1, assuming that $\varphi$ has sufficient decay for $\hat{\varphi}$ to be differentiable $2L$ times; one easily checks that both $F_1(\omega) = |\hat{\varphi}(\omega)|^2$ and $F_2(\omega) = |\omega\hat{\varphi}(\omega)|^2$ satisfy the necessary condition. Consequently

$$\begin{aligned}C_2 \leq C_3 = &\frac{\alpha}{2\pi^2}\frac{1}{(2L-2)!}\sup_{\xi}\left|\sum_{k\neq 0}(|\hat{\varphi}|^2)^{(2L-2)}(\xi + 2\pi k)\right| \\ &+ \frac{\alpha}{2\pi^2}\frac{1}{(2L)!}\sup_{\xi}\left|\sum_{k\neq 0}(|\xi\hat{\varphi}|^2)^{(2L)}(\xi + 2\pi k)\right|.\end{aligned}$$

Finally, putting things together

$$\|s - I_h s\|_2 \leq \|E_1\| + \|E_2\| \leq h^L \cdot \|s^{(L)}\|_2 \cdot \{C_1 + \sqrt{C_3}\}. \quad \square$$

### B. Improved Least Squares Estimate

By considering the block diagram in Fig. 2, we obtain the Fourier representation of the least squares approximation of $s$

$$(P_h s)^\wedge(\omega) = \hat{\varphi}(h\omega)\sum_{k\in Z} \overline{\hat{\mathring{\varphi}}(h\omega + 2\pi k)}\hat{s}\left(\omega + \frac{2\pi}{h}k\right).$$

Using the explicit transfer function of the prefilter $\mathring{\varphi}$ [cf. (17)], we get

$$(P_h s)^\wedge(\omega) = \hat{\varphi}(h\omega)\sum_{k\in Z} \frac{\overline{\hat{\varphi}(h\omega + 2\pi k)}}{\hat{a}_\varphi(h\omega)} \cdot \hat{s}\left(\omega + \frac{2\pi}{h}k\right) \tag{45}$$

where $\hat{a}_\varphi(\omega)$, which is $2\pi$ periodic, is defined by (6). The approximation error in the Fourier domain can therefore be written as

$$\hat{s}(\omega) - (P_h s)^\wedge(\omega) = \hat{e}_1(\omega) + \hat{e}_2(\omega) \tag{46}$$

where

$$\hat{e}_1(\omega) = \left[1 - \frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}\right]\hat{s}(\omega) \tag{47}$$

$$\hat{e}_2(\omega) = \sum_{k\neq 0}\frac{\hat{\varphi}(h\omega)}{\hat{a}_\varphi(h\omega)} \cdot \overline{\hat{\varphi}(h\omega + 2\pi k)} \cdot \hat{s}\left(\omega + \frac{2\pi}{h}k\right). \tag{48}$$

Using this decomposition, we can establish the following new error bound.

*Theorem 4.3:* If $\hat{\varphi}^{(m)}(2\pi k) = 0, k \in Z, k \neq 0$ and $m = 0, \cdots, L-1$, then

$$\begin{aligned}\forall s \in W_2^L, \quad \|s - P_h s\|_2 &\leq K_{\varphi,2L} \cdot h^{2L} \cdot \|s^{(2L)}\|_2 \\ &+ K_{\varphi,2L}^{1/2} \cdot h^L \cdot \|s^{(L)}\|_2\end{aligned} \tag{49}$$

where

$$K_{\varphi,2L} = \frac{1}{(2L)!} \cdot \frac{1}{A} \cdot \sup_{\xi}\left|\sum_{k\neq 0}(|\hat{\varphi}|^2)^{(2L)}(\xi + 2\pi k)\right| \tag{50}$$

and where $A = \inf_\omega[\hat{a}_\varphi(\omega)]$ is the lower Riesz bound in (6).

Note that a sufficient condition for $K_{\varphi,2L}$ to be finite is that $\varphi$ has exponential decay.

*Proof:* Using Parseval's relation (3), we evaluate the first error term

$$\|e_1\| = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{s}(\omega)|^2 \left[1 - \frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}\right]^2 d\omega\right)^{1/2}. \tag{51}$$

To estimate the second term, we use a different technique:

$$\begin{aligned}\|e_2\| &= \sup_{f\in L_2, \|f\|=1} \langle e_2, f\rangle \\ &= \sup_{f\in L_2, \|f\|=1} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\hat{f}(\omega)} \sum_{k\neq 0}\frac{\hat{\varphi}(h\omega)}{\hat{a}_\varphi(h\omega)^{1/2}} \\ &\qquad \cdot \frac{\hat{\varphi}(h\omega + 2\pi k)}{\hat{a}_\varphi(h\omega)^{1/2}}\hat{s}\left(\omega + \frac{2\pi}{h}k\right) d\omega.\end{aligned}$$

Rearranging the factors and using the Cauchy–Schwarz inequality over the sum and over the integral, we get

$$\begin{aligned}\|e_2\| \leq \sup_f \Bigg[&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\sum_{k\neq 0}|\hat{f}(\omega)|^2 \frac{|\hat{\varphi}(h\omega + 2\pi k)|^2}{\hat{a}_\varphi(h\omega)}\right)^{1/2} \\ &\cdot \left(\sum_{k\neq 0}\frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}\left|\hat{s}\left(\omega + \frac{2\pi}{h}k\right)\right|^2\right)^{1/2} d\omega\Bigg]\end{aligned}$$

$$\begin{aligned}\|e_2\| \leq \sup_f &\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k\neq 0}|\hat{f}(\omega)|^2 \frac{|\hat{\varphi}(h\omega + 2\pi k|^2}{\hat{a}_\varphi(h\omega)} \, d\omega\right]^{1/2} \\ &\cdot \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k\neq 0}\frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}\left|\hat{s}\left(\omega + \frac{2\pi}{h}k\right)\right|^2 d\omega\right]^{1/2}.\end{aligned}$$

The first factor is bounded by

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k\neq 0}|\hat{f}(\omega)|^2\frac{|\hat{\varphi}(h\omega+2\pi K|^2}{\hat{a}_\varphi(h\omega)}\,d\omega &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\hat{f}(\omega)|^2\sum_{k\in Z}\frac{|\hat{\varphi}(h\omega+2\pi k)|^2}{\hat{a}_\varphi(h\omega)}\,d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\hat{f}(\omega)|^2\,d\omega\end{aligned}$$

because of (6). Consequently

$$\|e_2\| \le \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{k\neq 0}\frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}\left|\hat{s}\left(\omega+\frac{2\pi}{h}k\right)\right|^2 d\omega\right]^{1/2}.$$

Next, we make the change of variable $\xi=\omega+2\pi k/h$, which yields

$$\|e_2\| \le \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\hat{s}(\xi)|^2\sum_{k\neq 0}\frac{|\hat{\varphi}(h\xi-2\pi k)|^2}{\hat{a}_\varphi(h\xi)}\,d\xi\right]^{1/2}. \tag{52}$$

We now need to estimate the quantity

$$1-\frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}=\sum_{\substack{k\in Z,\\ k\neq 0}}\frac{|\hat{\varphi}(h\omega+2\pi k)|^2}{\hat{a}_\varphi(h\omega)}$$

which plays a special role in both cases. It is easy to check that the function $F(\omega)=|\varphi(\omega)|^2$ (or $F(\omega)=|\varphi(\omega)|^2/\hat{a}_\varphi(\omega)$ since $\hat{a}_\varphi(\omega)$ is nonvanishing) has zeros of multiplicity $2L$ at all nonzero frequencies that are integer multiples of $2\pi$. Hence, we can use Lemma 4.1 to get the estimate

$$\begin{aligned}\left|1-\frac{|\hat{\varphi}(h\omega)|^2}{\hat{a}_\varphi(h\omega)}\right| &= \sum_{\substack{k\in Z,\\ k\neq 0}}\frac{|\hat{\varphi}(h\omega+2\pi k)|^2}{\hat{a}_\varphi(h\omega)}\\ &\le \frac{(h\omega)^{2L}}{(2L)!}\,\frac{1}{A}\sup_{\xi}\left|\sum_{k\neq 0}(|\hat{\varphi}|^2)^{(2L)}(\xi+2\pi k)\right|\end{aligned} \tag{53}$$

where we have used the lower Riesz bound $A\le\hat{a}_\varphi(\omega)$ to get rid of the denominator. Using this relation in (51), we find that

$$\|e_1\| \le h^{2L}\cdot K_{\varphi,2L}\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\omega|^{4L}|\hat{s}(\omega)|^2\,d\omega\right)^{1/2} \tag{54}$$

where $K_{\varphi,2L}$ is given by (50). Similarly, we estimate the right side of (52)

$$\|e_2\| \le h^{L}\cdot\sqrt{K_{\varphi,2L}}\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\omega|^{2L}|\hat{s}(\omega)|^2\,d\omega\right)^{1/2}. \tag{55}$$

Putting things together, we end up with

$$\begin{aligned}\|s-P_hs\|_2 \le \|e_1\|_2+\|e_2\|_2 &\le K_{\varphi,2L}\cdot h^{2L}\cdot\|s^{2L)}\|_2\\ &\quad+\sqrt{K_{\varphi,2L}}\cdot h^{L}\cdot\|s^{(L)}\|_2. \qquad\square\end{aligned}$$

### C. Results and Discussion

The $L_2$ bounds in Proposition 4.2 and Theorem 4.3 are both consistent with the Strang–Fix condition (22). However, the new bound for the least squares case provides a finer characterization of the error. It is made up of two distinct terms that represent the so-called in-band $(e_1)$ and out-of-band $(e_2)$ contributions of the error, respectively [cf. (46)–(48)]. For smaller values of $h$, the first part of the error becomes negligible, and the bound is dominated by the second $O(h^L)$ term. The corresponding constant $K_{\varphi,2L}^{1/2}$ turns out to be smaller than the constant $C$ in the Strang–Fix inequality (22), or $C_{\varphi,L}$ in Proposition 4.2, probably reflecting the fact that this term represents a portion of the error only [cf. (55)]. In other words, we have an improved bound for smaller values of $h$. This is a first indication that there is a true advantage in using least squares over interpolation.

In addition, we note that the first part of the least squares bound has the characteristic form of the error for an interpolator of order $2L$. This is the predominant term for larger values of $h$. Although we should not overinterpret this result, we can at least identify empirical conditions under which the performance of the least squares estimation is comparable to that of an interpolator with twice the order. For this purpose, we observe that the function $\hat{\varphi}_{2L}(\omega)=|\hat{\varphi}(\omega)|^2/\hat{a}_\varphi(\omega)$ in (51) represents the frequency response of an interpolator of order $2L$. The in-band error $\|e_1\|$, which is given by (51), therefore turns out to be exactly the same as the corresponding error $\|E_1\|$ [cf. (42)] for this "augmented" interpolator. Thus, under the condition that $\|e_1\| \gg \|e_2\|$, the least squares solution of order $L$ should perform as well (or even better if $\|E_2\|>\|e_2\|$) as the corresponding interpolator with twice the order. This condition typically arises for larger $h$ when the signal is somewhat undersampled (not bandlimited). In this case, the out-of-band error primarily depends on the decay of $\hat{\varphi}(\omega)$—aliasing tends to be reduced because of the prefiltering with $\mathring{\varphi}$.

This general behavior of the error as a function of $h$ was verified experimentally using several test functions. Some of these results are shown Fig. 3. These graphs were obtained by applying various polynomial spline approximation procedures to a test function and evaluating the corresponding $L_2$ error by numerical integration. The upper and lower error curves correspond to a linear interpolation and a cubic spline quasi-interpolation [cf. (14)], respectively. They all exhibit the characteristic $O(h^L)$ behavior predicted by the theory. The thicker curve corresponds to the least squares linear spline approximation. Interestingly, it first matches the cubic quasi-interpolant curve very closely and then progressively switches to its $O(h^2)$ asymptotic regime. Asymptotically, the least squares approach achieves a near constant 7.7 dB improvement over the interpolation method.

Note that the pseudo-equivalence between the least squares approximation and an interpolation with twice the order is consistent with the experimental results reported in [14] for image reduction and enlargement. An exact equivalence has also been demonstrated recently for the simpler task of signal translation [40].

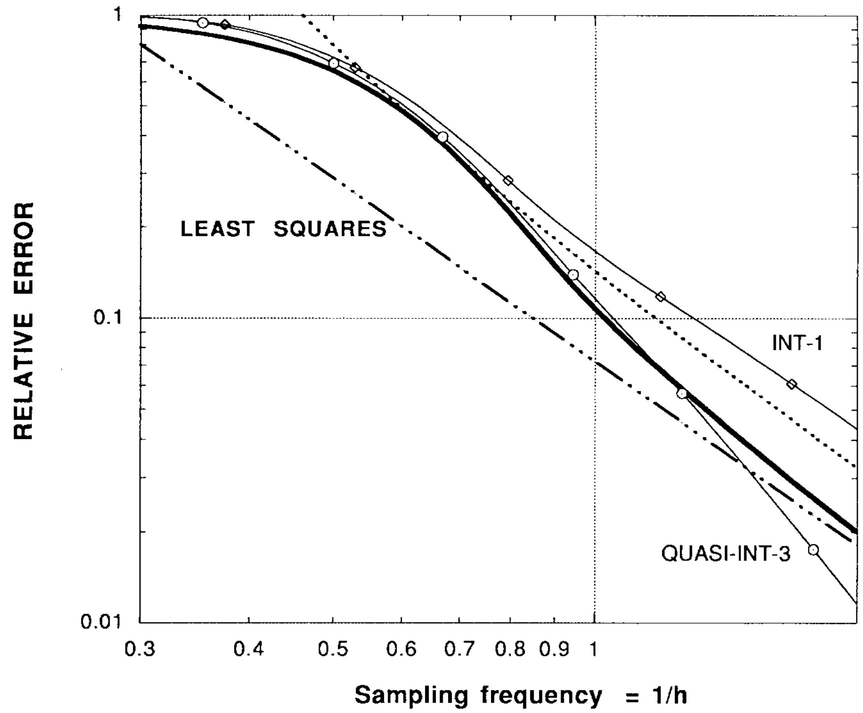


Fig. 3. Comparison of experimental error curves for the quasi-interpolation (solid fine lines—diamonds: INT-1, and circles: QUASI-INT-3) and least squares spline approximation (solid bold) of the function $\Psi_1(x) = -x \cdot e^{-x^2/2}$ (first derivative of a Gaussian). The theoretical $O(h^2)$ (linear splines) asymptotic trend is represented in mixed lines.

For the special case of least squares polynomial splines approximation, we can be more qualitative and provide simple bounds for the constants in Theorem 4.3. For this purpose, we use the following lemma.

*Lemma 4.4:* Let $\hat{f}(\omega)$ be the Fourier transform of a real-valued positive function $f(x) \geq 0, x \in R$. Then

$$\sum_{k=1}^{+\infty}[\hat{f}(2\pi k) + \hat{f}(-2\pi k)] \leq \sup_{\xi}\left|\sum_{k\neq 0}\hat{f}(\xi + 2\pi k)\right| \leq \sum_{k=0}^{+\infty}[\hat{f}(2\pi k) + \hat{f}(-2\pi k)] \tag{56}$$

holds.

*Proof:* We start by writing the two inequalities

$$\left|\sum_{k\neq 0}\hat{f}(\omega + 2\pi k)\right| \leq \left|\sum_{k\in Z}\hat{f}(\omega + 2\pi k)\right| + |\hat{f}(\omega)|$$

$$\sup_{\xi}\left|\sum_{k\in Z}\hat{f}(\xi + 2\pi k)\right| \leq \sup_{\xi}\left|\sum_{k\neq 0}\hat{f}(\xi + 2\pi k)\right| + \sup_{\xi}|\hat{f}(\xi)|$$

which together imply that

$$\sup_{\xi}\left|\sum_{k\in Z}\hat{f}(\xi + 2\pi k)\right| - \sup_{\xi}|\hat{f}(\xi)| \leq \sup_{\xi}\left|\sum_{k\neq 0}\hat{f}(\xi + 2\pi k)\right| \leq \sup_{\xi}\left|\sum_{k\in Z}\hat{f}(\xi + 2\pi k)\right| + \sup_{\xi}|\hat{f}(\xi)|. \tag{57}$$

We also note that $\Sigma_{k\in Z}\ \hat{f}(\omega + 2\pi k)$ is the discrete Fourier transform of the sequence $f(k)$. Using the fact that $f$ is positive for all $x$, we then bound the Fourier transforms on either side of (57) by their values at the origin, which yields the required suprema

$$\begin{aligned}|\hat{f}(\omega)| &= \left|\int_{-\infty}^{+\infty} f(x)e^{-j\omega x}dx\right| \leq \int_{-\infty}^{+\infty}|f(x)|dx \\ &= \int_{-\infty}^{+\infty} f(x)dx = \hat{f}(0)\end{aligned}$$

$$\begin{aligned}\left|\sum_{k\in Z}\hat{f}(\omega + 2\pi k)\right| &= \left|\sum_{k\in Z} f(k)e^{j\omega k}\right| \leq \sum_{k\in Z}|f(k)| \\ &= \sum_{k\in Z} f(k) = \sum_{k\in Z}\hat{f}(2\pi k).\end{aligned}$$

□

For the B-splines of degree $n$, we consider $\hat{f}(\omega) = (-1)^{n+1}(|\hat{\varphi}_n|^2)^{(2n+2)}(\xi)$, which is the Fourier transform of $f(x) = x^{2n+2} \cdot \varphi_{2n+1}(x)$, where $\varphi_{2n+1} = \varphi_n * \varphi_n$ is the B-spline of degree $2n+1$. This function satisfies the required positivity constraint, and we can use Lemma 4.4 to get an upper and lower bound for the spline constant in (49)

$$A_L^{-1/2} \cdot C_L^- \leq \sqrt{K_{\varphi,2L}} \leq A_L^{-1/2} \cdot C_L^+. \tag{58}$$

The various quantities involved in this inequality are the Riesz bound $A_L$ for the centered B-spline of order $L$ (or degree $n = L - 1$)

$$\begin{aligned}A_L &= \sum_{l\in Z}\operatorname{sinc}^{2L}\left(\frac{1}{2} + l\right) = \sum_{l\in Z}\frac{1}{((2l+1)\pi)^{2L}} \\ &= \frac{|B_{2L}|}{(2L)!}(4^{2L} - 4^L)\end{aligned} \tag{59}$$

TABLE I
VARIOUS BOUND AND LIMIT CONSTANTS FOR POLYNOMIAL SPLINES OF DEGREE $n = 0, \cdots, 5$

| $n = L-1$ | $A_L^{-1/2}$ | $C_L^+$ | $C_L^- = C_{0,L}^{LS}$ | $C_{0,L}^{int}$ | $C_{0,L}^{QI}$ |
|---|---|---|---|---|---|
| 0 | 1 | 0.5 | 0.288675 | 0.288675 | 0.288675 |
| 1 | 1.73205 | $1.62447\times10^{-1}$ | $3.72678\times10^{-2}$ | $9.12871\times10^{-2}$ | $9.12871\times10^{-2}$ |
| 2 | 2.73861 | $6.53137\times10^{-2}$ | $5.75055\times10^{-3}$ | $5.75055\times10^{-3}$ | $5.75055\times10^{-3}$ |
| 3 | 4.30458 | $2.75036\times10^{-2}$ | $9.09241\times10^{-4}$ | $1.66004\times10^{-3}$ | $2.91808\times10^{-2}$ |
| 4 | 6.76209 | $1.17897\times10^{-2}$ | $1.44488\times10^{-4}$ | $1.44488\times10^{-4}$ | $1.44488\times10^{-4}$ |
| 5 | 10.6219 | $5.10458\times10^{-3}$ | $2.29874\times10^{-5}$ | $4.02736\times10^{-5}$ | $6.67992\times10^{-3}$ |

and the spline constants $C_L^-$ and $C_L^+$, which can be evaluated as

$$C_L^- = \left(\frac{1}{(2L)!}\sum_{k=1}^{+\infty} 2\hat{f}(2\pi k)\right)^{1/2} = \left(2\sum_{k=1}^{+\infty}\frac{1}{(2\pi k)^{2L}}\right)^{1/2} = \sqrt{\frac{|B_{2L}|}{(2L)!}} \tag{60}$$

$$C_L^+ = \left(\frac{2\hat{f}(0)}{(2L)!} + (C_L^-)^2\right)^{1/2} = \sqrt{\frac{2b_{2L}+|B_{2L}|}{(2L)!}}. \tag{61}$$

$|B_{2L}|$ is Bernoulli's number of order $2L$, and $b_{2L} = \int x^{2L}\varphi_{2L-1}\,dx$ is the $(2L)$th moment of a B-spline of degree $2L-1$, which also corresponds to the $2L$th derivative of its Fourier transform at the origin. Specific numerical values for these various splines constants are given in Table I for $n = 0, \cdots, 5$.

Rigorously, we should only consider the larger constant $C_L^+$ which is the only one that guarantees that the bound in Theorem 4.3 is preserved. In practice, however, the corresponding estimate turns out to be rather conservative. The smaller constant $C_L^-$, on the other hand, leads to an estimate that is closer to the mark, even though it is not a true bound. This choice is not unreasonable because the same constant also appears in the asymptotic trend: $\|s - P_h s\| = C_L^- \cdot h^L \cdot \|s^{(L)}\|$ as $h \to 0$, which is derived in Section V-B. Hence, the estimate even provides an asymptotic margin of security that is precisely $A_L^{-1/2}$. It is also interesting to note that $A_{2L}^{-1/2} \cdot C_{2L}^- \approx (A_L^{-1/2} \cdot C_L^-)^2$, which is an approximate relation that improves with increasing $L$ since the various sums involved have essentially one dominant term at $k = 1$. This observation suggests that $K_{\varphi,2L} \approx \sqrt{K_{\varphi,4L}}$; in other words, the proportionality factor in the first error term in (49) has approximately the same order of magnitude as the constant that would appear in the second (asymptotic) term for an approximation with twice the order.

For all cases that we tested, we found the less conservative estimate (i.e., (49) with $\sqrt{K_{2L}^-} = A_L^{-1/2} \cdot C_L^-$) to be a good predictor of the true error curve. The results for the least squares approximation of a Mexican hat function are shown in Fig. 4. Except for an uncharacteristic dip that occurs at a lower sampling rate, the experimental error points are tightly sandwiched between the upper bound and the asymptotic trend predicted by the theory. This example also illustrates the fact that at a given $h$, it is possible to reduce the error quite substantially by switching to a higher order representation.

## V. ASYMPTOTIC ERROR ANALYSIS

For smaller sampling steps, we can be more precise and obtain a near-exact characterization of the $L_2$-error using the pointwise estimates in Section III.

### A. Quasi-Interpolant Asymptotics

For $h$ sufficiently small, the $O(h^{L+1})$ terms in (30) become negligible, and

$$\lim_{h\to 0}(\|s(x) - (I_h s)(x)\|_2/h^L) = \frac{1}{L!}\lim_{h\to 0}\left\|s^{(L)}(x)\cdot E_L\left(\frac{x}{h}\right)\right\|_2. \tag{62}$$

The function $E_L(x)$, which is defined by (29), is simply the periodized version of $f(x) = (-x)^L\varphi(x)$. Its Fourier series representation can therefore be determined by sampling the continuous Fourier transform $\hat{f}(\omega) = (-j)^L\hat{\varphi}^{(L)}(\omega)$ at multiples of $2\pi$:

$$E_L(x) = (-j)^L\sum_{k\in Z}\hat{\varphi}^{(L)}(2\pi k)e^{j2\pi kx}. \tag{63}$$

The function $E_L(x/h)$ oscillates with a periodicity $h$. As $h$ goes to zero, the signal term $s^{(L)}(x)$ in (62) can be locally represented by a constant within the duration of each of these oscillations. Assuming that $s^{(L)}$ is continuous, we write

$$\begin{aligned}\lim_{h\to 0}&\left\|s^{(L)}(x)\cdot E_L\left(\frac{x}{h}\right)\right\|^2\\ &= \lim_{h\to 0}\sum_{k\in Z}\int_{kh}^{(k+1)\cdot h}|s^{(L)}(kh)|^2\left|E_L\left(\frac{x}{h}\right)\right|^2 dx\\ &= \lim_{h\to 0}\sum_{k\in Z}h|s^{(L)}(kh)|^2\cdot\frac{1}{h}\int_0^h\left|E_L\left(\frac{x}{h}\right)\right|^2 dx\\ &= \|s^{(L)}\|_2^2\cdot\|E_L\|_2^2\end{aligned}$$

where the right-most factor is the mean square modulus of $E_L(x)$. This quantity, which is independent of $h$, is determined as

$$\|E_L\|_2^2 = \frac{1}{h}\int_0^h |E_L(x/h)|^2\,dx = \int_0^1 |E_L(x)|^2\,dx = \sum_{k\in Z}|\hat{\varphi}^{(L)}(2\pi k)|^2. \tag{64}$$

Hence, we get the following result.

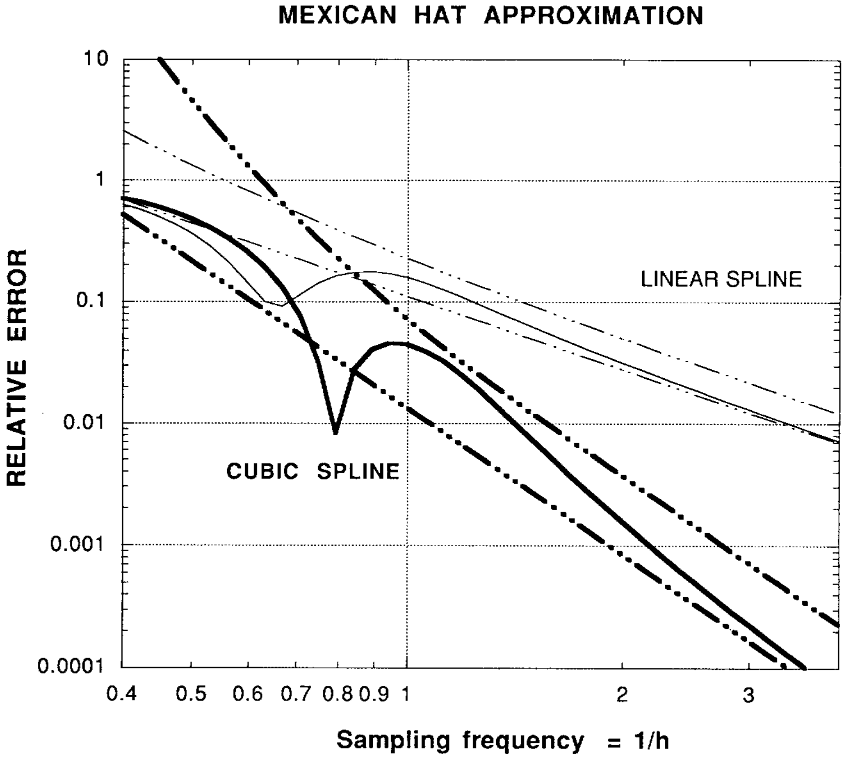


Fig. 4. Error bounds and asymptotic trends for the piecewise linear (fine) and cubic spline (bold) least squares approximations of the function $\Psi_2(x) = (1 - x^2) \cdot e^{-x^2/2}$ (second derivative of a Gaussian). The corresponding experimental error curves are represented in solid lines.

*Proposition 5.1:* If $\varphi$ is a quasiinterpolant of order $L$ with sufficient decay, then

$$\begin{aligned}\forall s \in W_2^L \cap W_\infty^{L+1}, \quad &\lim_{h\to 0}(\|s - I_h s\|_2 h^L) \\ &= C_{0,L}^{\mathrm{QI}} \cdot \|s^{(L)}\|_2 \end{aligned} \tag{65}$$

where

$$C_{0,L}^{\mathrm{QI}} = \frac{1}{L!}\left(\sum_{k\in Z} |\hat{\varphi}^{(L)}(2\pi k)|^2\right)^{1/2}.$$

## B. Least Squares Asymptotics

We can apply the same procedure to determine the asymptotic behavior for the least square case, except that we need to consider the function $e_L(x)$ instead of $E_L(x)$. For this derivation, we need an explicit representation of this function, which is obtained through a manipulation similar to the one in Proposition 3.2:

$$\begin{aligned} e_L(x) &= (-1)^L \left(\sum_{k\in Z}(x-k)^L \varphi(x-k) - \int_{-\infty}^{+\infty} y^L \varphi(y)\, dy\right) \\ &= (-j)^L \sum_{k\neq 0} \hat{\varphi}^{(L)}(2\pi k) e^{j2\pi k x}. \end{aligned} \tag{66}$$

Interestingly, this formula turns out to be the nonbiased (or zero mean) version of (63). Based on (66) and (37), we get the least squares counterpart of Proposition 5.1.

*Proposition 5.2:* If $V(\varphi)$ has and $L$th-order of approximation and $\varphi$ is such that $\hat{\varphi}(0) = 1$, then

$$\forall s \in W_2^L \cap W_\infty^{L+1}, \quad \lim_{h\to 0}(\|s - P_h s\|_2 / h^L) = C_{0,L}^{\mathrm{LS}} \cdot \|s^{(L)}\|_2 \tag{67}$$

where

$$C_{0,L}^{\mathrm{LS}} = \frac{1}{L!}\left(\sum_{k\neq 0} |\hat{\varphi}^{(L)}(2\pi k)|^2\right)^{1/2}.$$

This proposition can also be obtained as a corollary of [41, Th. 4.1], which covers the more general case of oblique projection operators. This recent paper also provides practical formulas for the determination of the asymptotic bound constant when $\varphi$ is specified indirectly in terms of a refinement filter (wavelet transform).

## C. Results and Discussion

The constants $C_{0,L}$ that appear in the asymptotic limit for the interpolation and least squares approximation are very similar, except that the latter one is usually smaller since the origin is excluded from the summation. These constants can be determined explicitly for polynomial splines. Specifically, for a spline quasi-interpolator of order $L$, we have

$$\begin{aligned} C_{0,L}^{\mathrm{QI}} &= \frac{1}{L!}\left(|\hat{\varphi}^{(L)}(0)|^2 + 2\sum_{k=1}^{+\infty}\left|\frac{L!}{(2\pi k)^L}\right|^2\right)^{1/2} \\ &= \sqrt{\left(\frac{m_\varphi^L}{L!}\right)^2 + \frac{|B_{2L}|}{(2L)!}} \end{aligned} \tag{68}$$

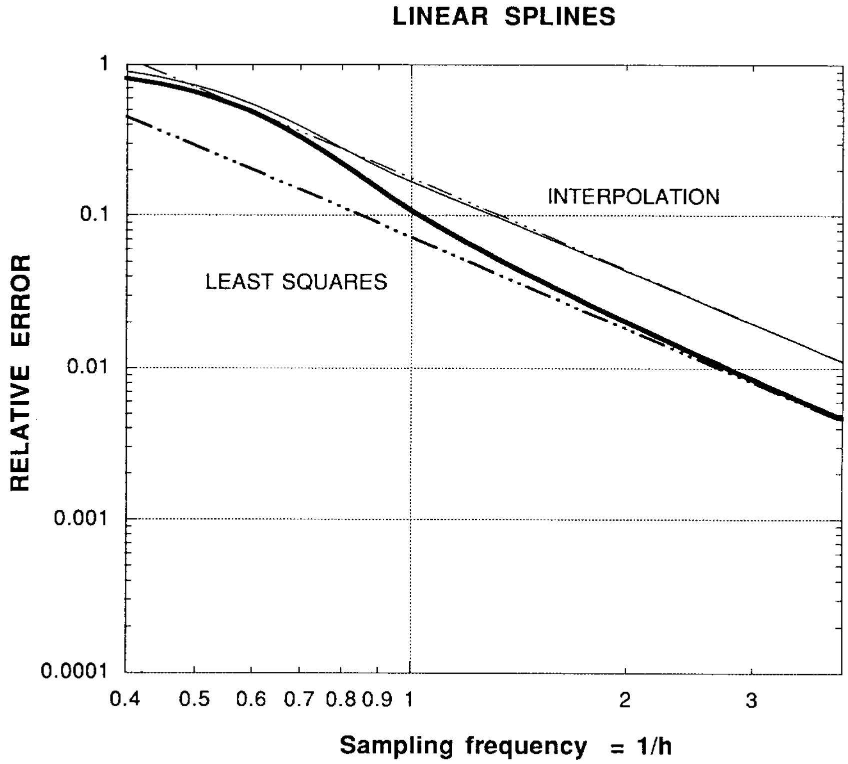


Fig. 5. Piecewise linear least squares approximation (bold) versus interpolation for the function $\Psi_1(x) = -x \cdot e^{-x^2/2}$. Comparison of experimental error curves (solid line) with the asymptotic trend (mixed line) predicted by the theory.

where $|B_{2L}|$ is Bernouilli's number of order $2L$, and where $m_\varphi^L$ is the $L$th moment of the interpolation kernel $\varphi$. For a least squares spline approximation of order $L$, there is no contribution at the origin, and $C_{0,L}^{\rm LS}$ is identical the constant $C_{\varphi,L}^{-}$ introduced earlier [cf. (50)]. These various spline constants are given in Table I for $n = 0, \cdots, 5$. The superscript "int" refers to the cardinal spline interpolator. The last column presents the values for the shortest quasiinterpolants of order $L$; the cubic spline solution is the function defined by (14). For a given order, the general tendency is $C_{0,L}^{\rm LS} < C_{0,L}^{\rm int} < C_{0,L}^{\rm QI}$, which is consistent with our expectation.

These asymptotic error predictions are in excellent agreement with all our experiments. In particular, we have represented these asymptotes on all the graphs presented so far using mixed dashed lines. Some additional comparisons between quasi-interpolation and least squares spline approximations are shown in Figs. 5 and 6. While the asymptotic rates are qualitatively equivalent, i.e., $O(h^L)$, the gap between the two solutions can be substantial. The general shape of the quasi-interpolation and least squares curves is also quite different. The former approaches its asymptote from below, whereas the latter reaches it from above. The least squares error curves typically exhibit a faster decay for intermediate values of $h$, which is a property that is consistent with the general form of the bound in Theorem 4.3.

Interestingly, for splines of even degree $n$, the least squares and cardinal interpolation constants are identical, suggesting that these algorithms are asymptotically equivalent. The reason for this is simply that the odd moments of a symmetric function are zero by construction. We also note that it is possible to construct quasi-interpolants with optimal asymptotic performance by adding the constraint that the $L$th moment be zero as well. One function that satisfies this requirement is the orthogonal Battle-Lemarié spline of order $L$ (cf. [36, Prop. 6]), but there are many other possibilities since the quasi-interpolation constraints are linear. While there may be cases such as these where the two basic schemes are asymptotically equivalent, the least squares procedure will always be superior, especially at coarser sampling rates.

## VI. Conclusion

In this paper, we have presented and illustrated the theory that explains the behavior of the error as a function of the sampling step $h$ for the two main convolution-based signal approximation methods. The performance of these algorithms is essentially determined by the ability of the representation to reproduce polynomials up to a certain degree $n = L - 1$. While the pointwise analysis brings out this connection and explains the general $O(h^L)$ behavior of the error, the more global Fourier and $L_2$-analyses turn out to be more useful in providing the answers to the questions formulated in the introduction. Based on those results, we can now answer these questions.

- *Least Squares versus Interpolation*: Although the (quasi-)interpolation and least squares methods achieve the same optimal rate, the least squares approach is superior. For larger values of $h$, our analysis suggests that the least squares solution roughly behaves like an interpolation with twice the order. For smaller values of $h$, all methods

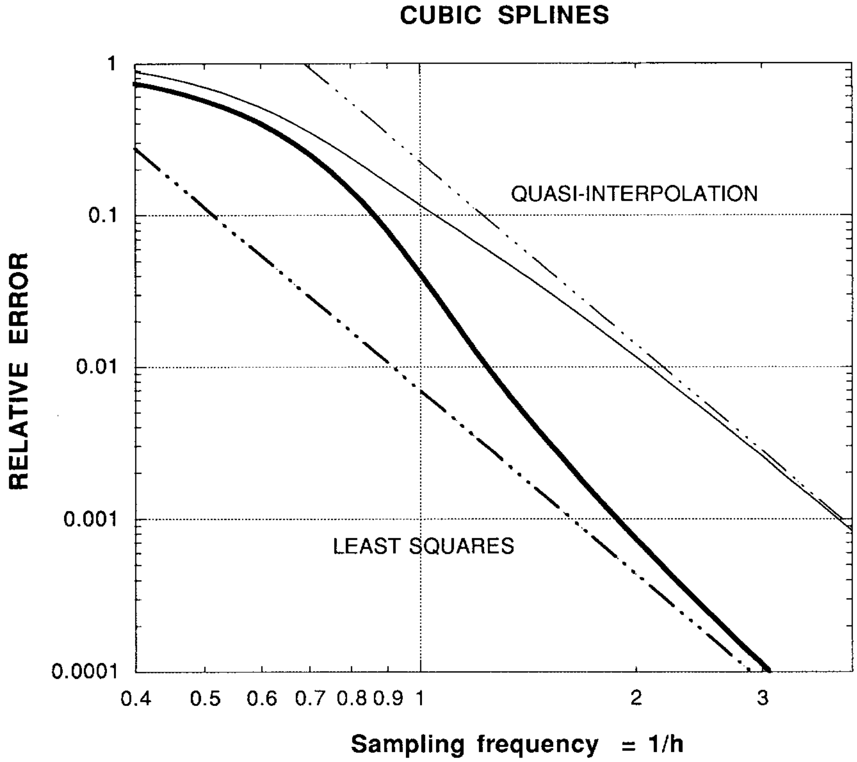


Fig. 6. Cubic spline least squares approximation (bold) versus quasi-interpolation for the function $\Psi_1(x) = -x \cdot e^{-x^2/2}$. Comparison of experimental error curves (solid line) with the asymptotic trend (mixed line) predicted by the theory.

exhibit the characteristic $O(h^L)$ decay, but the asymptotic $L_2$-error is usually smaller (by a known proportion) in the least squares case.

- *Sampling Step and Order Selection*: For a given signal $s(x)$, the sampling step $h$ and the approximation algorithm should be selected such that the approximation error is below a certain tolerance threshold. The simplest design procedure is to use the asymptotic formula $\varepsilon(s,h) = C_0 \cdot h^L \cdot \|s^{(L)}\|$ as $h \to 0$, which has the advantage that the constant $C_0$ has been specified explicitly for all algorithms. However, one should be aware of the fact that this formula usually underestimates the error, especially when the algorithm is asymptotically optimal (i.e., $C_0 = C_0^{\mathrm{LS}}$ is the smallest possible constant for the given representation space). A much safer approach would be to base the design on the general $L_2$ bounds derived in Section IV. While some of these estimates may turn out to be too conservative to be of much practical use, we can at least rely on our improved least squares bound

$$\varepsilon_{\mathrm{LS}}(s,h) \le (K_L)^2 \cdot h^{2L} \cdot \|s^{(2L)}\| + K_L \cdot h^L \cdot \|s^{(L)}\|.$$

In particular, we have derived a practical bound constant for splines ($K_L^- = C_0^{\mathrm{LS}} \cdot A_L^{-1/2}$) that provides a reasonable estimate of the true error curve, even though it is not absolutely safe. In this way, we get a safety factor of at least $A_L^{-1/2} \ge 1$ over the corresponding asymptotic formula.

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**Michael Unser** (M'89–SM'94) was born in Zug, Switzerland, on April 9, 1958. He received the M.S. (summa cum laude) and Ph.D. degrees in electrical engineering in 1981 and 1984, respectively, from the Swiss Federal Institute of Technology, Lausanne, Switzerland.

He has been with the Biomedical Engineering and Instrumentation Program, National Institutes of Health, Bethesda, MD, since 1985, where is presently a Visiting Scientist, head of the Image Processing Group. His research interests include the application of image processing and pattern recognition techniques to various biomedical problems, multiresolution algorithms, wavelet transforms, and the use of splines in signal processing. He is the author of over 60 published journal papers in these areas.

Dr. Unser serves as an Associate Editor for the IEEE SIGNAL PROCESSING LETTERS, and is a member of the Image and Multidimensional Signal Processing Committee of the IEEE Signal Processing Society. He is also on the editorial boards of SIGNAL PROCESSING, PATTERN RECOGNITION, and was a former Associate Editor (from 1992 to 1995) for the IEEE TRANSACTIONS ON IMAGE PROCESSING. He co-organized the 1994 IEEE-EMBS Workshop on *Wavelets in Medicine and Biology* and serves as conference co-chair for SPIE's *Wavelet Applications in Signal and Image Processing*, which has been held annually since 1993. He received the Dommer prize for excellence from the Swiss Federal Institute of Technology in 1981, the research prize of the Brown-Boweri Corporation (Switzerland) for his thesis in 1984, and the IEEE Signal Processing Society's 1995 Best Paper Award (in the IMDSP technical area) for a Transactions paper with A. Aldroubi and M. Eden on B-spline signal processing.

**Ingrid Daubechies** (M'89) received both the Bachelor's and Ph.D. degrees, in 1975 and 1980, respectively, from the Free University, Brussels, Belgium.

She held a research position at the Free University until 1987. From 1987 to 1994, she was a Member of the Technical Staff at AT&T Bell Laboratories, Murray Hill, NJ, during which time she took leaves to spend six months, in 1990, at the University of Michigan, Ann Arbor, and two years from 1991 to 1993, at Rutgers University, Piscataway, NJ. She is now at the Mathematics Department and the Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ. Her research interests focus on the mathematical aspects of time-frequency analysis, particularly wavelets, as well as applications.

Dr. Daubechies was awarded a Leroy P. Steele prize for exposition in 1994 for her book *Ten Lectures on Wavelets* (Philadelphia, PA: SIAM) by the American Mathematical Society. She was also awarded the 1997 Ruth Lyttle Satter Prize. Since 1992, she has been a fellow of the John D. and Catherine T. MacArthur Foundation. She is a member of the American Academy of Arts and Sciences, the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.