SIAM J. MATH. ANAL.
Vol. 46, No. 3, pp. 1843–1858

APPROXIMATION PROPERTIES OF SOBOLEV SPLINES AND THE CONSTRUCTION OF COMPACTLY SUPPORTED EQUIVALENTS*

JOHN PAUL WARD† AND MICHAEL UNSER†

Abstract. In this paper, we construct compactly supported radial basis functions that satisfy optimal approximation properties. Error estimates are determined by relating these basis functions to the class of Sobolev splines. Furthermore, we derive new rates for approximation by linear combinations of nonuniform translates of the Sobolev splines. Our results extend previous work as we obtain rates for basis functions of noninteger order, and we address approximation with respect to the L^∞ norm. We also use bandlimited approximation to determine rates for target functions with lower order smoothness.

Key words. radial basis functions, error estimates, bandlimited approximation

AMS subject classifications. 41A25, 41A30, 41A63, 42B10

DOI. 10.1137/130924615

1. Introduction.

The goals of this work are to establish novel approximation results for Sobolev splines on nonuniform grids and to propose compactly supported perturbations that maintain the same approximation properties.

Our model is the differential equation $Tf = g$, where T is a differential operator. A solution to this equation can be obtained by first deriving the Green's function of T. The Green's function (or fundamental solution) G satisfies

$$f(x) = \int_{\mathbb{R}^d} g(y)G(x-y)dy$$

for all f in a smoothness class defined by T. The solution to the differential equation can then be approximated as

$$f(x) \approx \sum_n a_n g(x_n)G(x-x_n),$$

where defining the coefficients a_n is equivalent to specifying a quadrature rule. This general setting of approximating functions f by scattered translates of a Green's function has been studied in [5, 7].

While Green's functions are theoretically well suited to solving approximation problems, they do not always have a nice structure for implementation. In an attempt to reduce the complexity of implementation while maintaining optimal approximation properties, perturbations of Green's functions have been proposed as appropriate basis functions. In the case of approximation on spheres, it has been observed that perturbations maintain approximation properties similar to those of the original Green's function [10]. Furthermore, perturbations may be chosen to have a simpler structure

*Received by the editors June 11, 2013; accepted for publication (in revised form) February 21, 2014; published electronically May 15, 2014. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement n° 267439. This work was also funded in part by the Swiss National Science Foundation under grant 200020-144355.
http://www.siam.org/journals/sima/46-3/92461.html

†Biomedical Imaging Group, Ecole polytechnique fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland (john.ward@epfl.ch, michael.unser@epfl.ch).

than G. In this paper, a perturbation of a Green's function G is defined by convolving it with a finite Borel measure. Specifically, we consider perturbations $G * \mu$, where μ is an invertible finite Borel measure. The Green's functions that we consider are the Sobolev splines, which correspond to operators of the form $(1 - \Delta)^{\gamma/2}$, $\gamma > 0$. We construct compactly supported perturbations as convolutions between thin-plate splines (basis functions corresponding to the operator Δ^m) and finite Borel measures. Importantly, the proposed perturbations exhibit the same approximation rates as the globally supported Green's functions. Furthermore, their compact support makes them an attractive alternative for implementation purposes.

Our construction uses Borel measures that are supported on spheres. The intuition behind the technique is that we are essentially transposing the finite-difference techniques for localizing Green's functions on the real line to the multidimensional setting. This is accomplished by employing spherical averages of thin-plate splines, and the approach effectively translates into a purely radial convolution of one-dimensional profiles. Previous work in this direction is contained in [3, 4, 8]. While the functions that we propose are a special case of those studied in [4], the interest of our approach is that we are able to exploit their particular structure to prove approximation rates that have not been available so far.

In the remainder of this section, we cover the background material that will be needed for our results. In section 2, we recall some definitions of radial basis functions and associated function spaces. There, we also cover the compactly supported basis functions and prove that they are indeed perturbations of the Sobolev splines. We present our main results concerning the approximation rates of the Sobolev splines and compactly supported basis functions in section 3. Finally, in section 4, we discuss a potential application to tomographic image reconstruction.

1.1. Notation. In this work, x denotes a point in the spatial domain $\mathbb{R}^d$, and t denotes a point in the (radial) spatial domain $[0, \infty)$. Likewise, ω and s denote points in transform domains. The variable Φ is used to denote radial basis functions defined on $\mathbb{R}^d$, and the lower case version $\phi : [0, \infty) \to \mathbb{R}$ denotes the function satisfying $\Phi(\cdot) = \phi(|\cdot|)$. The variables Ψ and ψ are used for the compactly supported functions that are defined in section 2.1.

The space of finite complex-valued Borel measures on $\mathbb{R}^d$ is denoted as $M(\mathbb{R}^d)$; cf. [18, section 5.3]. We view the space of integrable functions, $L^1(\mathbb{R}^d)$, as being embedded in $M(\mathbb{R}^d)$. The norm of $\mu \in M(\mathbb{R}^d)$ is defined as its total variation

$$\|\mu\|_{M(\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\,|\mu|\,(x).$$

The Fourier transform of μ is given by

$$\widehat{\mu}(\omega) = \int_{\mathbb{R}^d} e^{-ix\cdot\omega} d\mu(x).$$

The notation $\mathcal{F}\{\mu\}$ is also used to denote the Fourier transform of a measure μ.

The approximation spaces $S_X(\Phi)$, depending on a point set $X \subset \mathbb{R}^d$ and a basis function Φ, are defined in the next section. Approximation rates depend on the smoothness of the function f being approximated, and smoothness is measured in a norm of a Bessel potential space $L^{\gamma,p}$. The error of approximation is denoted as $\mathcal{E}(f, S_X(\Phi))_{L^p}$ or $\mathscr{E}(f, S_X(\Phi))_{L^p}$, and these error measures are defined in section 2 and Proposition 3.5, respectively.

1.2. Preliminaries. The Fourier transform of a radial $L^1(\mathbb{R}^d)$ function $\Phi(x) = \phi(|x|)$ is

$$\widehat{\Phi}(\omega) = (2\pi)^{d/2} \int_0^\infty t^{d-1}\phi(t)(|\omega|\, t)^{-(d-2)/2} J_{(d-2)/2}(|\omega|\, t)dt,$$

where J_ν denotes the order ν Bessel function of the first kind; cf. [18, Theorem 5.26].

The convolution of two measures $\mu, \nu \in M(\mathbb{R}^d)$ is denoted by $\mu * \nu$ and in the Fourier domain satisfies

$$\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}.$$

The space $M(\mathbb{R}^d)$ together with the operation of convolution is a commutative Banach algebra. If it exists, the inverse of $\mu \in M(\mathbb{R}^d)$ is a measure ν for which $\widehat{\mu}\widehat{\nu} = 1$.

A subalgebra of $M(\mathbb{R}^d)$ is the algebra of radial Borel measures $M_r(\mathbb{R}^d)$. A radial measure is one that is invariant with respect to rotations. Related to $M_r(\mathbb{R}^d)$ is the collection of finite Borel measures on $I = [0, \infty)$ with norm

$$\|\mu\|_{M(I)} = \int_0^\infty d\,|\mu|\,(t).$$

The Hankel–Stieltjes transform of a measure $\mu \in M(I)$ is given by

$$\mathcal{H}_d\{\mu\}(s) = 2^{(d-2)/2}\Gamma\,(d/2) \int_0^\infty (st)^{-(d-2)/2} J_{(d-2)/2}(st)d\mu(t).$$

A measurable function ϕ on I that satisfies

$$\int_0^\infty |\phi(t)|\, t^{d-1}dt < \infty$$

is associated with the measure whose Hankel–Stieltjes transform is

$$\int_0^\infty t^{d-1}\phi(t)(st)^{-(d-2)/2} J_{(d-2)/2}(st)dt.$$

The convolution of two measures $\mu, \nu \in M(I)$ is denoted as $\mu \circ \nu$, and this operation satisfies

$$\mathcal{H}_d\{\mu \circ \nu\} = \mathcal{H}_d\{\mu\}\mathcal{H}_d\{\nu\}.$$

With multiplication defined in this way, $M(I)$ is also a Banach algebra. Furthermore, there is an algebraic isometry

$$S : M(I) \to M_r(\mathbb{R}^d) \tag{1.1}$$

satisfying

$$\mathcal{F}\{S\mu\}(\omega) = \mathcal{H}_d\{\mu\}(|\omega|);$$

cf. [13, section 3]. In Figure 1, we illustrate our notation and the relationship between these transforms. The value of this association is that many problems can be simplified by working with $M(I)$ rather than $M(\mathbb{R}^d)$.

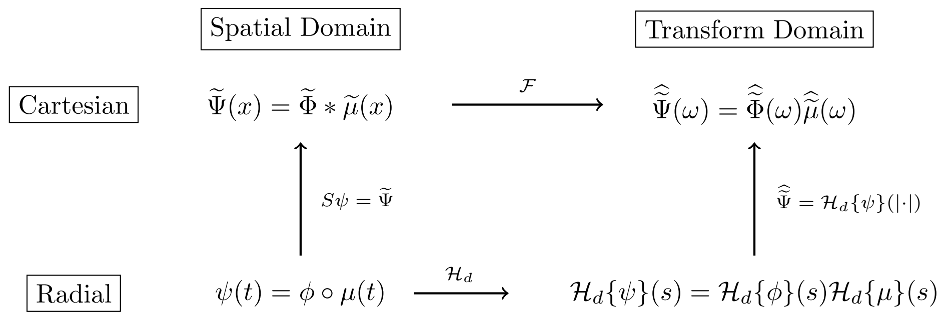


FIG. 1. *Schematic diagram relating the Fourier and Hankel–Stieltjes transforms of radial functions and measures.*

The algebra of measures $M(I)$, with multiplication defined by means of $\mathcal{H}_d$, has been studied in [13]. Here, we recall those results necessary for our purpose.

LEMMA 1.1 (see [13, Lemma 4.3]). *If $\mu \in M(I)$ and $d > 1$, then*

$$\mathcal{H}_d\{\mu\}(\infty) := \lim_{s\to\infty} \mathcal{H}_d\{\mu\}(s)$$

exists and satisfies $\mathcal{H}_d\{\mu\}(\infty) = \mu(\{0\})$.

LEMMA 1.2 (cf. [13, Lemma 5.2]). *If $\mu \in M(I)$, $d > 1$, and $\mathcal{H}_d\{\mu\}(s) \neq 0$ for $s \in I^* = [0,\infty]$, then there exists $\nu \in M(I)$ such that $\mathcal{H}_d\{\mu\}\mathcal{H}_d\{\nu\} = 1$ on I^*.*

Proof. Theorem 4.4 of [13] characterizes the complex homomorphisms of $M(I)$ as point evaluations of Hankel–Stieltjes transforms. Theorem 11.5 of [12] states that an element of a commutative Banach algebra is invertible if and only if the application of every complex homomorphism is a nonzero number. □

The following proposition gives an example of an invertible measure. Its Fourier transform involves the Fourier transform of the Sobolev splines, and we shall need it for deriving approximation rates.

PROPOSITION 1.3. *If $d > 1$ and $\gamma > 0$, then*

$$\frac{1 + |\omega|^\gamma}{(1 + |\omega|^2)^{\gamma/2}}$$

is the Fourier transform of an invertible measure in $M(\mathbb{R}^d)$.

Proof. Notice that we can write

$$\frac{1 + |\omega|^\gamma}{(1 + |\omega|^2)^{\gamma/2}} = \frac{1}{(1 + |\omega|^2)^{\gamma/2}} + \frac{|\omega|^\gamma}{(1 + |\omega|^2)^{\gamma/2}}.$$

The first term is the Fourier transform of the Sobolev spline of order γ, which is an L^1 function; cf. [14, section 5.3.1]. The second term is known to be the Fourier transform of a finite Borel measure; cf. [14, section 5.3.2]. Therefore there is a measure $\mu \in M(\mathbb{R}^d)$ whose Fourier transform is the function above. The invertibilty of μ now follows from Lemma 1.2. □

In order to verify compact support of the perturbations of the Sobolev splines, we use a Paley–Wiener theorem for Hankel–Stieltjes transforms. The result that we need was originally proved in [6], and a more general version can be found in [2, 16]. Below, we state the L^2 version that is sufficient for our purpose.

THEOREM 1.4 (cf. [16, Theorem 1]). *Let f be an even entire function of exponential type 1 such that $(\cdot)^{(d-1)/2} f \in L^2(0,\infty)$ for $d > 1$. Then f can be represented* *by*

$$f(z) = \int_0^1 (tz)^{-(d-2)/2} J_{(d-2)/2}(tz) g(t) dt \quad (z \in \mathbb{C})$$

with $(\cdot)^{(-d+1)/2} g \in L^2(0,1)$. *Conversely, if* f *has this representation and* $(\cdot)^{(-d+1)/2} g \in L^2(0,1)$, *then* f *is an even entire function of exponential type* 1 *such that* $(\cdot)^{(d-1)/2} f \in L^2(0,\infty)$.

2. Radial basis functions. In this paper, we focus on the Sobolev splines and their perturbations. The Sobolev spline Φ_γ is the Green's function for the operator $(1-\Delta)^{\gamma/2}$, where $\gamma > 0$. Therefore, a representation in the Fourier domain is

$$\mathcal{F}\{\Phi_\gamma\}(\omega) = (1 + |\omega|^2)^{-\gamma/2}.$$

A related family of functions is the collection of thin-plate splines. For a positive integer m, the thin-plate spline of order $2m$ on $\mathbb{R}^d$ is given by

$$\Phi_{2m}^{\text{tps}}(x) = \begin{cases} |x|^{2m-d} & \text{for } d \text{ odd}, \\ |x|^{2m-d} \log(|x|) & \text{for } d \text{ even}. \end{cases}$$

In both cases, the generalized Fourier transform is of the form

$$\mathcal{F}\{\Phi_{2m}^{\text{tps}}\}(\omega) = C_{d,m} |\omega|^{-2m};$$

cf. [18, section 8.3].

The approximation spaces $S_X(\Phi)$ associated with a radial basis function $\Phi : \mathbb{R}^d \to \mathbb{R}$ depend on discrete sets X. In particular, $S_X(\Phi)$ consists of all finite linear combinations of the elements of $\{\Phi(\cdot - x) : x \in X\}$ that lie in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. For $1 \le p < \infty$, the L^p error of approximation is

$$\mathcal{E}(f, S_X(\Phi))_{L^p} := \inf_{g \in S_X(\Phi)} \|f - g\|_{L^p}.$$

For $p = \infty$, we consider the error of approximation on compact sets $\Omega \subset \mathbb{R}^d$ and take the supremum over Ω:

$$\mathcal{E}(f, S_X(\Phi))_{L^\infty} := \sup_{\Omega \subset \mathbb{R}^d} \inf_{g \in S_X(\Phi)} \|f - g\|_{L^\infty(\Omega)}.$$

Approximation rates are given in terms of the fill distance h_X of the set X

$$h_X := \sup_{y \in \mathbb{R}^d} \inf_{x \in X} |x - y|.$$

The separation radius q_X for X is

$$q_X := \inf_{x,y \in X; x \ne y} |x - y|,$$

and we restrict our attention to quasi-uniform sets X, i.e., sets X for which h_X/q_X is bounded above by a constant. The smoothness of the functions being approximated is measured by a Bessel potential operator. The Bessel potential spaces $L^{\gamma,p}(\mathbb{R}^d)$ are defined by the norm

$$\|f\|_{L^{\gamma,p}} = \left\| \mathcal{F}^{-1}\left\{ (1 + |\cdot|^2)^{\gamma/2} \widehat{f} \right\} \right\|_{L^p}.$$

If $1 < p < \infty$ and γ is a positive integer, then $L^{\gamma,p}$ is equivalent to a standard Sobolev space [14, section 5.3.3]. However, these spaces are, in general, not equivalent for $p = 1, \infty$ [14, section 5.6].

2.1. Compactly supported radial basis functions. Our construction is closely related to the one presented in [3, 4]. In those papers, the author observes that the thin-plate spline Φ_{2m}^{tps} has a Fourier transform that is analytic on $\mathbb{C}^d\backslash\{0\}$. Therefore, if there is a rotationally invariant measure μ with a Fourier transform that is entire and has a zero of suitable order at the origin, then

$$\widehat{\Phi_{2m}^{\mathrm{tps}} * \mu} = \widehat{\Phi}_{2m}^{\mathrm{tps}}\widehat{\mu}$$

is an entire function of finite exponential type. The Paley–Wiener theorem then implies that the inverse Fourier transform $\Phi_{2m}^{\mathrm{tps}} * \mu$ is compactly supported.

The focus of [3, 4] was a general characterization of measures μ that ensures compact support of the convolution. Here, our purpose is to determine the approximation order of such functions, which is possible by restricting ourselves to the class of measures that produce perturbations of the Sobolev splines Φ_γ upon convolution with a thin-plate spline. This relationship with the Sobolev splines implies that our constructed functions retain the approximation properties of the thin-plate splines, while having the added benefit of compact support.

The measures that we consider are radial Borel measures. Two particular examples of radial measures are the Dirac measure that is concentrated at the origin and the collection of measures that are supported on spheres centered at the origin. Let σ be the standard surface measure of $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Then we define the surface supported measure $\sigma_r \in M_r(\mathbb{R}^d)$ by

$$\sigma_r(E) = \sigma(E \cap \mathbb{S}^{d-1}).$$

Since these measures and the radial basis functions that we convolve them with are radial, we use the isomorphism $S : M(I) \to M_r(\mathbb{R}^d)$ (defined in (1.1)) to work with them in a one-dimensional setting. In particular, notice that $S^{-1}\sigma_r$ is a Dirac measure located at 1. Hence its Hankel–Stieltjes transform is a constant multiple of

$$s^{-(d-2)/2}J_{(d-2)/2}(s);$$

cf. [15, section 8.3]. Such measures play a central role for us; we use linear combinations of them in our construction. The following definition specifies the admissible linear combinations. Also, note that these measures correspond to those of [3, Theorem 3.3], which are produced by taking the spherical average of linear combinations of thin-plate splines.

Definition 2.1. *For each positive integer $m > d/2$, we define the collection of functions $\Lambda(2m)$ to be those $\lambda : [0,\infty) \to \mathbb{R}$ satisfying the following:*

1. *For some finite sequence $0 < r_1 < r_2 < \cdots < r_K \leq 1$ and $\{a_k\}_{k=0}^K \subset \mathbb{R}$,*

$$\lambda(s) = 1 + \sum_{k=1}^{K} a_k (r_k s)^{(2-d)/2} J_{(d-2)/2}(r_k s). \tag{2.1}$$

2. $\lim_{s\to 0} s^{-2m}\lambda(s)$ *exists as a positive real number.*
3. $\lambda(s) > 0$ *for* $s > 0$.

In the next proposition, we show that convolving a thin-plate spline Φ_{2m}^{tps} with a measure $S\mu$ that satisfies $\mathcal{H}_d\{\mu\} \in \Lambda(2m)$ gives a compactly supported function. The odd dimensional case is covered in [4, Theorem 2.4]. In fact the referenced result establishes necessary and sufficient conditions on ν for $\Phi_{2m}^{\mathrm{tps}} * \nu$ to have compact support. Following this result, we show that the resulting function is a perturbation of the order $2m$ Sobolev spline.

PROPOSITION 2.2. *If $\lambda \in \Lambda(2m)$ for $d > 1$, then $(\cdot)^{-2m}\lambda$ is the Hankel–Stieltjes transform of a compactly supported function ψ. Furthermore, the function $S\psi$ is continuous and positive definite on $\mathbb{R}^d$.*

Proof. Let λ be a function in $\Lambda(2m)$ of the form (2.1). Then λ is an even entire function, which can be verified by considering the power series expansion

$$s^{-\nu} J_\nu(s) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(\nu+1+l)l!} \left(\frac{s}{2}\right)^{2l}.$$

Then the second condition of Definition 2.1 implies that $(\cdot)^{-2m}\lambda$ is entire. We also have $(\cdot)^{(d-1)/2-2m}\lambda \in L^2(0,\infty)$, so Theorem 1.4 implies that $(\cdot)^{-2m}\lambda$ is the Hankel–Stieltjes transform of a compactly supported function ψ. Continuity of $S\psi$ follows from the fact that it can be defined as a convolution of the order $2m$ thin-plate spline with a finite Borel measure. Finally, positive-definiteness follows from the positivity of λ. ∎

LEMMA 2.3. *If $\lambda \in \Lambda(2m)$ for $d > 1$, then $\Psi_{2m} := \mathcal{F}^{-1}\{|\cdot|^{-2m}\lambda(|\cdot|)\}$ is a perturbation of the Sobolev spline Φ_{2m}, i.e., there is an invertible $\mu \in M(\mathbb{R}^d)$ such that $\Psi_{2m} = \mu * \Phi_{2m}$.*

Proof. To prove this, we show that $\widehat{\Psi}_{2m}/\widehat{\Phi}_{2m}$ is the Fourier transform of an invertible finite Borel measure. To simplify the notation, we omit the subscripts, and we see that

$$\begin{aligned}\frac{\widehat{\Psi}}{\widehat{\Phi}} &= \frac{\lambda(|\omega|)}{|\omega|^{2m}}(1+|\omega|^2)^m \\ &= \left(\frac{\lambda(|\omega|)}{|\omega|^{2m}}(1+|\omega|^{2m})\right)\left(\frac{(1+|\omega|^2)^m}{(1+|\omega|^{2m})}\right).\end{aligned}$$

The first function can be written as

$$\frac{\lambda(|\omega|)}{|\omega|^{2m}}(1+|\omega|^{2m}) = \frac{\lambda(|\omega|)}{|\omega|^{2m}} + \lambda(|\omega|),$$

which is the Fourier transform of a finite Borel measure. As the function is positive and bounded away from 0, Lemma 1.2 implies that the measure is invertible.

The second function is the Fourier transform of an invertible measure by Proposition 1.3. ∎

We determine measures μ_{2m} whose Fourier transforms are in $\Lambda(2m)$ by considering the series expansion of J_ν. The corresponding compactly supported function is then $\Psi_{2m} = \Phi_{2m}^{\text{tps}} * \mu_{2m}$, where Φ_{2m}^{tps} is an appropriately normalized thin-plate spline. In $\mathbb{R}^2$, a formula for Ψ_{2m} can be found by directly computing the convolution. In odd dimensional spaces, the order of the Bessel function in the kernel of the radial Fourier transform is half of an odd integer. Therefore it has an expression in terms of simple functions, and Ψ_{2m} can be computed by inverting the Fourier transform.

Here, we provide some examples of perturbations of Sobolev splines that are supported on the ball of radius 1 centered at the origin. We use the notation $\psi_{2m}(|\cdot|) = \Psi_{2m}(\cdot)$ to define the radial component of the functions. Also note that the expressions have been normalized by $\psi_{2m}(0) = 1$.

Example 2.4. $\{r_k\} = \{\frac{1}{2}, 1\}$, $\{a_k\} = \{-\frac{4}{3}, \frac{1}{3}\}$, and $d = 2$:

$$\psi_4(t) = \begin{cases} \frac{3}{\ln(2)} t^2(\ln(t) - 1) + 4t^2 + 1, & 0 \le t < \frac{1}{2}, \\ -\frac{1}{\ln(2)} \left(t^2(\ln(t) - 1) + \ln(t) + 1 \right), & \frac{1}{2} \le t < 1, \\ 0, & t \ge 0. \end{cases}$$

Example 2.5. $\{r_k\} = \{\frac{1}{2}, 1\}$, $\{a_k\} = \{-\frac{4}{3}, \frac{1}{3}\}$, and $d = 3$:

$$\psi_4(t) = \begin{cases} \frac{7}{3} t^2 - 3t + 1, & 0 \le t < \frac{1}{2}, \\ -\frac{1}{3} \left(t^2 - 3t + 3 - t^{-1} \right), & \frac{1}{2} \le t < 1, \\ 0, & t \ge 0. \end{cases}$$

Example 2.6. $\{r_k\} = \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$, $\{a_k\} = \left\{-\frac{3}{2}, \frac{3}{5}, -\frac{1}{10}\right\}$, and $d = 3$:

$$\psi_6(t) = \begin{cases} -\frac{333}{10} t^4 + 45t^3 - 18t^2 + 1, & 0 \le t < \frac{1}{3}, \\ \frac{1}{30} \left(216t^4 - 675t^3 + 810t^2 - 450t + 105 - 5t^{-1} \right), & \frac{1}{3} \le t < \frac{2}{3}, \\ -\frac{9}{10} \left(t^4 - 5t^3 + 10t^2 - 10t + 5 - t^{-1} \right), & \frac{2}{3} \le t < 1, \\ 0, & t \ge 0. \end{cases}$$

3. Approximation properties. In this section, we derive approximation rates for the Sobolev splines and their perturbations. The proofs rely on properties of bandlimited approximation in $L^{\gamma,p}$.

3.1. Bandlimited approximation. A bandlimited approximant to an L^p function can be constructed by convolution with an approximate identity. As we are working with radial functions, we require the approximate identity to be of the same form. Let us define K to be a radial Schwartz class function with Fourier transform $\widehat{K}(\omega) = \kappa(|\omega|)$ satisfying the following:

1. κ is nonincreasing on $[0, \infty)$.
2. $\kappa(s) = 1$ for $s \le 1$ and $\kappa(s) = 0$ for $s \ge 2$.

We then define K_σ by $\widehat{K}_\sigma(\omega) = \widehat{K}(\omega/\sigma)$. The following lemmas address properties concerning approximation of L^p functions f by $K_\sigma * f$.

Lemma 3.1. *If $f \in L^{\alpha,p}(\mathbb{R}^d)$ for $\alpha \ge 0$, $\gamma > 0$, $1 \le p \le \infty$, and $d > 1$, then*

$$\|f * K_\sigma\|_{L^{\gamma+\alpha,p}} \le C\sigma^\gamma \|f\|_{L^{\alpha,p}}.$$

Proof. Young's inequality implies that $\|f * K_\sigma\|_{L^{\gamma+\alpha,p}} \le \|f\|_{L^{\alpha,p}} \|K_\sigma\|_{L^{\gamma,1}}$, so we must bound $\|K_\sigma\|_{L^{\gamma,1}}$. By Proposition 1.3,

$$\frac{1 + |\omega|^\gamma}{(1 + |\omega|^2)^{\gamma/2}}$$

is the Fourier transform of an invertible, finite Borel measure. We also have

$$\widehat{K}(\omega/\sigma)(1 + |\omega|^2)^{\gamma/2} = \widehat{K}(\omega/\sigma)(1 + |\omega|^\gamma) \frac{(1 + |\omega|^2)^{\gamma/2}}{(1 + |\omega|^\gamma)}.$$

Therefore

$$\begin{aligned} \|K_\sigma\|_{L^{\gamma,1}} &\le C \left\| \mathcal{F}^{-1}\left\{ \widehat{K}(\cdot/\sigma)(1+|\cdot|^\gamma) \right\} \right\|_{L^1} \\ &\le C\left(\|K_\sigma\|_{L^1} + \left\| \mathcal{F}^{-1}\left\{ \widehat{K}(\cdot/\sigma)\, |\cdot|^\gamma \right\} \right\|_{L^1} \right) \\ &\le C\sigma^\gamma \left\| \mathcal{F}^{-1}\left\{ \widehat{K}(\cdot/\sigma)\, |\cdot/\sigma|^\gamma \right\} \right\|_{L^1} \\ &\le C\sigma^\gamma. \qquad \square \end{aligned}$$

LEMMA 3.2. *If* $f \in L^{\gamma,p}(\mathbb{R}^d)$ *for* $1 \le p \le \infty$, $d > 1$, *and* $\gamma > 0$, *then*

$$\|f - (K_\sigma * f)\|_{L^p} \le C\sigma^{-\gamma} \|f\|_{L^{\gamma,p}} .$$

Proof. First, we have

$$\mathcal{F}\{f - (f * K_\sigma)\} = \widehat{\Phi}_\gamma (1 - \widehat{K}_\sigma)(\widehat{f}/\widehat{\Phi}_\gamma),$$

where Φ_γ is the Sobolev spline of order γ. Young's inequality then implies

$$\|f - (f * K_\sigma)\|_{L^p} \le \|\Phi_\gamma - (K_\sigma * \Phi_\gamma)\|_{L^1} \|f\|_{L^{\gamma,p}} .$$

We write the Fourier transform of the first term as

$$\mathcal{F}\{\Phi_\gamma - (\Phi_\gamma * K_\sigma)\}(\omega) = \sigma^{-\gamma} \frac{1 - \widehat{K}(\omega/\sigma)}{|\omega/\sigma|^\gamma} \frac{|\omega|^\gamma}{(1 + |\omega|^2)^{\gamma/2}}.$$

The second fraction on the right-hand side is the Fourier transform of a finite Borel measure [14, section 5.3.2]. So, it remains to show that

$$\frac{1 - \widehat{K}(\omega)}{|\omega|^\gamma}$$

is the Fourier transform of an L^1 function. To prove this, we decompose the function as

$$\begin{aligned} \frac{1 - \widehat{K}(\omega)}{|\omega|^\gamma} &= \frac{1 - \widehat{K}(\omega)}{|\omega|^\gamma} \frac{(1 + |\omega|^2)^{\gamma/2}}{(1 + |\omega|^2)^{\gamma/2}} \\ &= \frac{1 - \widehat{K}(\omega)}{\widehat{K}(2\omega) + \frac{|\omega|^\gamma}{(1+|\omega|^2)^{\gamma/2}}} \frac{1}{(1 + |\omega|^2)^{\gamma/2}}. \end{aligned}$$

Now,

$$\widehat{\mu}(\omega) := \widehat{K}(2\omega) + \frac{|\omega|^\gamma}{(1 + |\omega|^2)^{\gamma/2}}$$

is the Fourier transform of an invertible finite Borel measure μ by Lemma 1.2. Denote the inverse of μ by ν, i.e., $\widehat{\mu}\widehat{\nu} = 1$. Then $\nu - \nu * K$ is a finite Borel measure, and

$$\frac{1 - \widehat{K}(\omega)}{|\omega|^\gamma}$$

is the Fourier transform of an L^1 function. $\square$

3.2. Approximation by Sobolev splines. Having established the bandlimited approximation results, we now prove approximation rates for the Sobolev splines. Rates are given for approximation spaces $S_X(\Phi_\gamma)$, where X is required to be quasi-uniform and h_X denotes the fill distance of X. L^p approximation rates were covered in [17], and the main result is stated in the following theorem. In the statement of the theorem, we use the notation $\mathcal{D}$ to represent the space of infinitely differentiable, compactly supported functions.

THEOREM 3.3 (see [17, Corollary 1]). *Suppose $f \in L^{2k,p}(\mathbb{R}^d)$ for $1 \le p \le \infty$ and $k > d/2$ is a positive integer. If there is a sequence $f_n \in \mathcal{D}$ converging to f in $L^{2k,p}$, then*

$$\inf_{g \in S_X(\Phi_{2k})} \|f - g\|_{L^p} \le C h_X^{2k} \|f\|_{L^{2k,p}} \quad \text{for } d \text{ odd},$$
$$\inf_{g \in S_X(\Phi_{2k})} \|f - g\|_{L^p} \le C h_X^{2k-1} \|f\|_{L^{2k,p}} \quad \text{for } d \text{ even},$$

where Φ_{2k} is the order $2k$ Sobolev spline.

Proof. This is a special case of the cited corollary, which uses the estimates of [17, section 3.1]. □

Our first approximation results concern the extension of the L^∞ rates to a larger class of functions. The following density property allows us to work with the space of test functions $\mathcal{D}$.

PROPOSITION 3.4. *Let f be in $L^\infty(\mathbb{R}^d)$, and let k be a positive integer. Then $f \in L^{2k,\infty}$ if and only if there is a sequence $\{f_n\}_{n=1}^\infty$ such that*

1. *each function f_n is an element of $\mathcal{D}$;*
2. *$f_n \to f$ uniformly on compact subsets of $\mathbb{R}^d$;*
3. $\sup_n \|f_n\|_{L^{2k,\infty}} < \infty$.

Proof. This result is stated for the standard Sobolev space $W^{k,\infty}$ in [14, section 5.6.2]. A similar statement for $1 < p < \infty$ is proved in [14, section 5.2.1]. Since k is an integer, $(1 - \Delta)^k$ is an ordinary derivative, and the proof can be adapted to $L^{2k,\infty}$ in a straightforward way. □

PROPOSITION 3.5. *Let $f \in L^{\gamma,\infty}$, and let f_n be a sequence of test functions converging to f as described in Proposition 3.4. Then the L^∞ error of approximation $\mathcal{E}(f, S_X(\Phi_\gamma))_{L^\infty}$ is bounded above by*

$$\mathscr{E}(f, S_X(\Phi_\gamma))_{L^\infty} := \limsup_{n\to\infty} \inf_{g \in S_X(\Phi_\gamma)} \|f_n - g\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. For any compact Ω and any $g \in S_X(\Phi_\gamma)$, we have

$$\begin{aligned}\|f - g\|_{L^\infty(\Omega)} &\le \|f - f_n\|_{L^\infty(\Omega)} + \|f_n - g\|_{L^\infty(\Omega)} \\ &\le \|f - f_n\|_{L^\infty(\Omega)} + \|f_n - g\|_{L^\infty(\mathbb{R}^d)},\end{aligned}$$

and taking the infimum over g gives

$$\inf_{g \in S_X(\Phi_\gamma)} \|f - g\|_{L^\infty(\Omega)} \le \|f - f_n\|_{L^\infty(\Omega)} + \inf_{g \in S_X(\Phi_\gamma)} \|f_n - g\|_{L^\infty(\mathbb{R}^d)}.$$

Now, taking the limit supremum of both sides, we have

$$\inf_{g \in S_X(\Phi_\gamma)} \|f - g\|_{L^\infty(\Omega)} \le \limsup_{n\to\infty} \inf_{g \in S_X(\Phi_\gamma)} \|f_n - g\|_{L^\infty(\mathbb{R}^d)},$$

and finally, taking the supremum over Ω verifies the result. □

LEMMA 3.6. *Let k be an integer satisfying $1 \leq d < 2k$, and let Φ_{2k} be the Sobolev spline of order $2k$. If $X \subset \mathbb{R}^d$ is a quasi-uniform set, then*

$$\mathcal{E}(f, S_X(\Phi_{2k}))_{L^\infty} \leq C_f h_X^{2k} \quad \text{for } d \text{ odd},$$
$$\mathcal{E}(f, S_X(\Phi_{2k}))_{L^\infty} \leq C_f h_X^{2k-1} \quad \text{for } d \text{ even}$$

for every $f \in L^{2k,\infty}$.

Proof. Let $f \in L^{2k,\infty}(\mathbb{R}^d)$ for an odd dimension d, and let $\{f_n\}$ be a sequence in $\mathcal{D}$ converging to f as described in Proposition 3.4. Then Theorem 3.3 implies

$$\inf_{g \in S_X(\Phi_{2k})} \|f_n - g\|_{L^\infty(\mathbb{R}^d)} \leq C h_X^{2k} \|f_n\|_{L^{2k,\infty}(\mathbb{R}^d)} .$$

Taking the limit supremum of the left-hand side gives $\mathscr{E}(f, S_X(\Phi_{2k}))_{L^\infty}$. For the right-hand side, we apply Condition 3 of Proposition 3.4. Therefore, Proposition 3.5 establishes the result.

The same proof also works for the even dimensional case. □

Notice that in the even dimensional case, there is a gap between the smoothness assumed on f and the rate of approximation. We would ideally like to have these numbers match, and it is likely that the size of the gap is an artifact of the proof. Despite the fact that the rates are not ideal, we believe that they are the best known rates for the functions considered. Also, these rates will be transferred to the compactly supported basis functions, and we believe that these results surpass the known approximation rates for any positive-definite, compactly supported radial basis function.

The next results concern approximation rates for Sobolev splines Φ_γ, where γ is allowed to be a noninteger real number. We first prove a simultaneous approximation lemma, which will be combined with the bandlimited approximation results to obtain the approximation rates for Φ_γ.

LEMMA 3.7. *Let k be an integer satisfying $1 < d < 2k$, and let $f \in L^{2k+2,p}(\mathbb{R}^d)$ for d odd and $1 \leq p < \infty$. If X is a quasi-uniform subset of $\mathbb{R}^d$, then there exists $g \in S_X(\Phi_{2k+2})$ and a constant $C > 0$ such that*

$$\|f - g\|_{L^p} \leq C h_X^{2k+2} \|f\|_{L^{2k+2,p}} ,$$
$$\|(1-\Delta)(f - g)\|_{L^p} \leq C h_X^{2k} \|f\|_{L^{2k+2,p}} .$$

Proof. We verify the result for $f \in \mathcal{D}$. The general statement then follows because $\mathcal{D}$ is dense in $L^{\gamma,p}$ for $1 \leq p < \infty$ and $\gamma \geq 0$; cf. [1, Theorem 7.63]. In [17], approximants to $f \in \mathcal{D}$ from $S_X(\Phi_{2k+2})$ were constructed as

$$g(x) = \int_{\mathbb{R}^d} ((1-\Delta)^{k+1} f)(y) \mathcal{K}(x,y) dy,$$

where the kernel is of the form

$$\mathcal{K}(x,y) = \mu_y(\Phi_{2k+2}(x - \cdot)).$$

Here, μ_y is a linear combination of Dirac deltas

$$\mu_y = \sum_{\xi \in X} a(\xi, y) \delta_\xi$$

that reproduces polynomials ($\mu_y(P) = P(y)$ for polynomials P up to a fixed degree) and $a(\xi, y)$ is 0 when ξ lies outside of a neighborhood of y. Hence the approximant is of the form

$$g(x) = \sum_{\xi \in X} \Phi_{2k+2}(x - \xi) \int_{\mathbb{R}^d} ((1 - \Delta)^{k+1} f)(y) a(\xi, y) dy.$$

The important point is that we can use the same μ_y to define an approximant to $(1 - \Delta)f$ from $S_X(\Phi_{2k})$:

$$\begin{aligned} \widetilde{g}(x) &= \sum_{\xi \in X} \Phi_{2k}(x - \xi) \int_{\mathbb{R}^d} ((1 - \Delta)^k (1 - \Delta) f)(y) a(\xi, y) dy \\ &= \sum_{\xi \in X} \Phi_{2k}(x - \xi) \int_{\mathbb{R}^d} ((1 - \Delta)^{k+1} f))(y) a(\xi, y) dy. \end{aligned}$$

Hence the coefficients of Φ_{2k+2} in g are the same as the coefficients of Φ_{2k} in $\widetilde{g}$, i.e., $\widetilde{g} = (1 - \Delta)g$. To finish the proof, we note that the error estimates in Theorem 3.3 were obtained using approximants of this form, so

$$\begin{aligned} \|f - g\|_{L^p} &\le C h_X^{2k+2} \|f\|_{L^{2k+2,p}}, \\ \|(1 - \Delta)f - \widetilde{g}\|_{L^p} &\le C h_X^{2k} \|(1 - \Delta)f\|_{L^{2k,p}}. \qquad \square \end{aligned}$$

COROLLARY 3.8. *Let $f \in L^{2k+2,p}(\mathbb{R}^d)$ for even $d < 2k$. Then there exists $g \in S_X(\Phi_{2k+2})$ and a constant $C > 0$ such that*

$$\begin{aligned} \|f - g\|_{L^p} &\le C h_X^{2k+1} \|f\|_{L^{2k+2,p}}, \\ \|(1 - \Delta)(f - g)\|_{L^p} &\le C h_X^{2k-1} \|f\|_{L^{2k+2,p}}. \end{aligned}$$

LEMMA 3.9. *Let $1 \le p < \infty$, and let k be an integer satisfying $1 < d < 2k$. If $2k < \gamma < 2k + 2$, then*

$$\begin{aligned} \mathcal{E}(f, S_X(\Phi_\gamma))_{L^p} &\le C h_X^{\gamma} \|f\|_{L^{\gamma,p}} \quad \text{for } d \text{ odd}, \\ \mathcal{E}(f, S_X(\Phi_\gamma))_{L^p} &\le C h_X^{\gamma-1} \|f\|_{L^{\gamma,p}} \quad \text{for } d \text{ even} \end{aligned}$$

for every $f \in L^{\gamma,p}(\mathbb{R}^d)$, where Φ_γ is the order γ Sobolev spline.

Proof. Fix $f \in L^{\gamma,p}(\mathbb{R}^d)$ for d odd, and define $g \in S_X(\Phi_{2k})$ to be an approximant satisfying

$$\begin{aligned} \left\| (1 - \Delta)^{(\gamma-2k)/2} f - g \right\|_{L^p} &\le C h_X^{2k} \left\| (1 - \Delta)^{(\gamma-2k)/2} f \right\|_{L^{2k,p}}, \\ \left\| \Phi_2 * ((1 - \Delta)^{(\gamma-2k)/2} f - g) \right\|_{L^p} &\le C h_X^{2k+2} \|f * \Phi_{2k+2-\gamma}\|_{L^{2k+2,p}}. \end{aligned}$$

For $E := f - g * \Phi_{\gamma-2k}$, we have

$$\|E\|_{L^p} \le \|K_\sigma * E\|_{L^p} + \|E - K_\sigma * E\|_{L^p}. \tag{3.1}$$

For the first term, we apply the Bernstein inequality of Lemma 3.1, followed by the approximation result Theorem 3.3:

$$\begin{aligned} \|K_\sigma * E\|_{L^p} &= \|K_\sigma * E * \Phi_{2k+2-\gamma}\|_{L^{2k+2-\gamma,p}} \\ &\le C \sigma^{2k+2-\gamma} \|E * \Phi_{2k+2-\gamma}\|_{L^p} \\ &\le C \sigma^{2k+2-\gamma} h_X^{2k+2} \|f * \Phi_{2k+2-\gamma}\|_{L^{2k+2,p}} \\ &\le C \sigma^{2k+2-\gamma} h_X^{2k+2} \|f\|_{L^{\gamma,p}}. \end{aligned}$$

Now, setting $\sigma = h_X^{-1}$ gives

$$\|K_\sigma * E\|_{L^p} \le C h_X^\gamma \|f\|_{L^{\gamma,p}}. \tag{3.2}$$

For the second term, we can use the bandlimited approximation result:

$$\begin{aligned}
\|E - K_\sigma * E\|_{L^p} &\le C\sigma^{2k-\gamma} \|E\|_{L^{\gamma-2k,p}} \\
&= C\sigma^{2k-\gamma} \left\|(1-\Delta)^{(\gamma-2k)/2} f - g\right\|_{L^p} \\
&\le C\sigma^{2k-\gamma} h_X^{2k} \left\|(1-\Delta)^{(\gamma-2k)/2} f\right\|_{L^{2k,p}} \\
&\le C\sigma^{2k-\gamma} h_X^{2k} \|f\|_{L^{\gamma,p}}.
\end{aligned}$$

Again, setting $\sigma = h_X^{-1}$ gives

$$\|E - K_\sigma * E\|_{L^p} \le C h_X^\gamma \|f\|_{L^{\gamma,p}}. \tag{3.3}$$

Using the estimates (3.2) and (3.3) in (3.1) finishes the proof for d odd.

The same proof applies in the even dimensional case. ∎

The last point to address is approximation rates for functions with lower smoothness, i.e., bounds for $\mathcal{E}(f, S_X(\Phi_\gamma))$, where $f \in L^{\alpha,p}$ with $\alpha < \gamma$. We again use the bandlimited approximation results to derive these rates.

THEOREM 3.10. *Let k be an integer and γ a real number satisfying $1 < d < 2k \le \gamma$. Also, let X be a quasi-uniform set in $\mathbb{R}^d$. If $f \in L^{\alpha,p}$ for $\alpha < \gamma$ and $1 \le p < \infty$, then*

$$\begin{aligned}
&\mathcal{E}(f, S_X(\Phi_\gamma))_{L^p} \le C h_X^\alpha \|f\|_{L^{\alpha,p}} \quad \text{for } d \text{ odd}, \\
&\mathcal{E}(f, S_X(\Phi_\gamma))_{L^p} \le C h_X^{\alpha-1} \|f\|_{L^{\alpha,p}} \quad \text{for } d \text{ even},
\end{aligned}$$

where Φ_γ is he Sobolev spline of order γ. A similar result holds when $p = \infty$ if γ is a positive even integer.

Proof. Let d be odd, and let $1 \le p < \infty$. Then, we can write

$$\inf_{g \in S_X(\Phi_\gamma)} \|f - g\|_{L^p} \le \|f - f * K_\sigma\|_{L^p} + \inf_{g \in S_X(\Phi_\gamma)} \|f * K_\sigma - g\|_{L^p}.$$

For the second term, we have

$$\begin{aligned}
\inf_{g \in S_X(\Phi_\gamma)} \|f * K_\sigma - g\|_{L^p} &\le C h_X^\gamma \|f * K_\sigma\|_{L^{\gamma,p}} \\
&\le C h_X^\gamma \sigma^{\gamma-\alpha} \|f\|_{L^{\alpha,p}},
\end{aligned}$$

where the second inequality follows from the Bernstein inequality of Lemma 3.1. A bound of the first term follows from Lemma 3.2:

$$\|f - f * K_\sigma\|_{L^p} \le C\sigma^{-\alpha} \|f\|_{L^{\alpha,p}}.$$

Together, these bounds imply

$$\inf_{g \in S_X(\Phi_\gamma)} \|f - g\|_{L^p} \le C\left(\sigma^{-\alpha} + h_X^\gamma \sigma^{\gamma-\alpha}\right) \|f\|_{L^{\alpha,p}}.$$

Setting σ to be proportional to h_X^{-1}, we obtain the result.

For $p = \infty$, let f_n be a sequence of test functions converging to f as defined in Proposition 3.4. Then by the same argument, we have

$$\inf_{g \in S_X(\Phi_\gamma)} \|f_n - g\|_{L^\infty} \le C h_X^\alpha \|f_n\|_{L^{\alpha,\infty}}$$

and taking the limit supremum gives the result by Proposition 3.5.

The same proof applies when d is even. ∎

3.3. Approximation by perturbations. Perturbations of the Sobolev splines inherit the approximation properties of the previous section. This follows from the fact that any finite Borel measure μ defines a bounded transformation $U : L^p \to L^p$, where $U(f) = f * \mu$. When μ is invertible ($\widehat{\mu}\widehat{\nu} = 1$), the transformation U is invertible, and hence it is an isomorphism.

THEOREM 3.11. *If μ is an invertible element of $M(\mathbb{R}^d)$ for $d \geq 1$ and Φ_γ is the Sobolev spline of order $\gamma > d$, then $\Phi_\gamma * \mu$ satisfies the same approximation properties as Φ_γ.*

Proof. Let $f \in L^p$, $\{x_n\}_{n=1}^N \subset \mathbb{R}^d$, and $\{a_n\}_{n=1}^N \subset \mathbb{R}$. Also let $\nu \in M(\mathbb{R}^d)$ be the inverse of μ. Then $f * \mu$ is an L^p function, and by Young's inequality

$$\begin{aligned}\left\| f - \sum_{n=1}^N a_n \Phi_\gamma(\cdot - x_n) \right\|_{L^p} &= \left\| \nu * \left(\mu * f - \sum_{n=1}^N a_n \mu * \Phi_\gamma(\cdot - x_n) \right) \right\|_{L^p} \\ &\leq C \left\| \mu * f - \sum_{n=1}^N a_n \mu * \Phi_\gamma(\cdot - x_n) \right\|_{L^p}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\| f - \sum_{n=1}^N a_n \mu * \Phi_\gamma(\cdot - x_n) \right\|_{L^p} &= \left\| \mu * \nu * \left(f - \sum_{n=1}^N a_n \mu * \Phi_\gamma(\cdot - x_n) \right) \right\|_{L^p} \\ &\leq C \left\| \nu * f - \sum_{n=1}^N a_n \Phi_\gamma(\cdot - x_n) \right\|_{L^p}. \qquad \square\end{aligned}$$

We apply this result to obtain approximation rates for the compactly supported radial basis functions that were defined in section 2.1.

COROLLARY 3.12. *Let Ψ_{2m} a compactly supported function as defined in Lemma 2.3. If $d > 1$, $1 \leq p \leq \infty$, and $f \in L^{\alpha,p}$ for $\alpha \leq 2m$, then*

$$\begin{aligned}\mathcal{E}(f, S_X(\Psi_{2m}))_{L^p} &\leq C_f h_X^\alpha \quad \textit{for } d \textit{ odd}, \\ \mathcal{E}(f, S_X(\Psi_{2m}))_{L^p} &\leq C_f h_X^{\alpha-1} \quad \textit{for } d \textit{ even}.\end{aligned}$$

4. Application. Here, we present a potential application to tomographic reconstruction. The mathematical principle behind tomographic imaging is the reconstruction of a function from its X-ray transform. Following the notation of [11], the X-ray transform $\mathcal{P}f : \mathbb{S}^{d-1} \times \mathbb{R}^d \to \mathbb{R}$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ is

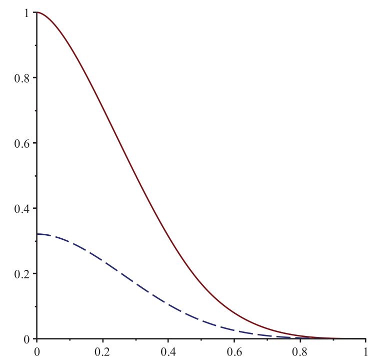
$$\mathcal{P}f(\theta, x) = \int_{\mathbb{R}} f(x + t\theta) \mathrm{d}t$$

for $x \in \{y \in \mathbb{R}^d : y \cdot \theta = 0\}$. When f is a radial function, $\mathcal{P}f$ does not depend on the direction θ, and $\mathcal{P}f(\theta, \cdot)$ depends only on the modulus of the argument. To be precise, let $\Phi(\cdot) = \phi(|\cdot|)$ on $\mathbb{R}^d$, and without loss of generality, set $\theta_0 = (0, \dots, 0, 1)$. Then the transform $\mathcal{P}\Phi$ is defined on

$$\{x = (x_1, x_2, \dots, x_{d-1}, 0) : x_1, \dots, x_{d-1} \in \mathbb{R}\},$$

and its value depends only on the value of $r^2 := x_1^2 + \cdots + x_{d-1}^2$. In particular,

$$\mathcal{P}\Phi(\theta_0, x) = \int_{\mathbb{R}} \phi\left(\sqrt{r^2 + t^2}\right) \mathrm{d}t. \tag{4.1}$$



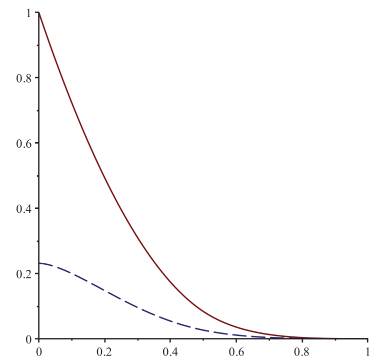


FIG. 2. *Plots of the compactly supported basis functions Ψ_{2m} and their X-ray transforms. The plot on the left exhibits the two-dimensional Ψ_4 along with its X-ray transform (dashed line), while the plot on the right exhibits the three-dimensional Ψ_4 along with its X-ray transform (dashed line).*

As in [9], we model a tomographic image using a linear combination of translates of a radial function Φ, so the X-ray transform of the image is modeled as a linear combination of the translates of the X-ray transform of Φ. Using (4.1), we compute the X-ray transform of the compactly supported basis functions Ψ_{2m} on $\mathbb{R}^d$ to obtain the basis functions in the transform domain. An iterative procedure can then be used to determine the coefficients defining the approximant.

Notice that the radial nature of the basis functions makes this construction user-friendly, since the X-ray transform satisfies directional independence. In [9], the author proposes using generalized Kaiser–Bessel window functions, which are also radial and compactly supported; however, the approximation properties of these functions seem to be unknown. Therefore, we propose using the compactly supported functions described in section 2.1, as they satisfy nearly optimal approximation rates.

4.1. Explicit formulas and plots.

Example 4.1. The X-ray transform of the function from Example 2.4 is

$$\mathcal{P}\Psi_4(\theta_0, x) = \begin{cases} \zeta(r) - \frac{1}{2}\zeta(2r), & 0 \le r < \frac{1}{2}, \\ \zeta(r), & \frac{1}{2} \le r < 1, \\ 0, & r > 1, \end{cases}$$

where

$$\zeta(r) = \frac{1}{9\ln(2)} \left(\left(4 + 11r^2\right) \sqrt{1 - r^2} - \left(9r + 6r^3\right) \arctan\left(\frac{\sqrt{1-r^2}}{r} \right) \right);$$

cf. Figure 2.

Example 4.2. The X-ray transform of the function from Example 2.5 is

$$\mathcal{P}\Psi_4(\theta_0, x) = \begin{cases} \zeta(r) - \zeta(2r), & 0 \le r < \frac{1}{2}, \\ \zeta(r), & \frac{1}{2} \le r < 1, \\ 0, & r > 1, \end{cases}$$

where

$$\zeta(r) = \left(\frac{1}{3} + \frac{r^2}{2}\right) \ln\left(\frac{\sqrt{1-r^2}+1}{r} \right) - \left(\frac{11}{18} + \frac{2r^2}{9}\right) \sqrt{1-r^2};$$

cf. Figure 2.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Pure Appl. Math. 65, Academic Press, New York, 1975.

[2] N. B. Andersen, *On Fourier and Hankel sampling*, Austral. Math. Soc. Gaz., 33 (2006), pp. 44–46.

[3] B. J. C. Baxter, *On spherical averages of radial basis functions*, Found. Comput. Math., 8 (2008), pp. 395–407.

[4] B. J. C. Baxter, *On kernel engineering via Paley–Wiener*, Calcolo, 48 (2011), pp. 21–31.

[5] R. A. DeVore and A. Ron, *Approximation using scattered shifts of a multivariate function*, Trans. Amer. Math. Soc., 362 (2010), pp. 6205–6229.

[6] J. L. Griffith, *Hankel transforms of functions zero outside a finite interval*, J. Proc. Roy. Soc. New South Wales, 89 (1955), pp. 109–115.

[7] T. Hangelbroek, *Error estimates for thin plate spline approximation in the disk*, Constr. Approx., 28 (2008), pp. 27–59.

[8] I. R. H. Jackson, *Convergence properties of radial basis functions*, Constr. Approx., 4 (1988), pp. 243–264.

[9] R. M. Lewitt, *Multidimensional digital image representations using generalized Kaiser–Bessel window functions*, J. Opt. Soc. Amer. A, 7 (1990), pp. 1834–1846.

[10] H. N. Mhaskar, F. J. Narcowich, J. Prestin, and J. D. Ward, *L^p Bernstein estimates and approximation by spherical basis functions*, Math. Comp., 79 (2010), pp. 1647–1679.

[11] F. Natterer, *The Mathematics of Computerized Tomography*, B. G. Teubner, Stuttgart, 1986.

[12] W. Rudin, *Functional Analysis*, Internat. Seri. Pure Appl. Math., 2nd ed., McGraw-Hill, New York, 1991.

[13] A. Schwartz, *The structure of the algebra of Hankel transforms and the algebra of Hankel–Stieltjes transforms*, Canad. J. Math., 23 (1971), pp. 236–246.

[14] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser. 30, Princeton University Press, Princeton, NJ, 1970.

[15] E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Ser. 43, Princeton University Press, Princeton, NJ, 1993.

[16] K. R. Unni, *Hankel transforms and entire functions*, Bull. Amer. Math. Soc., 71 (1965), pp. 511–513.

[17] J. P. Ward, *L^p error estimates for approximation by Sobolev splines and Wendland functions on $\mathbb{R}^d$*, Adv. Comput. Math., 38 (2013), pp. 873–889.

[18] H. Wendland, *Scattered data approximation*, vol. 17 of Cambridge Monogr. Appl. Comput. Math., Cambridge University Press, Cambridge, 2005.