# SPLINE WAVELETS WITH FRACTIONAL ORDER OF APPROXIMATION

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We extend Schoenberg's family of polynomial splines with uniform knots to all fractional degrees $\alpha > -\frac{1}{2}$. These splines, which involve linear combinations of the one sided power functions $x_+^\alpha = \max\{0,x\}^\alpha$, are $\alpha$-Hölder continuous for $\alpha \geq 0$. We construct the corresponding B-splines by taking fractional finite differences and provide an explicit characterization in both time and frequency domains. We show that these functions satisfy most of the properties of the traditional B-splines, including the convolution property, and a generalized fractional differentiation rule that involves finite differences only. We characterize the decay of the fractional B-splines which are not compactly supported for non-integral $\alpha$'s. The fractional splines' most notable idiosyncrasies are:

- Fractional splines, as their name suggests, have a fractional order of approximation, a property that does not appear to have been encountered before in approximation theory. Specifically, the approximation error decays like $\|f - P_a f\| = O(a^{\alpha+1})$ as $a \to 0$. We give the asymptotic development of the $L_2$-error and provide quantitative error bounds to substantiate this claim.
- For non-integer $\alpha$, the fractional splines do not satisfy the Strang-Fix theory which states the equivalence between the reproduction of polynomials of degree $n$ and the order of approximation which is one more than the degree ($L$=$n$+1). Specifically, we show that fractional splines reproduce polynomials of degree $n$ with $n-1 < \alpha \leq n$ (or $n = \lceil \alpha \rceil$), while their order of approximation is $\alpha + 1$ (and not $\lceil \alpha \rceil + 1$ as one would expect).
- The fractional B-splines generate valid multiresolution analyses of $L_2$ for $\alpha > -\frac{1}{2}$. However, for $-\frac{1}{2} < \alpha < 0$, their refinement filters $H(z)$ do not have the factor $(1+z)$ which is usually required for the construction of valid wavelet bases. Yet, the filters have the right vanishing property: $H(e^{j\pi}) = 0$, which guarantees the partition of unity condition (almost everywhere, except at the knots).

These functions satisfy all the requirements for a multiresolution analysis of $L_2$ (Riesz bounds, two scale relation) and may therefore be used to build new families of wavelet bases with a continuously-varying order parameter.