From Sample Probabilities to Random Processes

Life as a (somewhat frustrated) measure non-theorist

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Random processes

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ALSO: want to compute global probabilities for the entire process.

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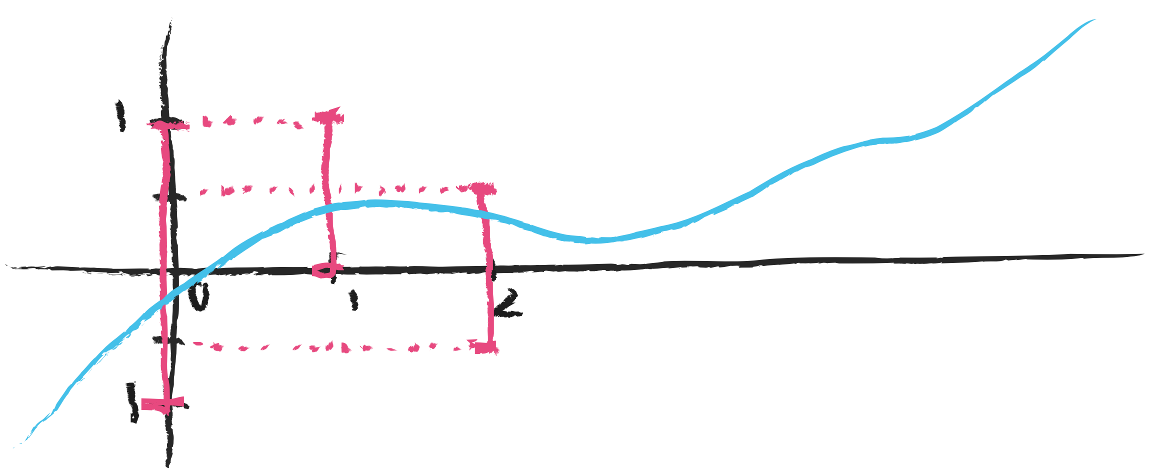
$\boxed{\text{Example:}}$ $T_1 = \{0, 1, 2\}$ $A = [-1,1] \times [0,1] \times [-1/2, 1/2]$



$\pi_{T_1}^{-1} A$ = set of all functions that satisfy

$f(0) \in [-1,1]$,

$f(1) \in [0,1]$,

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Precise formulation of the problem:

We have a finitely additive set function

on the algebra $\mathcal{A} = \bigcup_{\substack{T_1 \subset T \\ \text{finite}}} \pi_{T_1}^{-1} \mathcal{G}_{T_1}$

Borel σ-algebra

defined as

$$\boxed{\mu(A) = \mu_{T_1}(\pi_{T_1} A) \quad \text{for} \quad A \in \pi_{T_1}^{-1} \mathcal{G}_{T_1}.}$$

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Kolmogorov's Extension Theorem (Kolmogorov, 1933):

for $\mathcal{G} = \sigma(\mathcal{A})$:

Yes (if the finite measures are consistent).

SMALL PRINT

μ_{T_i}'s must possess an approximating compact class, which all Borel measures on $\mathbb{R}^{T_i}$, T_i finite, do possess.

Consistency : Given $T_2 \subset T_1 \subset T$ with $\# T_1 < \infty$,

$$\boxed{\mu_{T_2}(A) = \mu_{T_1}(\pi^{-1}_{T_1 \to T_2} A) \qquad \forall A \in \mathcal{G}_{T_2}}$$

where $\pi_{T_1 \to T_2} : \mathbb{R}^{T_1} \to \mathbb{R}^{T_2}$ is the natural projection $\mathbb{R}^{T_1} \to \mathbb{R}^{T_2}$.

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sets in $\sigma(\mathcal{A})$ are countably determined

(σ - combinations of finitely determined cylinder sets) ...

Many interesting sets are left out
(cannot compute their probability) :

e.g. $\{ f : |f| < a \}$ $\quad C := \{ f$ continuous on $\mathbb{R} \}$

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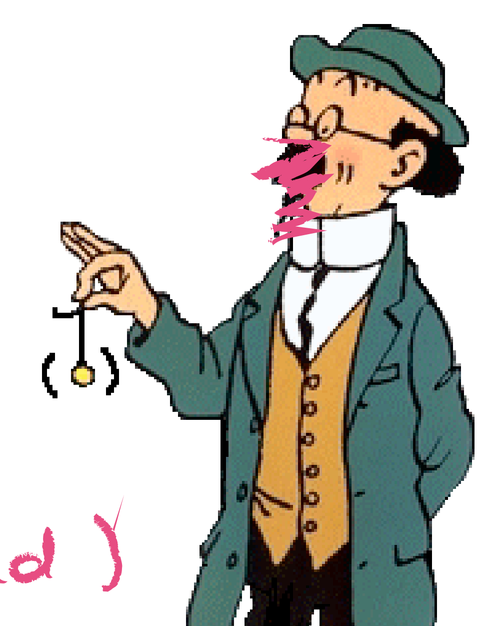

(remove the beard)

Measures on linear spaces

Random processes on linear spaces

E : linear space ;

f : random element of E^*,
the algebraic dual of E
(= random linear functional on E) ;

μ : measure on σ-algebra of Borel cylinder
sets in E^* associated with f.

notation: $f(e_i) \equiv \langle e_i, f \rangle \rightsquigarrow$ R.V.

$\boxed{\text{How to characterize } \mu\text{ ?}}$

RECALL:

Given

any lin. indep. vector $E_1 = (e_1, \ldots, e_n) \in E^n$

$\Downarrow$

Define

Projection $\pi_1 : E^* \longrightarrow \mathbb{R}^n : f \longmapsto (f(e_1), \ldots, f(e_n))$

For $\Delta \in \mathrm{Borel}(\mathbb{R}^n)$,

$\pi^{-1}\Delta$ is a Borel cylinder set.

Consistency for linear processes: Given

$E_1 = (e_1, \dots, e_n) \in E^n$ and

$\tilde{E}_1 = (\tilde{e}_1, \dots, \tilde{e}_m) = M E_1$,

$M \to$ matrix $\in \mathbb{R}^{n \times m}$

$$\boxed{\mu_{\tilde{E}_1}(A) = \mu_{E_1}(M^{-1}A) \quad \forall A \in \mathrm{Borel}(\mathbb{R}^m)}$$

Equivalently:

$$f\left(\sum a_i e_i\right) = \sum \alpha_i \overbrace{f(e_i)}^{\text{R.V.}}$$

$=$ in distribution

Zorn's lemma ($\equiv$ Axiom of Choice) $\Rightarrow$ E has a (Hamel) basis T

f is uniquely determined by $f(t)$, $t \in T$

Problem is reduced to Kolmogorov's thm.

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Enter Generalized Random Processes

of Gel'fand & Co.

Generalized Random Process:

Family of R.V.s $\boxed{\{f(e) \mid e \in E\}}$

where E is a topological vector space.

- consistency defined as before;
- New notion: Continuity:

$$\boxed{\left.\begin{array}{l} e_{1,k} \to e_1 \\ \vdots \\ e_{n,k} \to e_n \end{array}\right\} \Rightarrow \left(f(e_{1,k}), \ldots, f(e_{n,k})\right) \longrightarrow \left(f(e_1), \ldots, f(e_n)\right)}$$

in E

in distribution

$\mu_{E_{1,k}} \longrightarrow \mu_{G_1}$

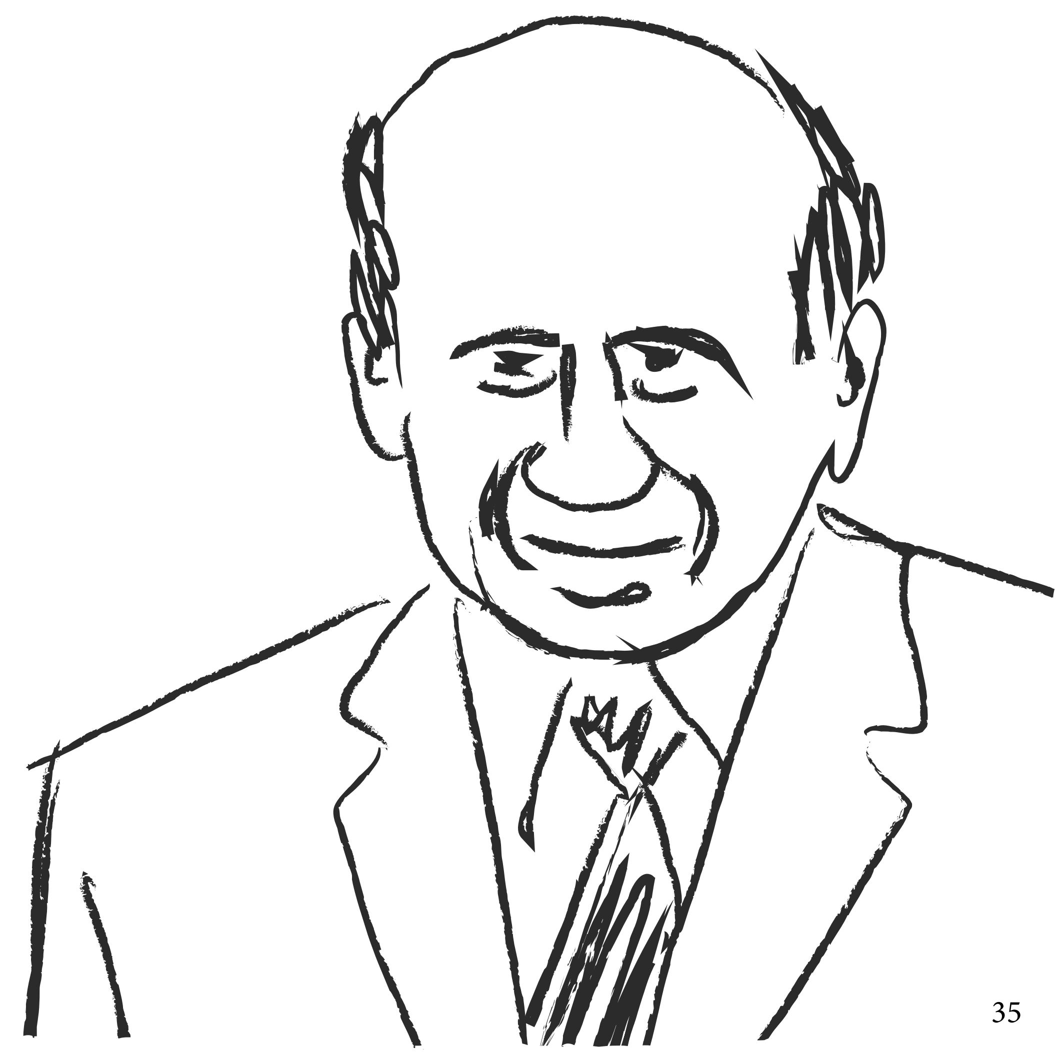
Naturally, if μ is a measure on E' (= continuous dual), consistency and continuity are satisfied.
① ②

Q: When are ①,② sufficient for extending a cylinder measure on E' to a σ-additive one?

(RECALL: consistency alone was sufficient to have a measure on E^*.)

Minlos (1958) — proof of Gel'fand's conjecture:

For finite-dim. measures w/ 1, 2 to uniquely extend to a measure on E', it is sufficient that E be nuclear.

Examples of nuclear spaces:

$\mathcal{S}$: Schwartz space

$\to$ random processes in $\mathcal{S}'$

$\mathcal{D}$: space of compactly supp. test functions

$\to$ random distributions

characteristic functionals:

one way to define/construct finite dim. measures fulfilling 1, 2.

$\boxed{\text{NOT TODAY ...}}$

QUESTIONS ?

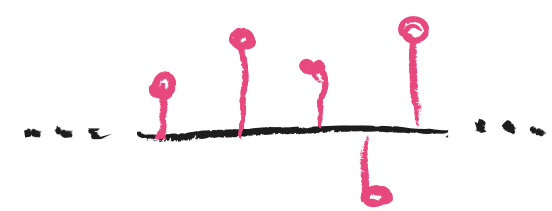
Note on notation:

$$A^B = \text{set of all functions } B \to A$$

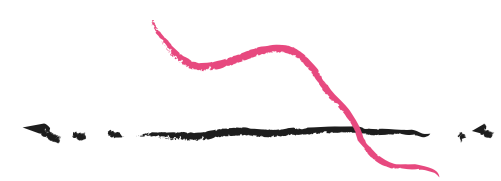
sets

Examples:

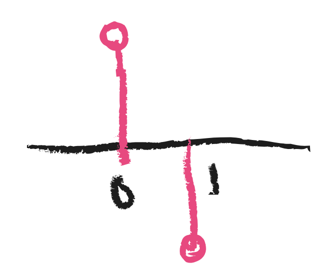
$\mathbb{R}^{\mathbb{Z}}$ = set of all real sequences



$\mathbb{R}^{\mathbb{R}}$ = set of all functions $\mathbb{R} \to \mathbb{R}$



$\mathbb{R}^{\{0,1\}}$ = set of all functions $\{0,1\} \to \mathbb{R}$



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$\mathfrak{S}$: set system $\subset \mathcal{P}(X)$

closed under $\cup$ $\cap$ $\setminus$ $\rightarrow$ algebra

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$\mu : \mathfrak{S} \longrightarrow [0,1]$ isotone, $\mu(\emptyset) = 0$

pairwise disjoint $\mathcal{A} \rightarrow \mu\left(\bigcup_{A \in \mathcal{A}} A\right) = \sum_{A \in \mathcal{A}} \mu(A)$

$\mathcal{A}$ finite $\longrightarrow$ additivity

$\mathcal{A}$ countable $\longrightarrow$ σ-additivity