GENERALIZED DAUBECHIES WAVELETS

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ABSTRACT

We present a generalization of the Daubechies wavelet family. The context is that of a *non-stationary* multiresolution analysis — i.e., a sequence of embedded approximation spaces generated by scaling functions that are not necessarily dilates of one another. The constraints that we impose on these scaling functions are: (1) orthogonality with respect to translation, (2) reproduction of a given set of exponential polynomials, and (3) minimal support. These design requirements lead to the construction of a general family of compactly-supported, orthonormal wavelet-like bases of L_2. If the exponential parameters are all zero, then one recovers Daubechies wavelets, which are orthogonal to the polynomials of degree $(N-1)$ where N is the order (vanishing-moment property). A fast filterbank implementation of the generalized wavelet transform follows naturally; it is similar to Mallat's algorithm, except that the filters are now scale-dependent. The new transforms offer increased flexibility and are tunable to the spectral characteristics of a wide class of signals.

1. INTRODUCTION

The wavelet transform constitutes a versatile tool for signal analysis and approximation. Part of its success is due to the remarkable property that it sparsely represents functions that are piecewise polynomial, a model that is well-suited to the description of a wide class of "natural signals" that are essentially lowpass. Another important aspect is its computational efficiency.

Our aim in this work is to introduce an extended type of wavelet bases that offer more flexibility and may be suitable for an even larger class of signals; for instance, those that contain some natural resonances and are not necessarily predominantly lowpass. For such signals, it is appropriate to develop an approximation theory that replaces the polynomials of classical wavelet theory by adequately modulated versions of these functions.

We thus focus on exponential polynomials, that is, functions of the form $P(t)e^{\alpha t}$, where $P(t)$ is a polynomial in t and α is a complex exponential parameter. We construct embedded approximation spaces which all contain the exponential polynomials corresponding to a given set of parameters and maximum degrees. This is possible in the generalized framework of non-stationary multiresolution analysis introduced by de Boor, DeVore and Ron [1], where each space is generated by a different scaling function. Here, these are chosen to be orthonormal and minimally supported, and lead to the specification of corresponding wavelet-like basis functions — the generalized Daubechies wavelets [2].

The proposed wavelet transforms are computationally attractive because of their short support property; they can be efficiently implemented using an adaptation of Mallat's algorithm [3] with scale-dependent FIR filters. Interestingly, there is also a relation between these generalized wavelets and non-stationary subdivision schemes [4] developed independently in computer graphics; in particular, a recent construct by Dyn et al. [5] that preserves exponential polynomials. We note, however, that these only correspond to the lowpass synthesis part of the wavelet algorithm and that the underlying basis functions are typically interpolating instead of being orthogonal, as is the case here.

2. NON-STATIONARY MULTIRESOLUTIONS

2.1. Definition

We start by defining the fundamental structure we consider: it is a set of embedded, shift-invariant approximation spaces. The important difference with stationary multiresolutions encountered in classical wavelet theory is that each space can be generated using a different function [1], depending on the scale parameter j. In addition, we impose that these so-called *scaling functions* are orthonormal to their integer translates.

Definition 1 (Non-stationary multiresolution). *Given the scaling functions $(\phi_j)_{j\in\mathbb{Z}}$, the spaces*

$$V_j = \operatorname{span}\left\{\phi_j\left(\frac{t-2^j k}{2^j}\right), k \in \mathbb{Z}\right\} \qquad (1)$$

define a non-stationary multiresolution if, for any $j \in \mathbb{Z}$:

- $\left(\frac{1}{\sqrt{2^j}}\phi_j\left(\frac{t-2^j k}{2^j}\right)\right)_{k\in\mathbb{Z}}$ *is an orthonormal basis of $V_j \cap L_2(\mathbb{R})$;*
- $V_{j+1} \subset V_j$*;*

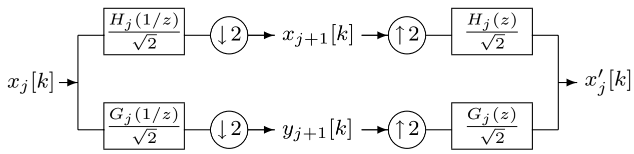


Fig. 1. Non-stationary filter bank.

and in addition $V_j \cap L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $\{0\}$ respectively as $j \to -\infty, +\infty$.

2.2. Basic properties

The embedding of the spaces V_j implies the existence of *scaling filters* $h_j[k]$ such that

$$\phi_{j+1}\left(\frac{t}{2^{j+1}}\right) = \sum_{k\in\mathbb{Z}} h_j[k]\phi_j\left(\frac{t-2^jk}{2^j}\right). \tag{2}$$

Conversely this *scaling relation* implies that any function in V_{j+1} can be expressed as a linear combination of the basis functions of V_j, hence providing a necessary and sufficient condition for $V_{j+1} \subset V_j$.

In the Fourier-domain, the scaling relation (2) writes

$$2\hat{\phi}_{j+1}(2\omega) = H_j\left(e^{i\omega}\right)\hat{\phi}_j(\omega). \tag{3}$$

A consequence of this relation is the *infinite-product formula*, which defines the scaling functions using the scaling filters only, provided $\lim_{j\to-\infty}\hat{\phi}_j\left(2^j\omega\right) = 1$. In this case

$$\hat{\phi}_j(\omega) = \prod_{\ell=1}^{+\infty}\frac{1}{2}H_{j-\ell}\left(e^{i2^{-\ell}\omega}\right). \tag{4}$$

3. MULTIRESOLUTION DECOMPOSITION

The embedded spaces $(V_j)_{j\in\mathbb{Z}}$ as defined above provide a practical structure for multiresolution signal approximation. Given a function $f = f(t) \in L_2(\mathbb{R})$, its best approximation in V_j is given by its orthogonal projection $P_{V_j}f$. Definition 1 ensures that the approximation error $\left\|f - P_{V_j}f\right\|$ tends to zero as $j \to -\infty$.

From a mathematical standpoint, the projection $P_{V_j}f$ can be expressed using simple scalar products with the functions $\left(\frac{1}{\sqrt{2^j}}\phi_j\left(\frac{t-2^jk}{2^j}\right)\right)_{k\in\mathbb{Z}}$, which are orthonormal. More generally, it is also possible to define bi-orthogonal non-stationary multiresolution analyses. This is done in the semi-orthogonal construction of Khalidov et al. using E-splines [6], with the important difference that the obtained scaling functions and wavelets are not necessarily compactly supported anymore.

The following simple property relates continuous-time multiresolution approximation and discrete orthogonal perfect-reconstuction filter banks. Notice that the scaling filters are assumed to be real.

Property 1 (Conjugate mirror filters). *If for any $j \in \mathbb{Z}$ the function $\phi_j(t)$ is orthonormal to its integer shifts, then*

$$H_j(z)H_j(z^{-1}) + H_j(-z)H_j(-z^{-1}) = 4 \tag{5}$$

must hold for all j.

(5) is also known as the *conjugate mirror filter* condition. Equivalently, it states that the discrete filter $h_j[k]/\sqrt{2}$ is orthonormal to its even translates. It makes the filter bank in Fig. 1 orthogonal and ensures perfect reconstruction, provided one sets $G_j(z) = z^{2n-1}H_j(-z^{-1})$. This structure efficiently computes coarser approximations of a function f from a given projection $P_{V_j}f$. More precisely, if $x_j[k]$ denotes the components of the projection at scale j; i.e.,

$$P_{V_j}f(t) = \sum_{k\in\mathbb{Z}} x_j[k]\frac{1}{\sqrt{2^j}}\phi_j\left(\frac{t-2^jk}{2^j}\right), \tag{6}$$

then $x_{j+1}[k]$, which represents $P_{V_{j+1}}f$, is obtained from the filter bank in Fig. 1.

Similarly we may introduce W_{j+1}, the orthogonal complement of V_{j+1} in V_j. Then, $y_{j+1}[k]$ corresponds to $P_{W_{j+1}}f$. These projections on the spaces W_j are expressed using the dyadic shifts of *wavelet functions* $\psi_j(t)$ respectively, which are defined by the following relation:

$$\psi_{j+1}(t/2) = \sum_{k\in\mathbb{Z}} g_j[k]\phi_j(t-k). \tag{7}$$

In the reminder of the paper, we constrain the scaling functions (and thus the wavelets) to be orthonormal and compactly supported. This implies that the scaling filters $H_j(z)$ must be FIR. In this way, the projection operator P_{V_j} is both orthogonal and local, allowing a simple and numerically efficient filter-bank implementation. Since the involved filters depend on the scale j, all that needs to be done is to precompute them up to the desired coarseness level.

4. REPRODUCTION OF EXPONENTIAL POLYNOMIALS

We shall now discuss the conditions under which all the approximation spaces V_j can contain a given set of exponential polynomials.

Definition 2 (Reproduction of exponential polynomials). *Given a vector $\vec{\alpha} \in \mathbb{C}^N$, we denote $\left(\alpha_{(m)}\right)_{m\in[1,N_d]}$ its distinct components and $N_{(m)}$ their respective multiplicities. The non-stationary multiresolution $(V_j)_{j\in\mathbb{Z}}$ reproduces the* *exponential polynomials corresponding to $\vec{\alpha}$ if and only if, for any $P(t)e^{\alpha_{(m)}t}, \deg P(t) < N_{(m)}$, and at any scale j, there exists a sequence $p_j[k]$ such that*

$$P(t)e^{\alpha_{(m)}t} = \sum_{k\in\mathbb{Z}} p_j[k]\phi_j\left(\frac{t-2^jk}{2^j}\right). \tag{8}$$

The following result describes how this property reflects on the scaling functions:

Theorem 1. *Let $(V_j)_{j\in\mathbb{Z}}$ be a non-stationary multiresolution analysis as in Def. 1; it reproduces the exponential polynomials corresponding to $\vec{\alpha}$ if and only if each filter $H_j(z)$ has a zero of order $N_{(m)}$ at $z = -e^{2^j\alpha_{(m)}}$, for $m \in [1, N_d]$.*

(The proof of this theorem as well as other mathematical results of this paper can be derived from [7].)

This means that, in order to reproduce the exponential polynomials corresponding to $\vec{\alpha}$, $H_j(z)$ must be divisible by $R_{2^j\vec{\alpha}}(z)$, where

$$R_{\vec{\alpha}}(z) = \prod_{n=1}^{N}\left(1+e^{\alpha_n}z^{-1}\right). \tag{9}$$

5. GENERALIZED DAUBECHIES WAVELETS

In the compactly supported case, the previous result implies the existence of an FIR filter $Q_j(z)$ such that $H_j(z) = R_{2^j\vec{\alpha}}(z)Q_j(z)$. Our goal is thus to design the shortest possible $Q_j(z)$ for the filters $H_j(z)$ to be orthonormal, which will lead to minimally supported scaling and wavelet functions through the infinite-product formula (4).

We impose that the scaling filters are real, which implies that the elements of $\vec{\alpha}$ should be real or appear in complex-conjugate pairs. We can then perform the standard change of variable $Z = (z + z^{-1})/2$ and get a polynomial $U(Z) \in \mathbb{R}[Z]$ such that

$$U\left(\frac{z+z^{-1}}{2}\right) = R_{\vec{\alpha}}(z)R_{\vec{\alpha}}(z^{-1}). \tag{10}$$

One can thus relate the orthonormality constraint (5) to the following well-known result:

Theorem 2 (Bézout). *Given $U(Z) \in \mathbb{R}[Z]$, there exists a polynomial $V(Z) \in \mathbb{R}[Z]$ such that*

$$U(Z)V(Z) + U(-Z)V(-Z) = 4 \tag{11}$$

if and only if $U(Z)$ has no pair of opposite roots.

The fundamental limitation here is *spectral factorization*. In fact, (5) is equivalent to the formulation in (11) provided the following condition is fulfilled:

Lemma 1 (Riesz). *For any polynomial $V(Z) \in \mathbb{R}[Z]$, there exists a polynomial $Q_j(z) \in \mathbb{R}[z]$ such that*

$$V\left(\frac{z+z^{-1}}{2}\right) = Q_j(z)Q_j(z^{-1}) \tag{12}$$

if and only if $V(Z) \geq 0$ for $Z \in [-1, 1]$.

We are thus looking for a solution of (11) that is positive on $[-1, 1]$ with lowest-possible degree. Therefore, it is useful to give a complete description of the solution set.

Property 2 (Bézout solution set). *If $U(Z)$ has no pair of opposite roots,*

1. *there exists a unique polynomial $V_0(Z) \in \mathbb{R}[Z]$ s. t.*

$$\begin{cases} \deg V_0(Z) < \deg U(Z) \\ U(Z)V_0(Z) + U(-Z)V_0(-Z) = 4. \end{cases} \tag{13}$$

2. *the set of all polynomials $V(Z) \in \mathbb{R}[Z]$ satisfying* (11) *is given by*

$$\left\{V_0(Z) + Z\lambda(Z^2)U(-Z), \lambda(Z) \in \mathbb{R}[Z]\right\}. \tag{14}$$

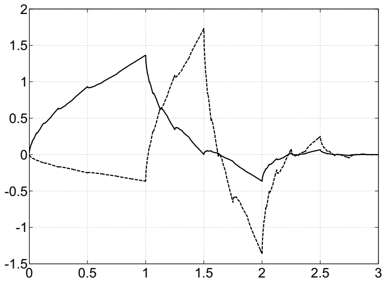
We provide examples to illustrate the previous propositions. Some plots of the resulting scaling functions and wavelets are shown in Fig. 2.

Example 1 (Daubechies case). *Daubechies' traditional construction corresponds to $\vec{\alpha} = \vec{0} \in \mathbb{C}^N$ (Fig. 2 (a)). In this case $V_0(Z)$ is positive on $[-\sqrt{2}, \sqrt{2}]$ [8]. Thus one can always use the shortest Bézout solution to derive the corresponding orthonormalized filter. Notice that we are in fact considering a stationary multiresolution that reproduces regular polynomials up to degree $(N-1)$, and one only needs to derive a single filter which is then the same for all scales.*

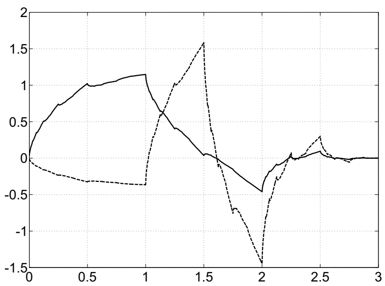
Example 2 (Harmonic case). *If we choose $\vec{\alpha}$ to be composed of a single imaginary conjugate pair $(i\omega_0, -i\omega_0)$ with multiplicity M, we can reproduce modulated polynomials of the form $P(t)\cos(\omega_0 t), \deg P(t) < M$. Here one shows that $V_0(Z) = V_{0\,D}\left(Z/\cos(\omega_0)\right)/\cos(\omega_0)^{2M}$, where the index D refers to the corresponding Daubechies case $\vec{\alpha} = \vec{0} \in \mathbb{C}^{2M}$. Thus, the shortest Bézout solution is guaranteed to be acceptable if $|\cos(\omega_0)| \geq 1/\sqrt{2}$. If the order M is low, this bound is even lower (Fig. 2 (b)).*

One potential problem is that $V_0(Z)$ is generally not guaranteed to be positive definite over $[-1, 1]$. This situation may occur when $|\cos(\omega_0)| < 1/\sqrt{2}$ and the order M is large or when $\vec{\alpha}$ includes distinct frequencies. One then needs to look for higher-order Bézout solutions from (14) (Fig. 2 (c)).

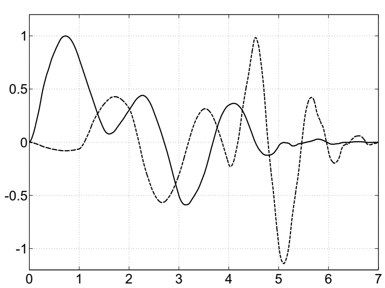
The following result ensures that there always exists a positive Bézout solution, thereby allowing to extend Daubechies' idea of looking for a minimal-length scaling filter:



(a) $\vec{\alpha} = [0 \quad 0]$



(b) $\vec{\alpha} = [i\pi/3 \quad -i\pi/3]$



(c) $\vec{\alpha} = [i7\pi/6 \quad -i7\pi/6 \quad i2\pi/5 \quad -i2\pi/5]$

Fig. 2. Scaling functions (continuous lines) and wavelets (dashed lines) corresponding to different $\vec{\alpha}$ parameters.

Property 3 (Positive Bézout solution). *If $U(Z)$ has no pair of opposite roots, then there exists a polynomial $V(Z) \in \mathbb{R}[Z]$ such that*

$$\begin{cases} V(Z) \geq 0, Z \in [-1, 1] \\ U(Z)V(Z) + U(-Z)V(-Z) = 4. \end{cases} \tag{15}$$

Note that the higher-order solutions are generally non-unique, which calls for additional design constraints.

6. CONVERGENCE

We complete our presentation with some results concerning the convergence of the infinite product formula (4) and the properties of the resulting scaling functions and multiresolution spaces.

Property 4 (Convergence). *If the filters $(H_j(z))_{j \in \mathbb{Z}}$ are always constructed using a Bézout solution with the lowest-possible degree, then for each $j \in \mathbb{Z}$, the infinite product on the right-hand side of* (4) *converges to a function $\hat{\phi}_j(\omega) \in L_2(\mathbb{R})$. Moreover, $\phi_j(t)$ is orthonormal to its integer translates.*

Property 5 ($L_2(\mathbb{R})$-density). *Let $f \in L_2(\mathbb{R})$ be a function whose Fourier transform has a compact support in $[-2^J\pi, 2^J\pi]$. Then,*

$$\lim_{j \to -\infty} \left\| f(t) - P_{V_j} f(t) \right\|^2 = 0 \tag{16}$$

7. CONCLUSION

The multiresolution approximation structure that was presented along with its discrete implementation is expected to be applicable to any type of signals that have their energy concentrated around specific frequencies. Indeed, for these modulated components, exponential polynomials have approximation properties that are equivalent to those of polynomials in the case of base-band signals. The ability to tune the filters to the considered application is a promising feature. In speech processing for instance, one may derive filters that are adapted to the pitch and harmonics of a specific speaker. Many algorithms may be designed to exploit the sparsity of the resulting representation. More generally, the structure that we described can be adjusted to fit the natural modes and responses of a wide class of differential systems. For example, in neurophysiology, it might be well-suited for the analysis of exponential pulses corresponding to different spiking neurons.

8. REFERENCES

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