# Self-Similar Vector Fields

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# Motivation

- Self-similar stochastic models (FBM, FSM, etc.) have applications in image processing and elsewhere
- Key property: Invariances
- Vector-field imaging modalities (flow-sensitive MRI, Doppler ultrasound, etc.) are becoming common-place
- Idea: **Vector** stochastic models **and** data-processing schemes **based on invariances**

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- First, an indicative characterization of FBM

# FBM Revisited

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Gaussian innovation

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spatially-independent

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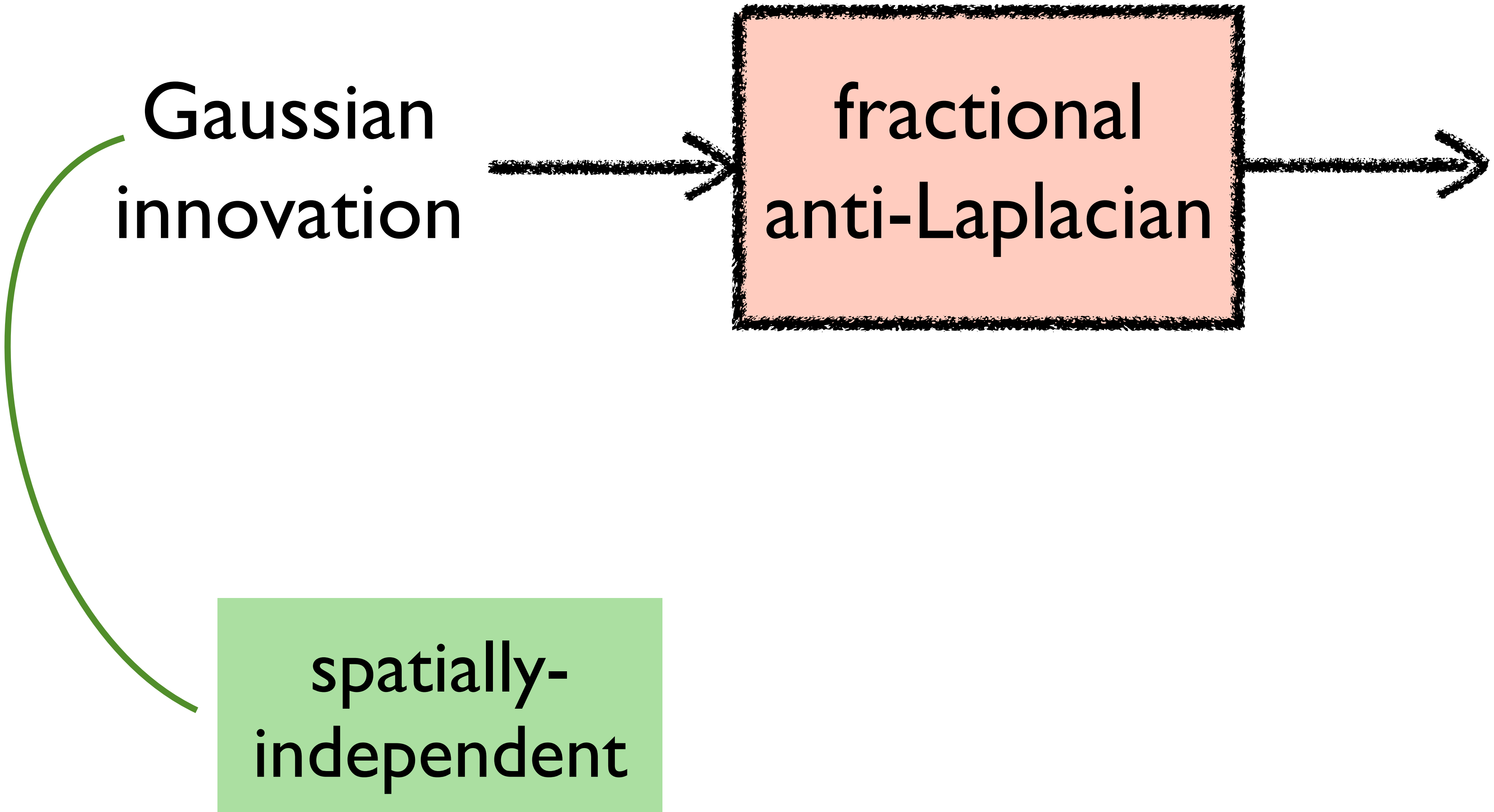

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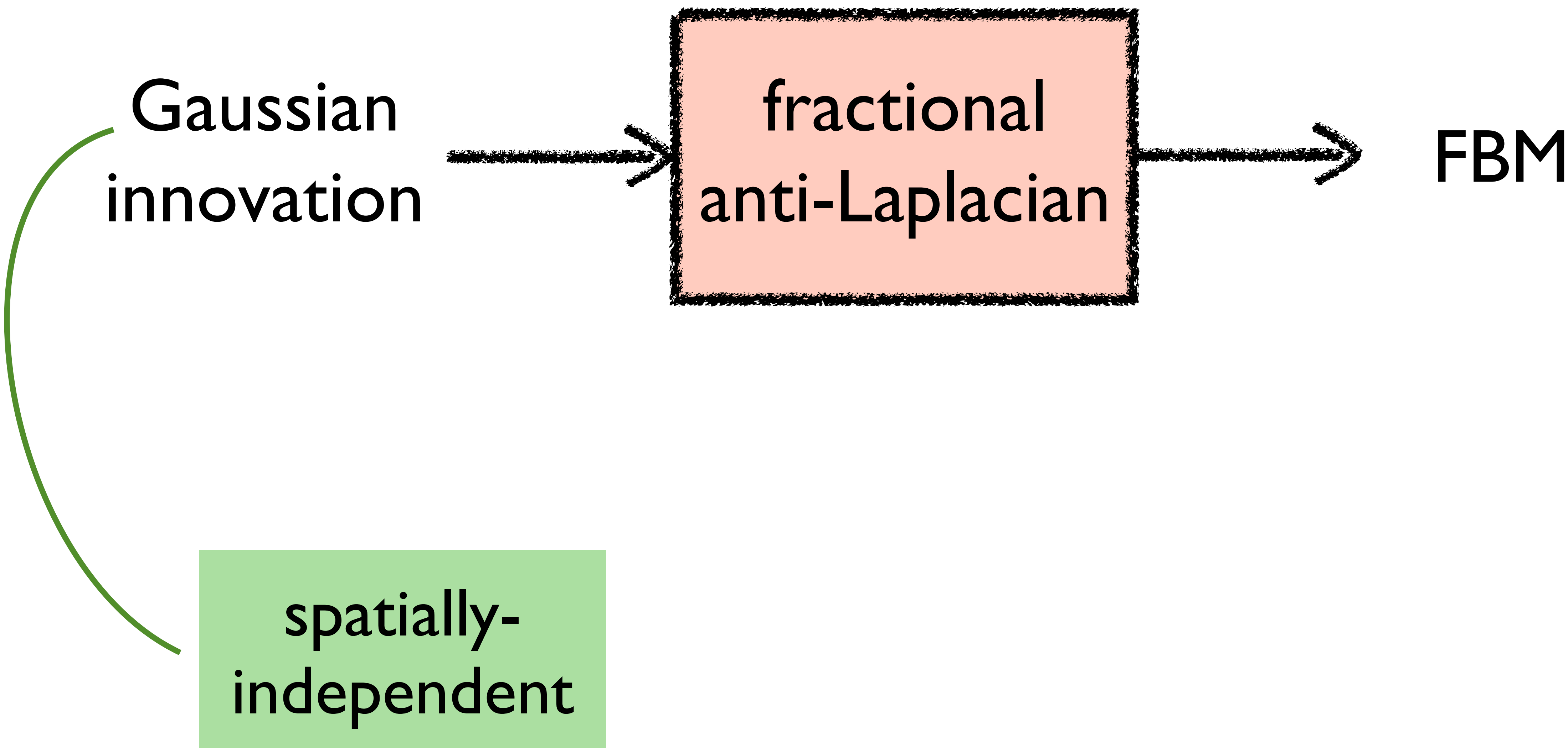

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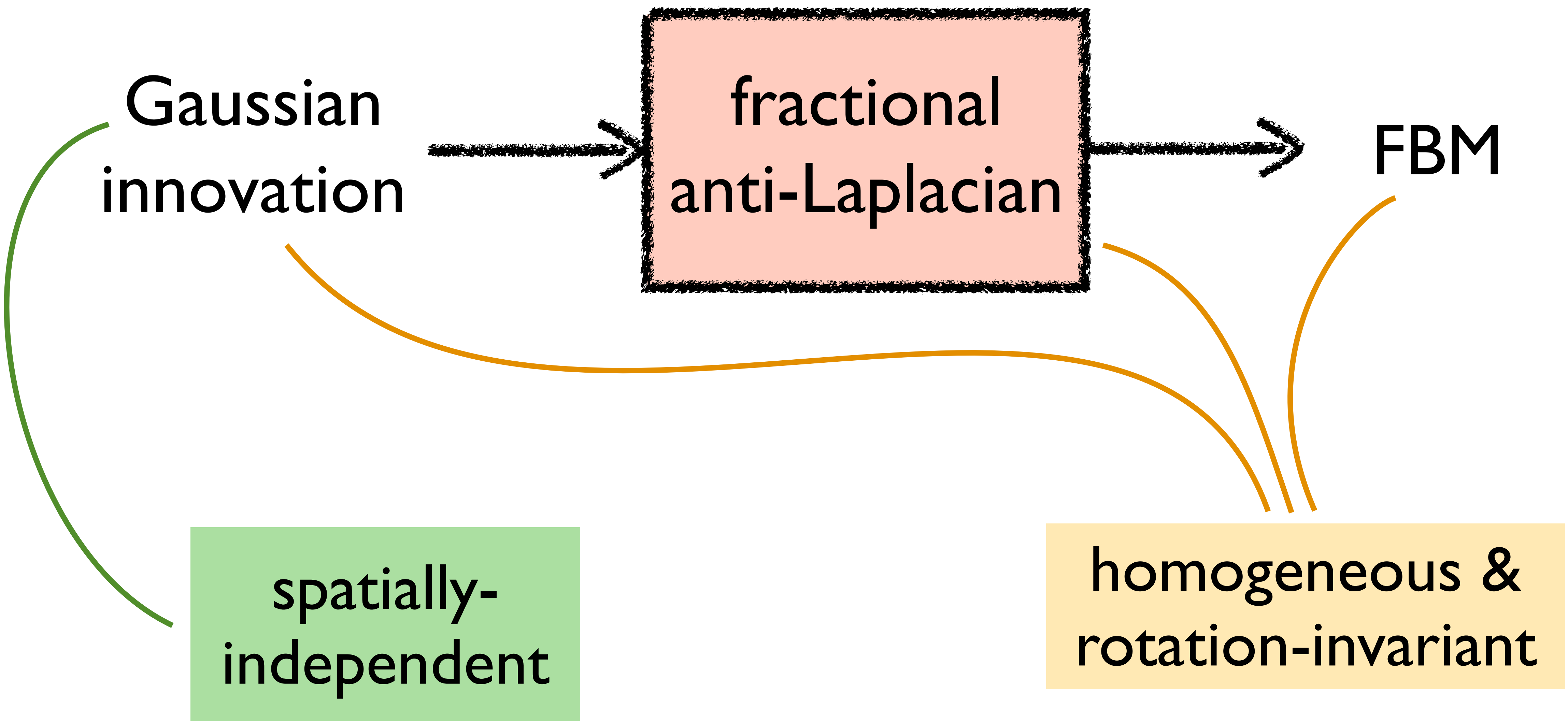
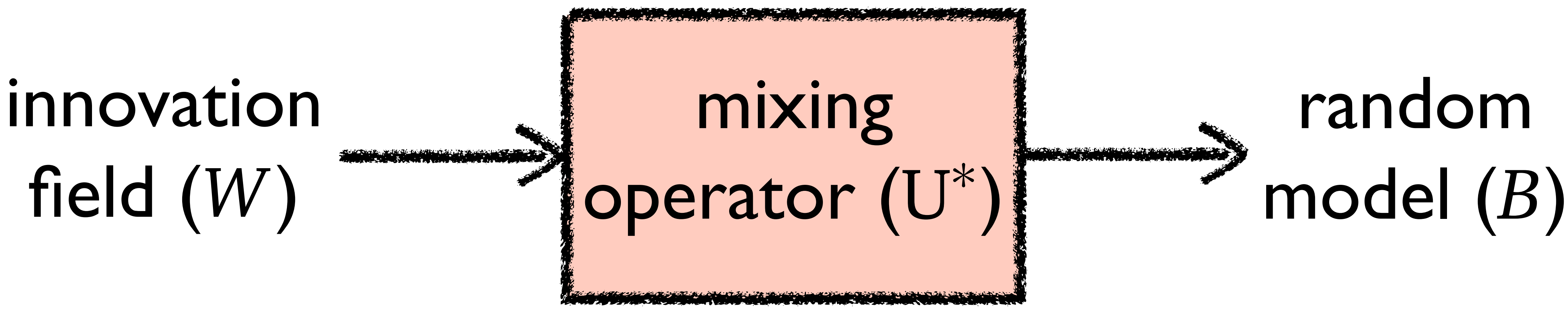
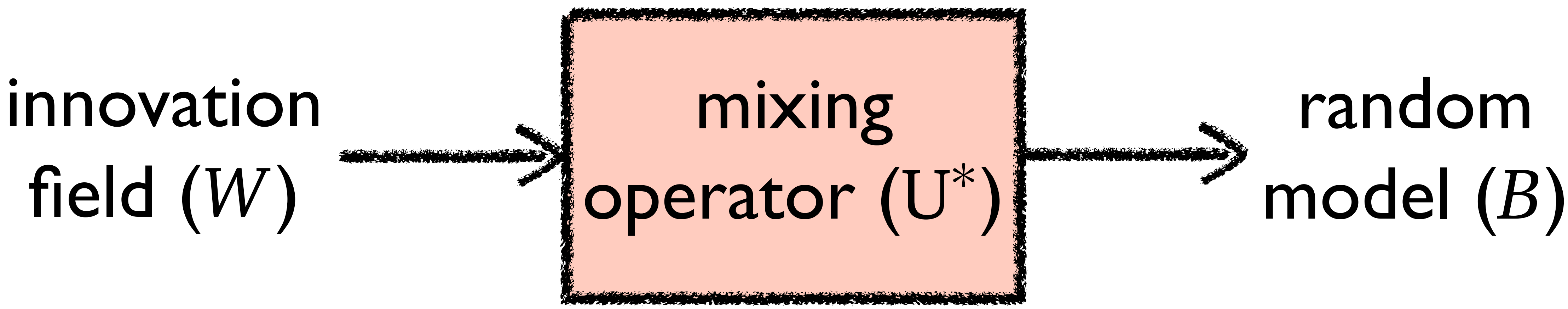


# Innovation Modelling (I)




1. Pick an innovation field
2. Pick an operator
3. Get a random model

# Innovation Modelling (1)

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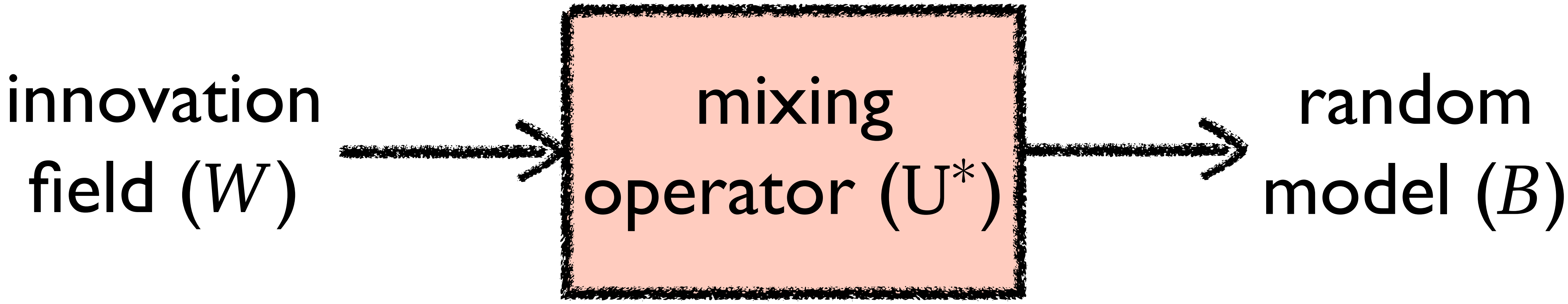
mathematical sense?

# Innovation Modelling (2)



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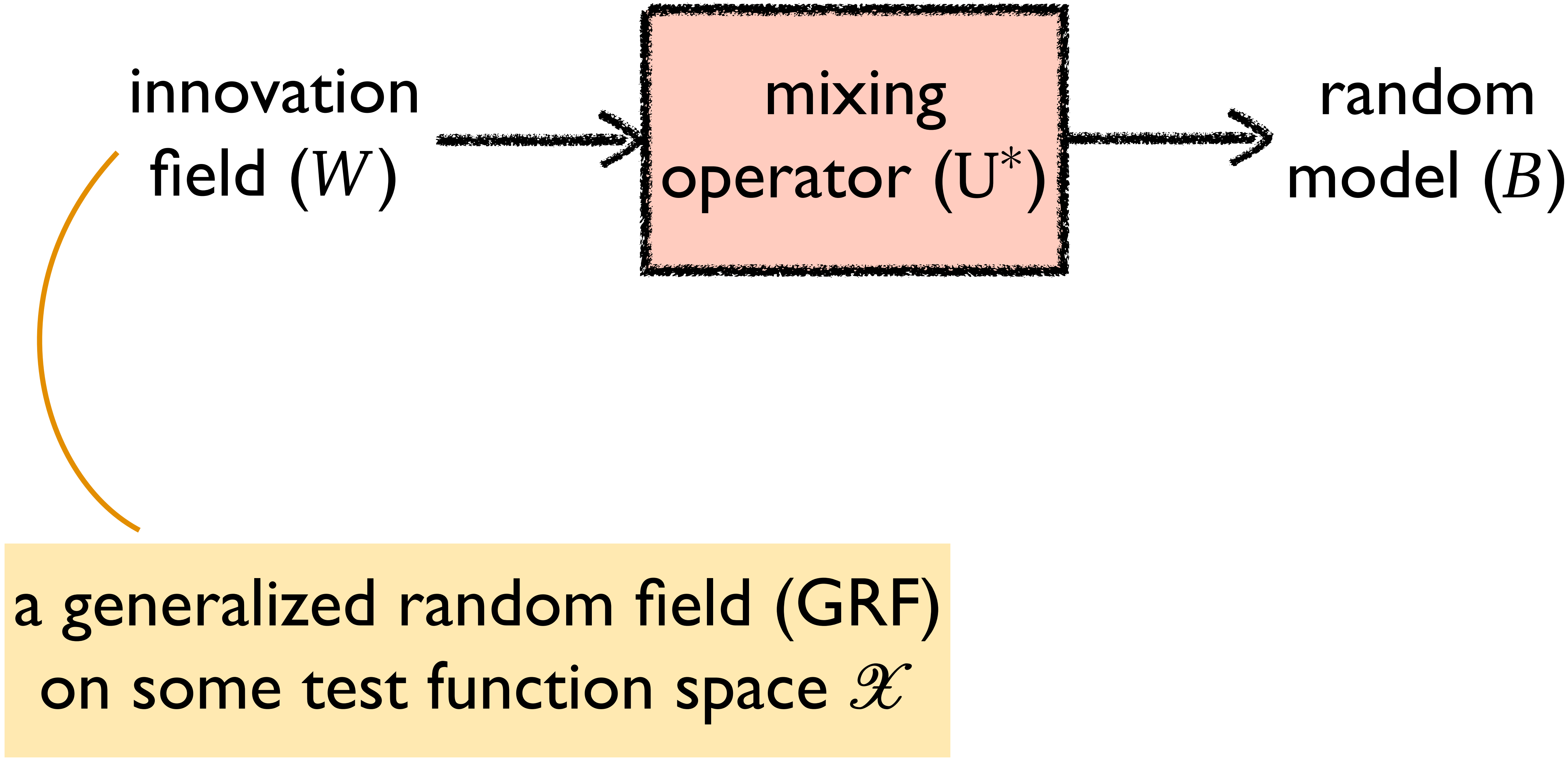

# Innovation Modelling (2)

innovation field ($W$) → mixing operator ($\mathrm{U}^*$) → random model ($B$)

a continuous linear operator $\mathscr{X}' \longrightarrow \mathscr{E}'$, where $\mathscr{E}$ is nuclear

a generalized random field (GRF) on some test function space $\mathscr{X}$

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their products, subspaces, quotients, etc.

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$$\langle \phi, W \rangle, \quad \phi \in \mathscr{X} \qquad\qquad \langle \phi, B \rangle, \quad \phi \in \mathscr{E}$$

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- **Characterized by** characteristic functionals:

$$\widehat{\mathscr{P}_B}(\phi) = \mathbb{E}\left\{ \mathrm{e}^{\mathrm{i}\langle \phi, B \rangle} \right\} = \int_{\mathscr{E}'} \mathrm{e}^{\mathrm{i}\langle \phi, b \rangle} \, \mathscr{P}_B(\mathrm{d}b)$$

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**fin.-additive** cylinder set measure (CSM)

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**fin.-additive** cylinder set measure (CSM)

CSM extends to count.-additive measure (sample path interpretation) if space is nuclear (Minlos)

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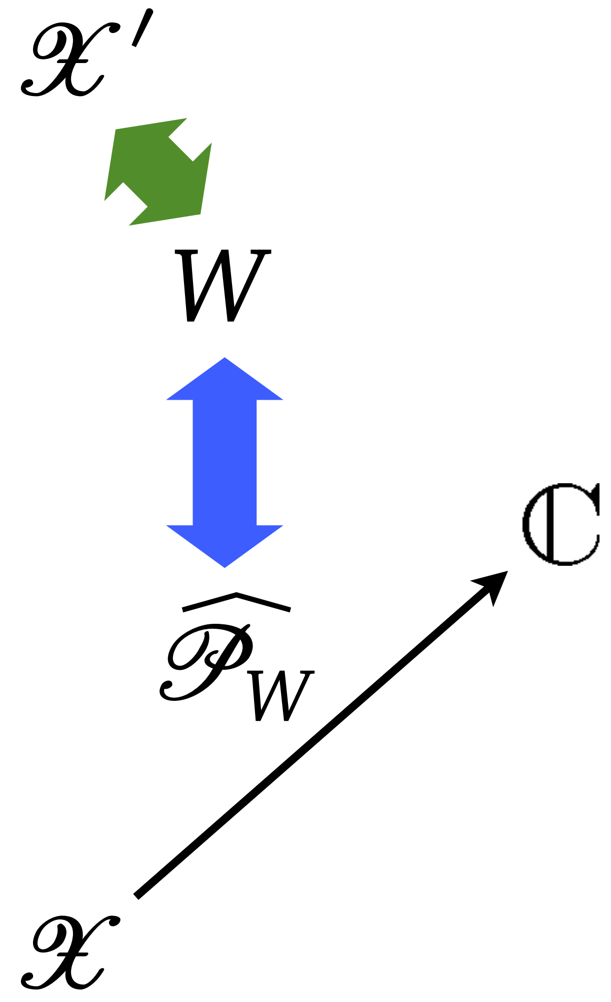

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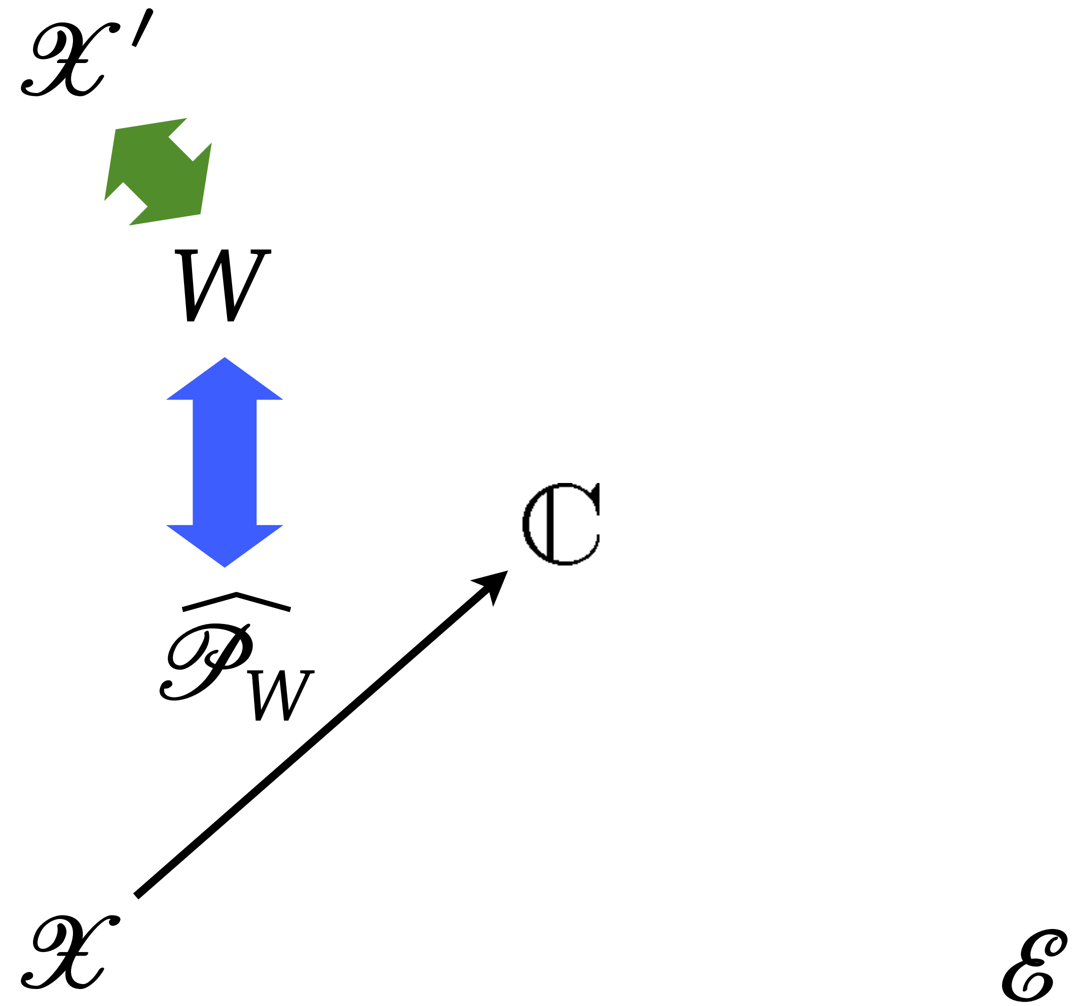

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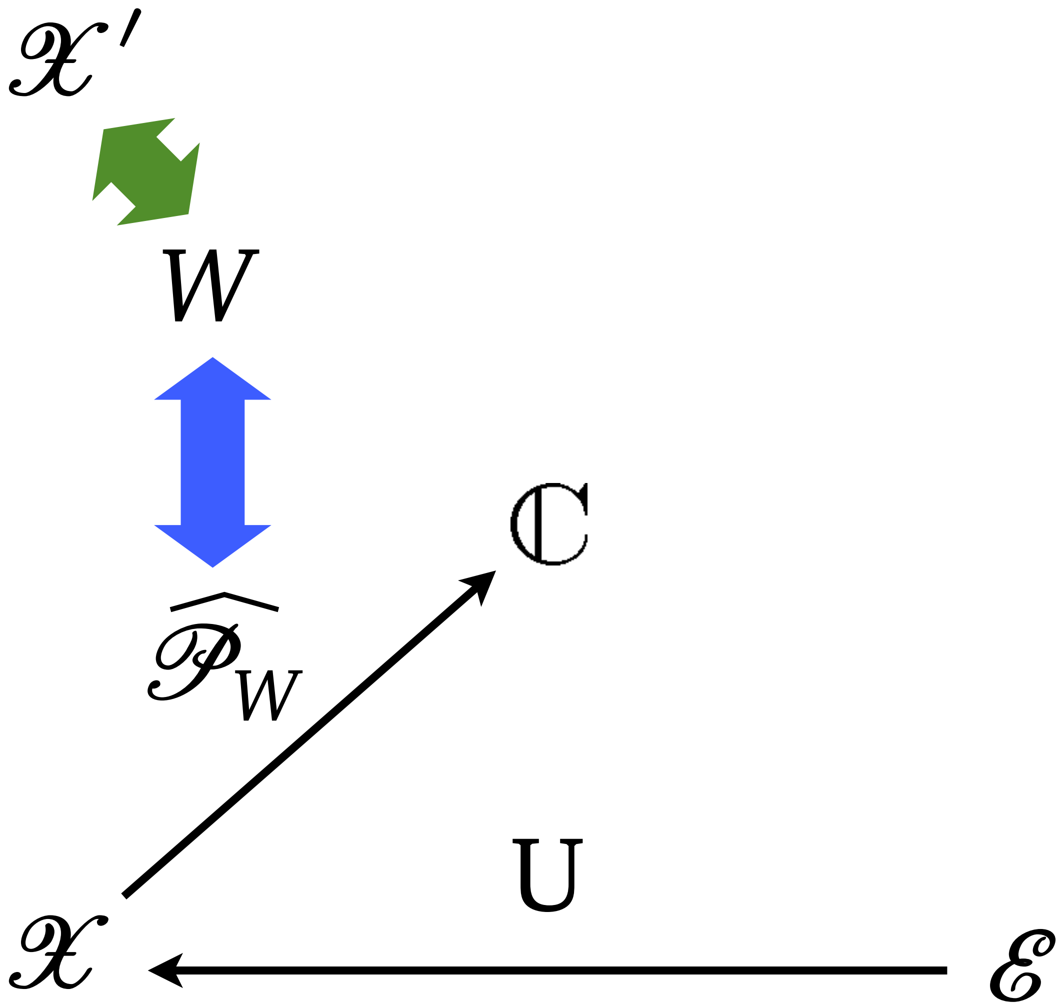

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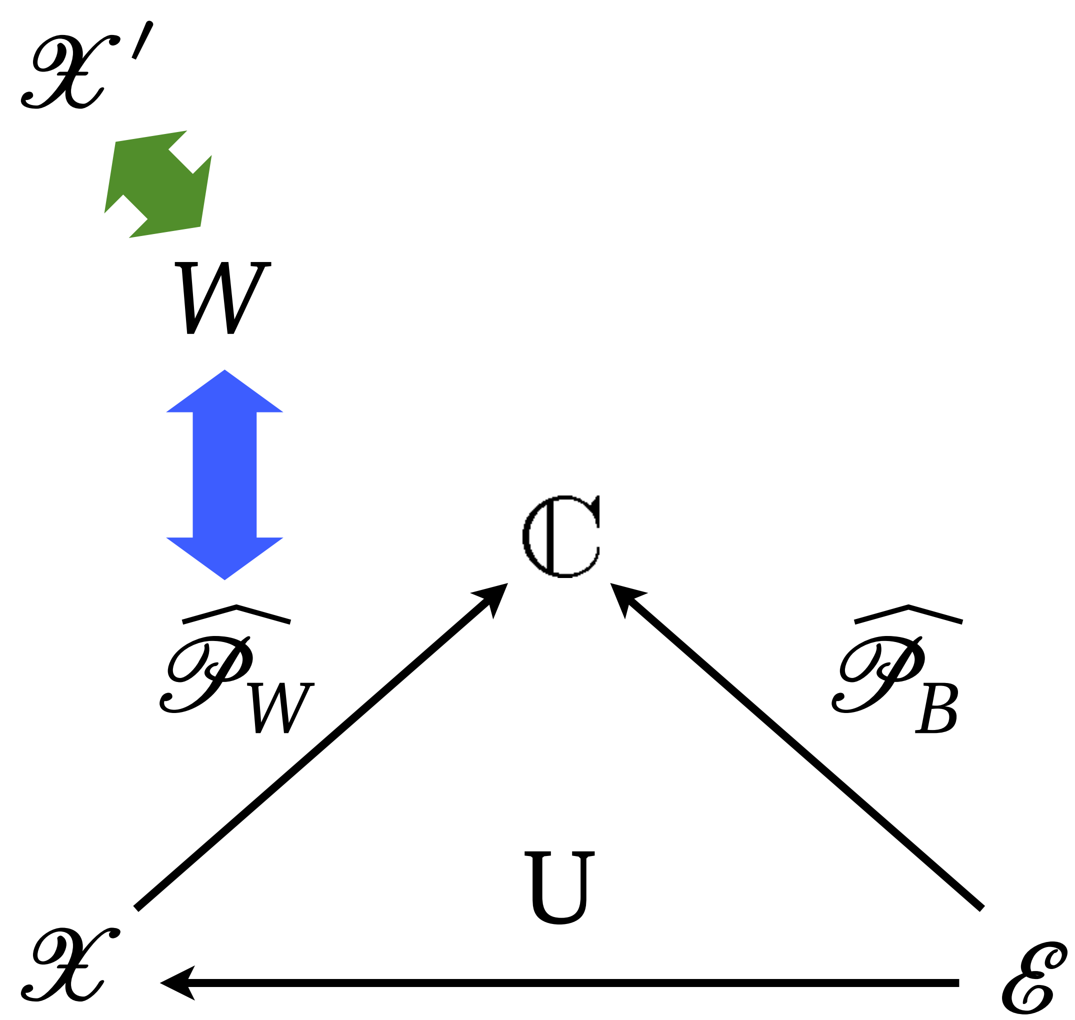


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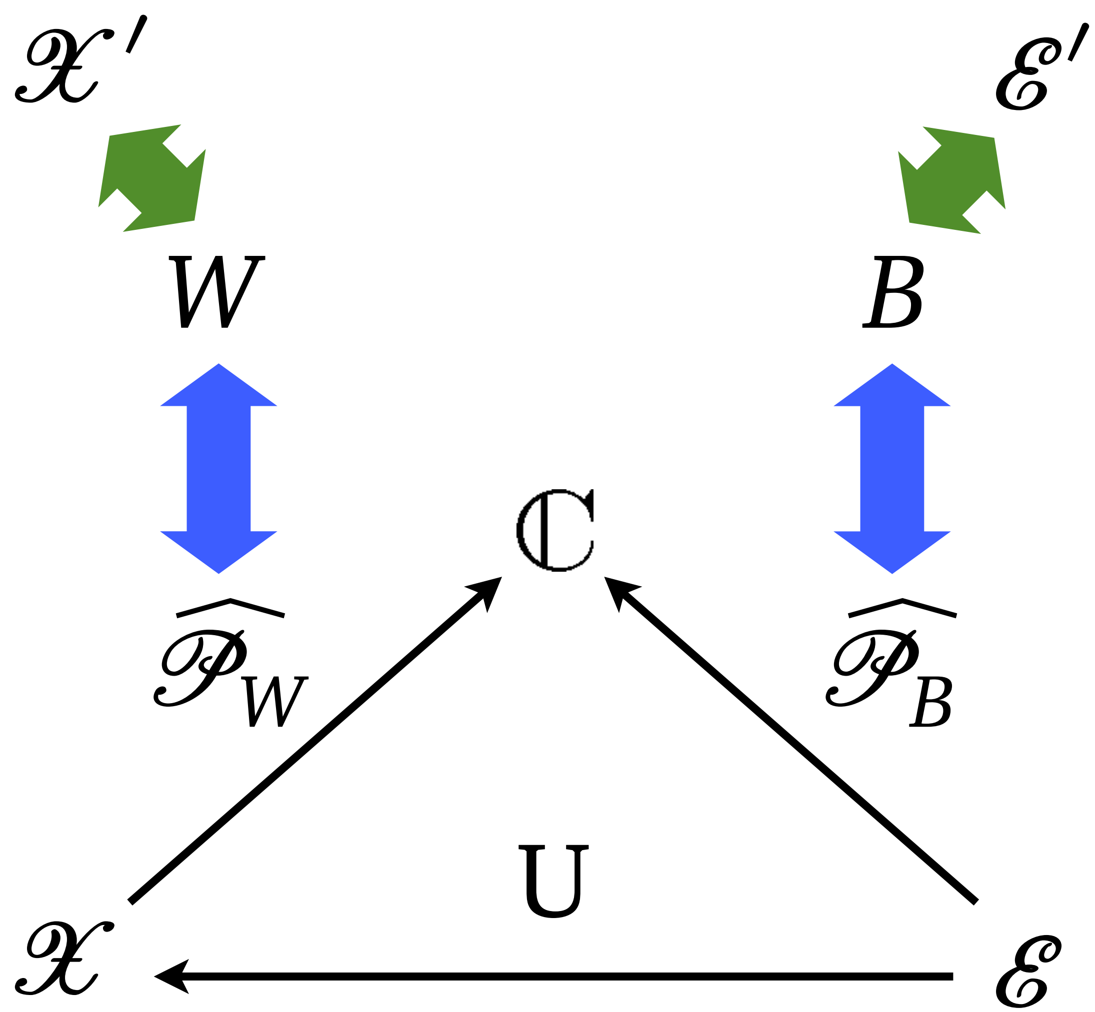


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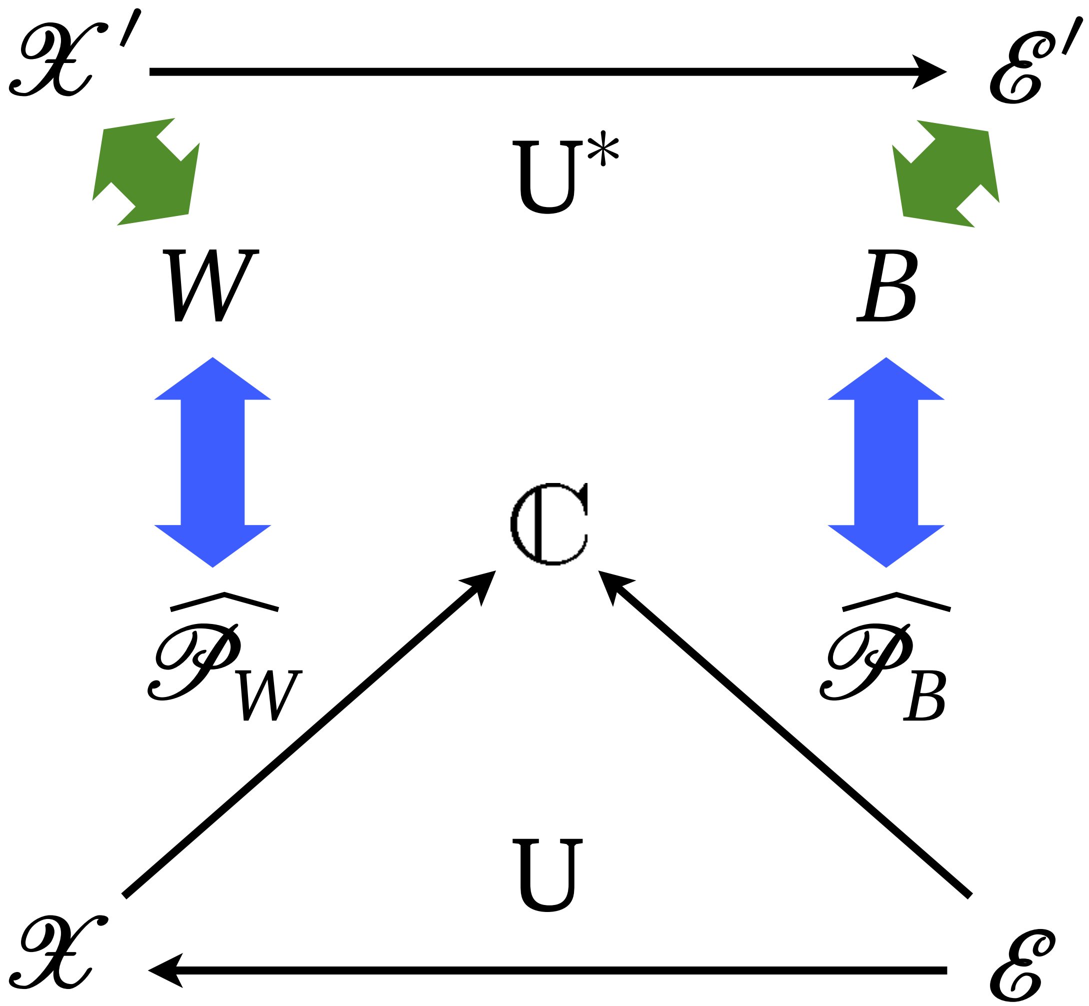


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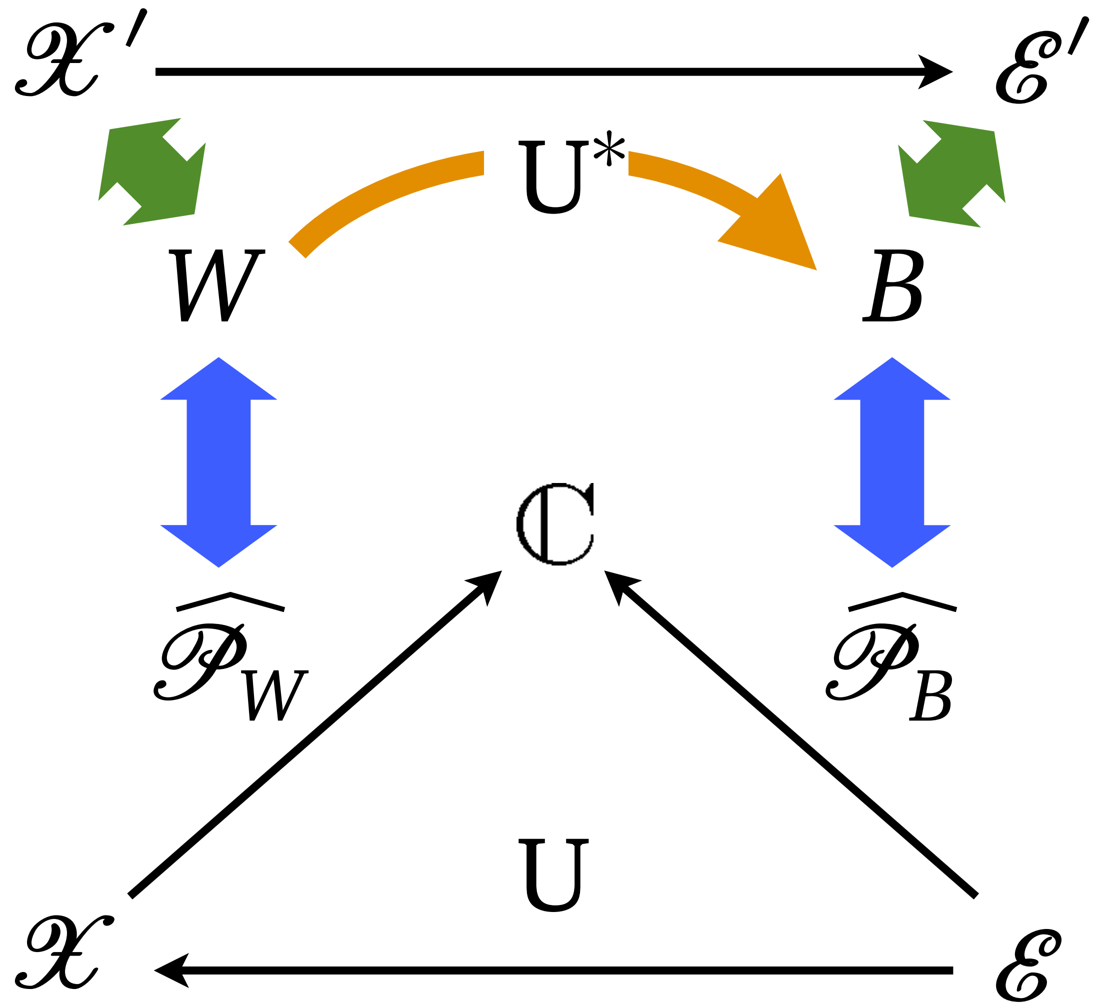


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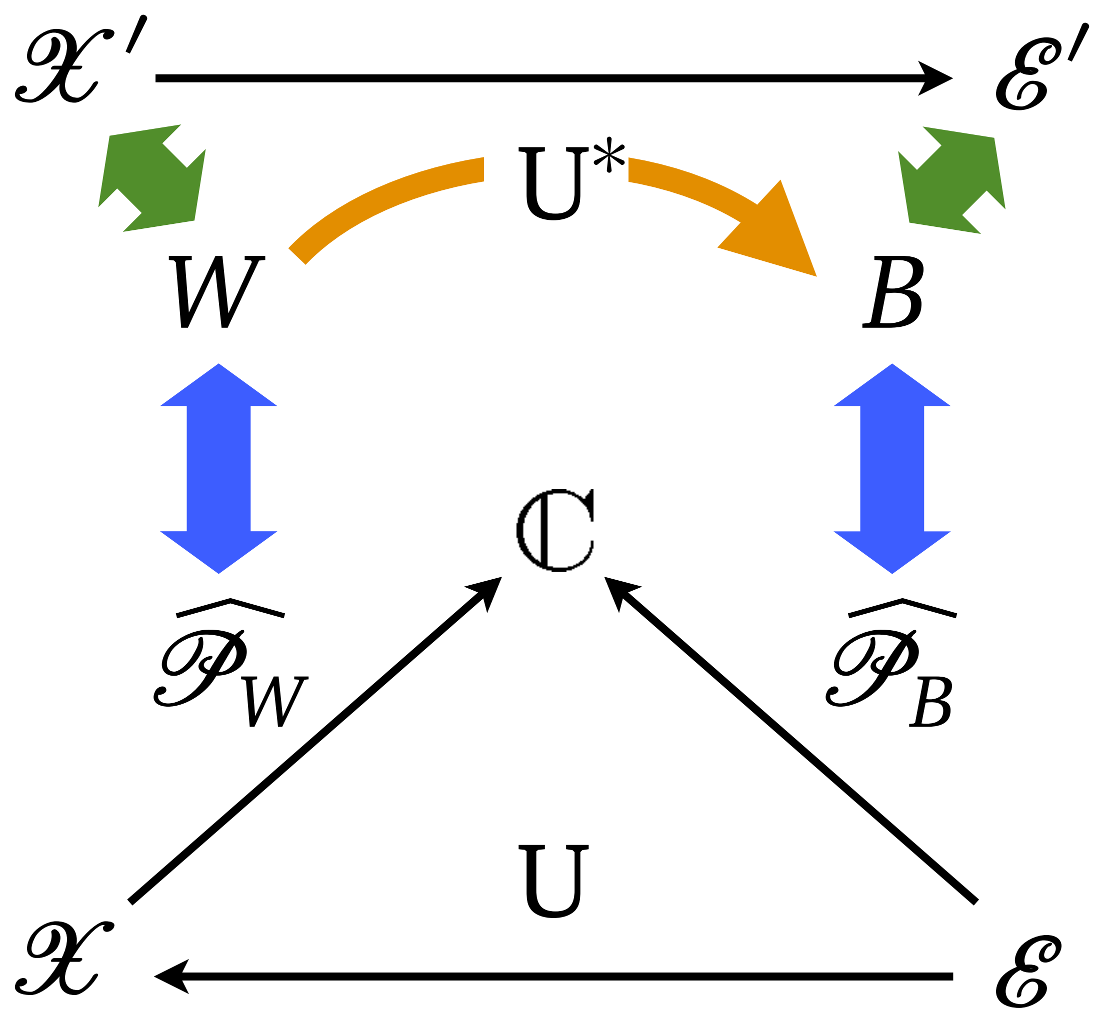


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$$\widehat{\mathscr{P}_B}(\phi) \iff \langle \phi, B \rangle$$

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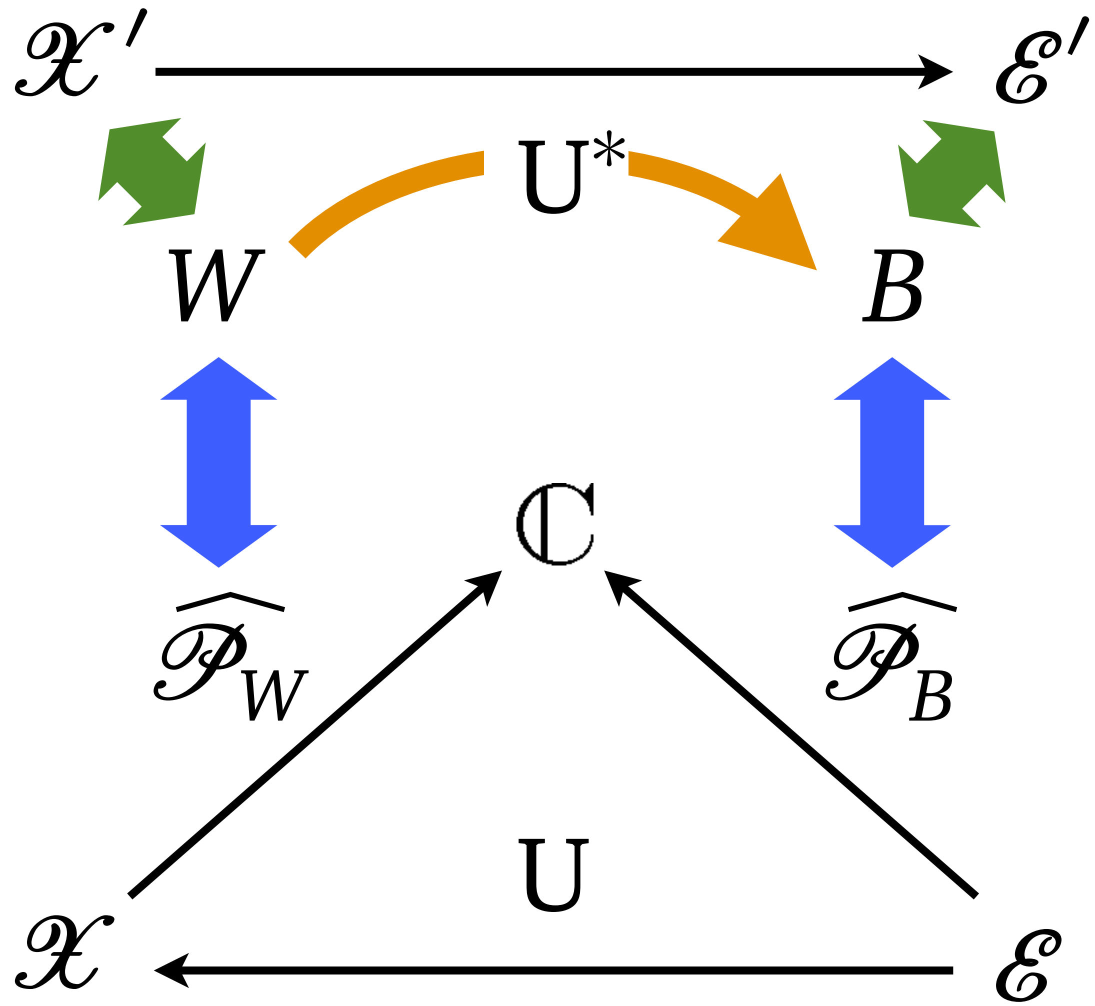


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$$\widehat{\mathscr{P}_B}(\phi) \Leftrightarrow \langle \phi, B \rangle = \langle \phi, \mathrm{U}^* W \rangle = \langle \mathrm{U}\phi, W \rangle$$

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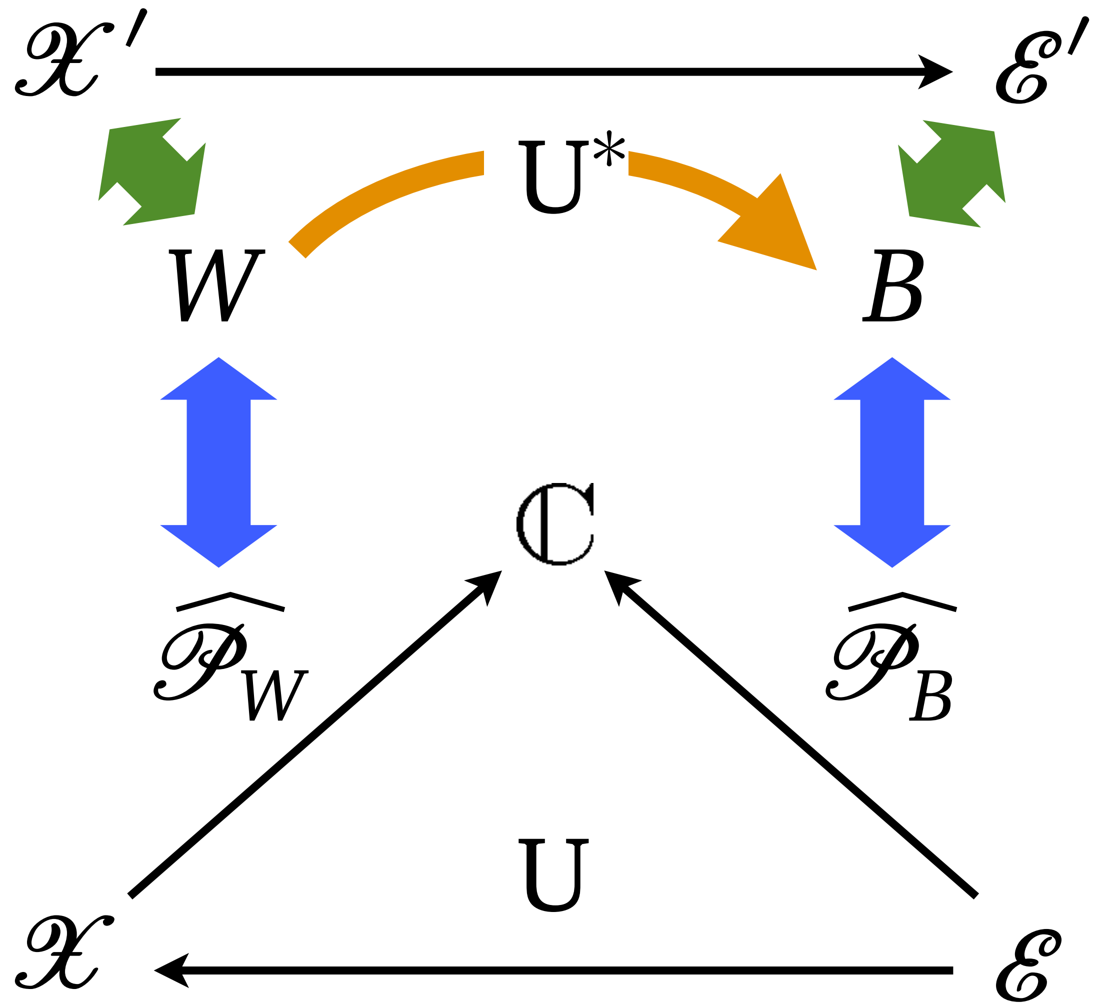


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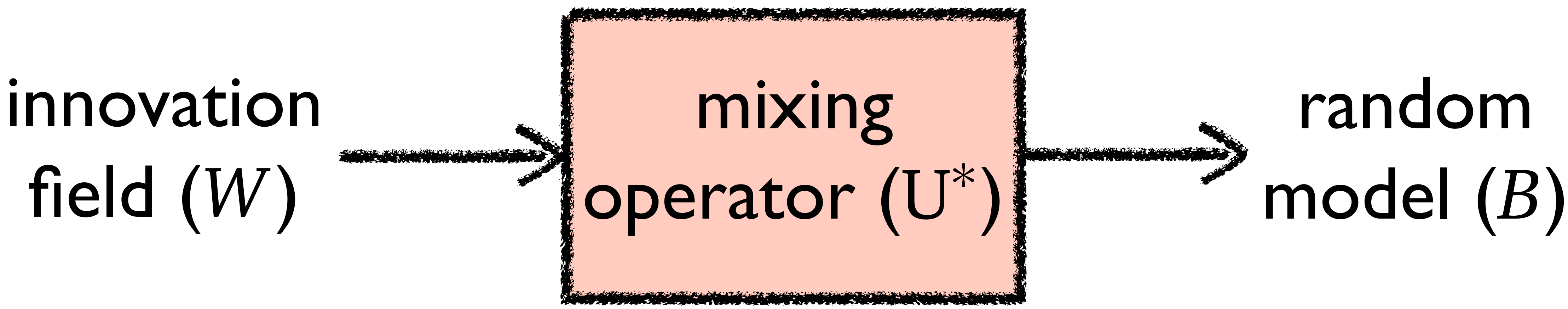
# Innovation Modelling

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# Back to FBM, FSM (1)

1. Innovation: Gaussian and stable innovations

$$\widehat{\mathscr{P}}_W(\phi) = \mathrm{e}^{-2^{-\frac{\alpha}{2}}\|\phi\|_\alpha^\alpha}, \quad \phi \in \mathscr{X} = \mathrm{L}_\alpha(\mathbb{R}^d)$$

- Continuous, normalized, pos.-definite (Lemma 1.t)
- Spatially independent:

  $\phi$, $\psi$ disjoint $\Rightarrow \langle \phi, W \rangle$, $\langle \psi, W \rangle$ independent

- Rotation-invariant, homogeneous (Lemma 1.W)

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i.e. fulfil required invariances

# Back to FBM, FSM (2)

## 2. Operator:

- Homogeneous isotropic distributions (§2.2.1)

$$\rho_c^\lambda := c\,\frac{|x|^\lambda}{2^{\frac{\lambda}{2}}\,\Gamma(\frac{\lambda+d}{2})}$$

  - Complete characterization (Theorem 2.ab)
  - Singularity regularized by analytic continuation in $\lambda$
  - Closed under rotation, scale, Fourier, Laplacian, products and convolutions (when defined)

- Convolution: $\mathrm{U}_c^{-\lambda} : \phi \mapsto \phi * \rho_c^{\lambda-d}$
  - Homogeneous, rotation-invariant (also shift-invariant) ✔
  - but not continuous $\mathscr{D}(\mathbb{R}^d) \to \mathrm{L}_\alpha(\mathbb{R}^d)$ ✘

# Back to FBM, FSM (3)

## 2. Operator (cont.):

- Modified operator (§2.2.2):

$$\mathrm{U}_{c,n}^{-\lambda} : \phi \mapsto \phi * \rho_c^{\lambda-d} - \mathrm{Reg}_{c,n}^{-\lambda}\,\phi$$

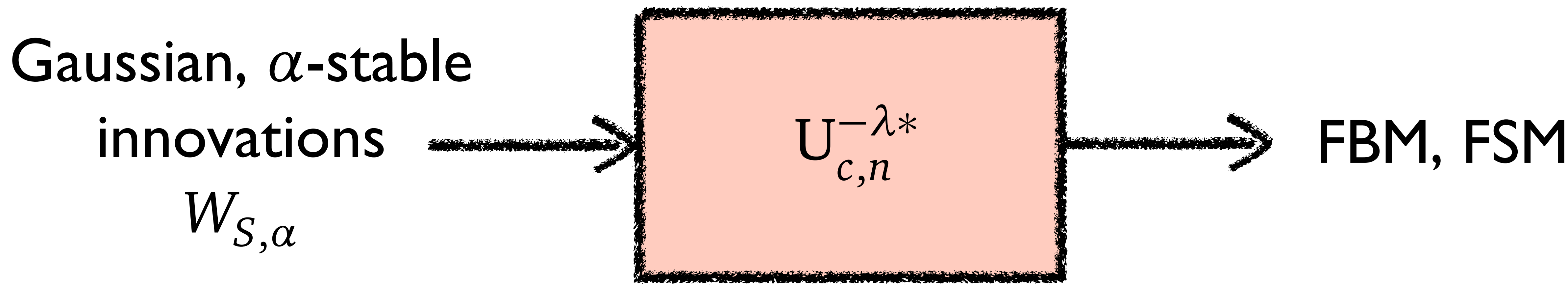
where

$$\mathrm{Reg}_{c,n}^{-\lambda} : \phi \mapsto \sum_{|k|\leq n} \frac{\partial_k \rho^{\lambda-d}}{k!} \int_{\mathbb{R}^d} (-y)^k \phi(y)\,\mathrm{d}y$$

  - Remains homogeneous, rot.-invariant (but not shift-invariant) (Lemma 2.al) ✔
  - Continuous $\mathscr{D}(\mathbb{R}^d) \to \mathrm{L}_\alpha(\mathbb{R}^d)$ for the right $n$ (Theorem 2.aq) ✔
  - $n+1$st finite diffs the same as those of $\mathrm{U}_c^{-\lambda}$ (Lemma 2.ao)

hence stationary (higher-order) increments

# Back to FBM, FSM (4)




$$\widehat{\mathscr{P}_B}(\phi) := \widehat{\mathscr{P}_W}(\mathrm{U}\phi)$$

# Vector Generalization

0. Define vector invariances
1. Find invariant vector innovations $(\widehat{\mathscr{P}_W}, \mathscr{X})$
2. Find invariant operators $(\mathrm{U}, \mathscr{E})$
    - Start with convolution operators $\mathscr{E} \to \mathscr{E}'$
    - Find continuous modification $\mathscr{E} \to \mathscr{X}$
3. Vector models defined by CHF

$$\widehat{\mathscr{P}_B}(\phi) := \widehat{\mathscr{P}_W}(\mathrm{U}\phi)$$

# 0. Vector Invariances

- Transformations:
  - Scaling same as before
  - Vector rotation:

$$R_{\omega,v} : f \mapsto \omega f(\omega^{T} \cdot)$$

re-expression of the same direction in the new coordinate system

# 1. Vector Innovations

- Gaussian and stable vector GRFs $\underline{W}_{S,\alpha}$ with CHFs on vector $\mathrm{L}_\alpha$ spaces:

$$\widehat{\mathscr{P}_W}(\phi) = \mathrm{e}^{-2^{-\frac{\alpha}{2}} \|\phi\|_\alpha^\alpha}, \qquad \phi \in \mathscr{X} = \mathrm{L}_\alpha^d(\mathbb{R}^d)$$

norm: $\|\phi\|_\alpha := \left(\int_{\mathbb{R}^d} (\phi^{\mathrm{H}}\phi)^{\frac{\alpha}{2}}\right)^{\frac{1}{\alpha}}$

  - Continuous, normalized, pos.-definite (Lemma 1.t) ✔
  - Spatially independent ✔
  - Rotation-invariant, homogeneous (Lemma 1.W) ✔

# 2. Vector Operators (1)

- Homogeneous and rot.-invariant matrix distributions (conv. kernels) (§2.3.1)

$$\left[P^{\lambda}_{(r_1,r_2)}\right]_{ij} := \delta_{ij}\rho^{\lambda}_{r_2} + \frac{1}{\lambda+2}\partial_{ij}\rho^{\lambda+2}_{r_1-r_2}$$

  - Complete characterization (Theorem 2.aw)
  - Two additional parameters beside homogeneity order (several parametrizations, 2.av)
  - Family closed under rotation, scale, Fourier, products and convolutions (when defined)

# 2. Vector Operators (2)

- Convolution:

$$\underline{\mathrm{U}}^{-\lambda}_{(r_1,r_2)} \ : \ \phi \longmapsto P^{\lambda-d}_{(\hat{r}_1,\hat{r}_2)} * \phi$$

✔ nuclear

- maps $\mathscr{D}^d(\mathbb{R}^d) \to (\mathscr{D}')^d(\mathbb{R}^d)$

✘ not right but we know how to fix it

- Homogeneous, rot.-invariant (also shift-invariant) ✔
- Helmholtz-type decomposition (2.bp):

$$\underline{\mathrm{U}}^{\lambda}_{(r_1,0)} \,\mathrm{Curl}^*\phi = 0, \quad \phi \in \mathscr{D}^{d\times d}$$

$$\underline{\mathrm{U}}^{\lambda}_{(0,r_2)} \,\mathrm{Div}^*\phi = 0, \quad \phi \in \mathscr{D}$$

defined in any number of dims

# 2. Vector Operators (3)

- Modified operator (§2.3.2):

$$\underline{\mathrm{U}}^{-\lambda}_{(r_1,r_2),n} = \underline{\mathrm{U}}^{-\lambda}_{(r_1,r_2)} - \underline{\mathrm{Reg}}^{-\lambda}_{(r_1,r_2),n}$$

  - maps $\mathscr{D}^d(\mathbb{R}^d) \to \mathrm{L}^d_\alpha(\mathbb{R}^d)$ (Theorem 2.aq) ✔
  - Homogeneous, rot.-invariant (Lemma 2.be) ✔
  - Helmholtz-type decomposition (Lemma 2.bq):

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  - Same $n+1$st finite diffs as $\underline{\mathrm{U}}^{-\lambda}_{(r_1,r_2)}$ (Lemma 2.bi)

hence, again, stationary (higher-order) increments

# Vector FBM, FSM (and Extensions)

Gaussian, $\alpha$-stable vec. innovations $\underline{W}_{S,\alpha}$ → $\underline{\mathrm{U}}^{-H-d+\frac{d}{\alpha}*}_{(r_1,r_2),\lfloor H \rfloor}$ → Vector FBM, FSM

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$\underline{W}_{S,\alpha_1}$

independent

$\underline{W}_{S,\alpha_2}$

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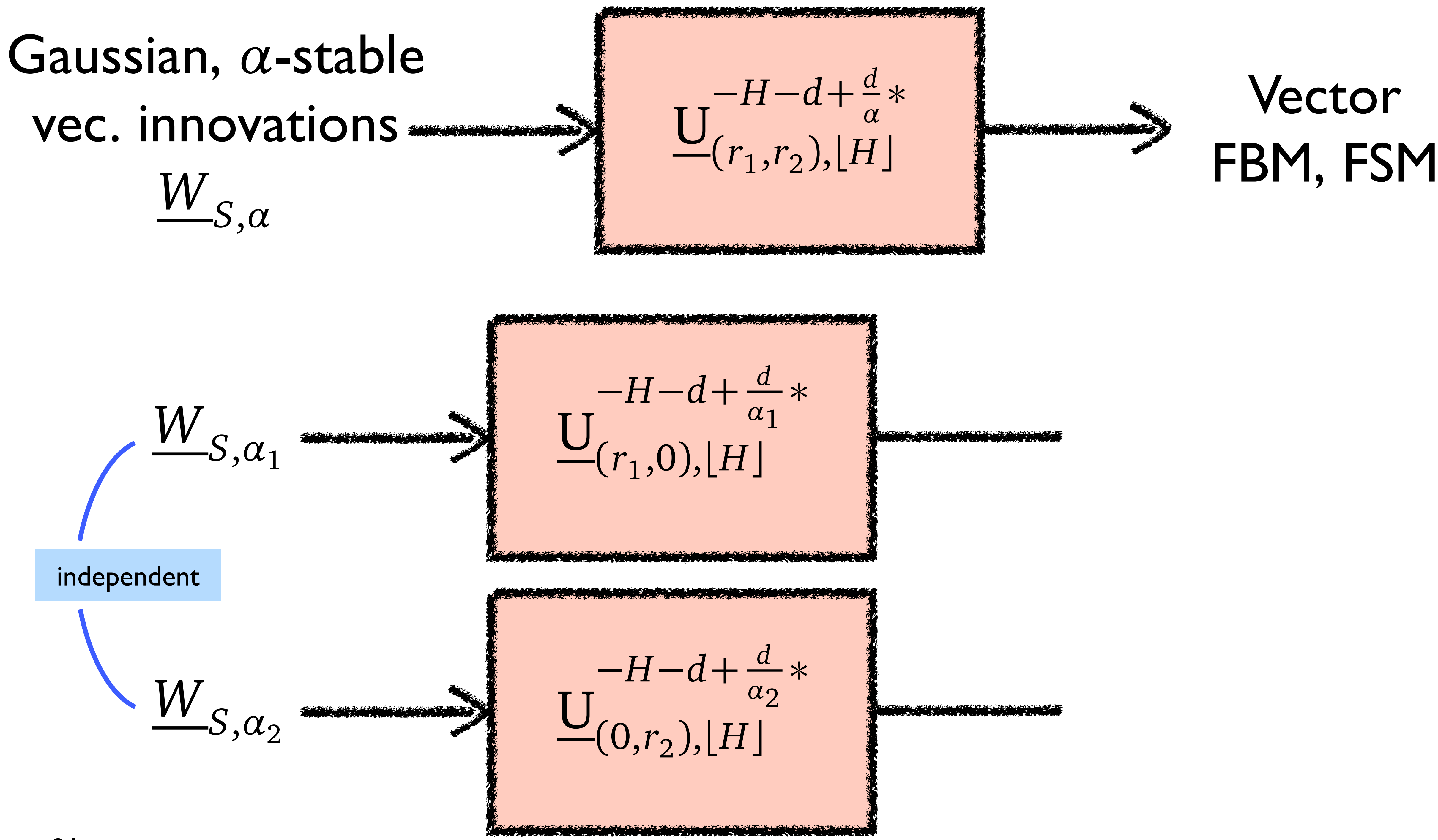

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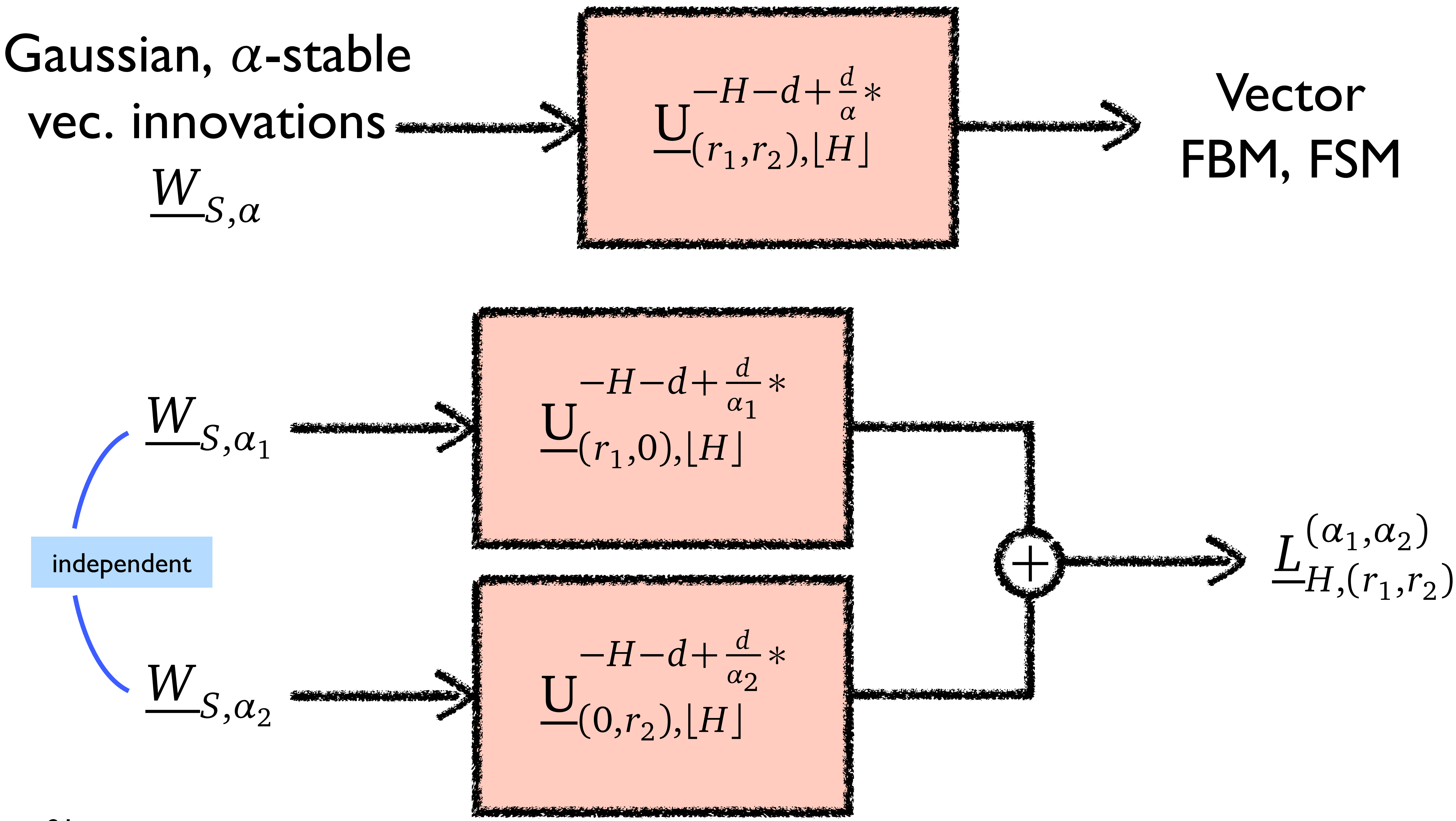

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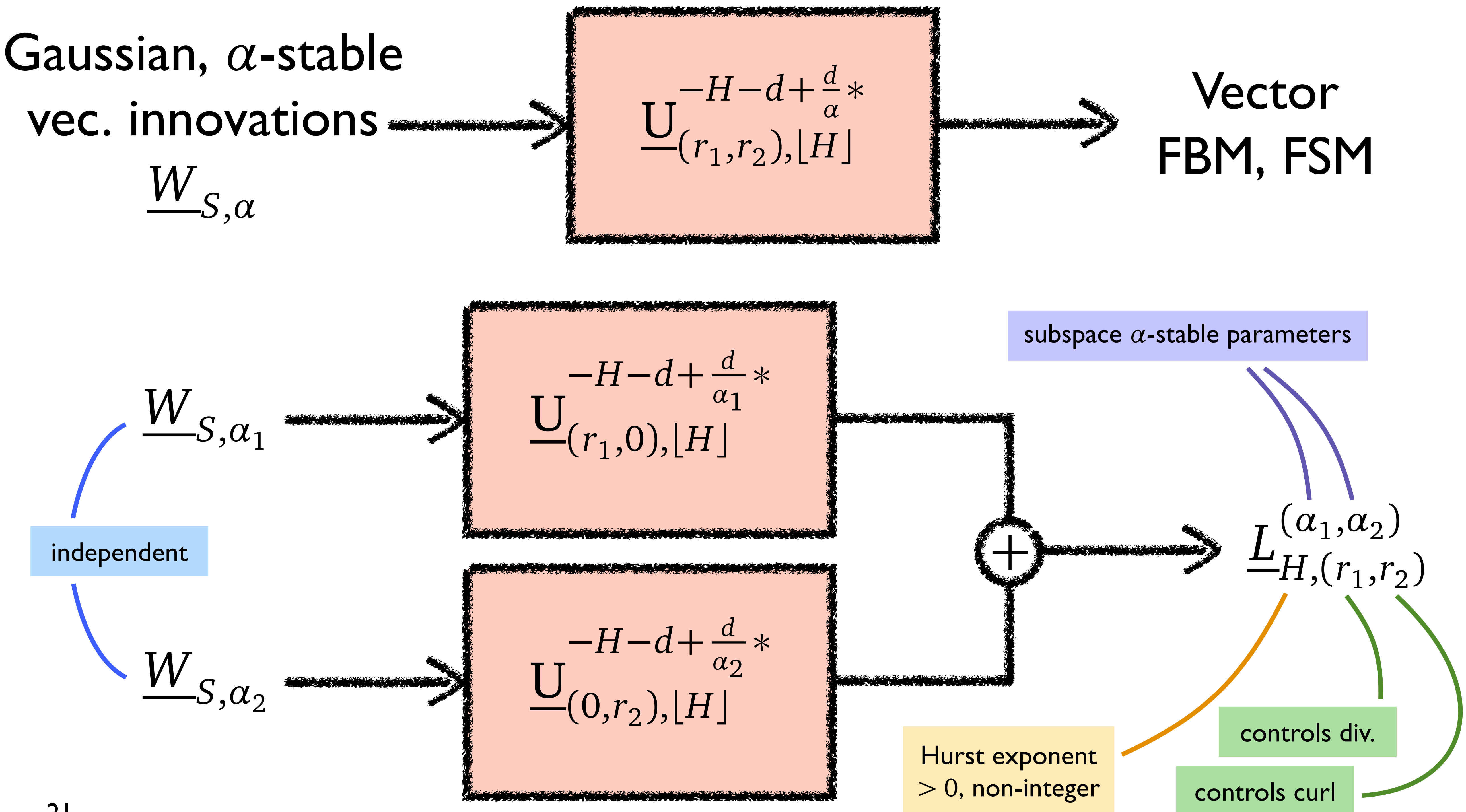


# Properties

- Self-similarity and rotation-invariance (3.o)
- Stationary $\lfloor H \rfloor + 1$st-order incerements (3.p)
- Helmholtz decomposition (3.t):

independent

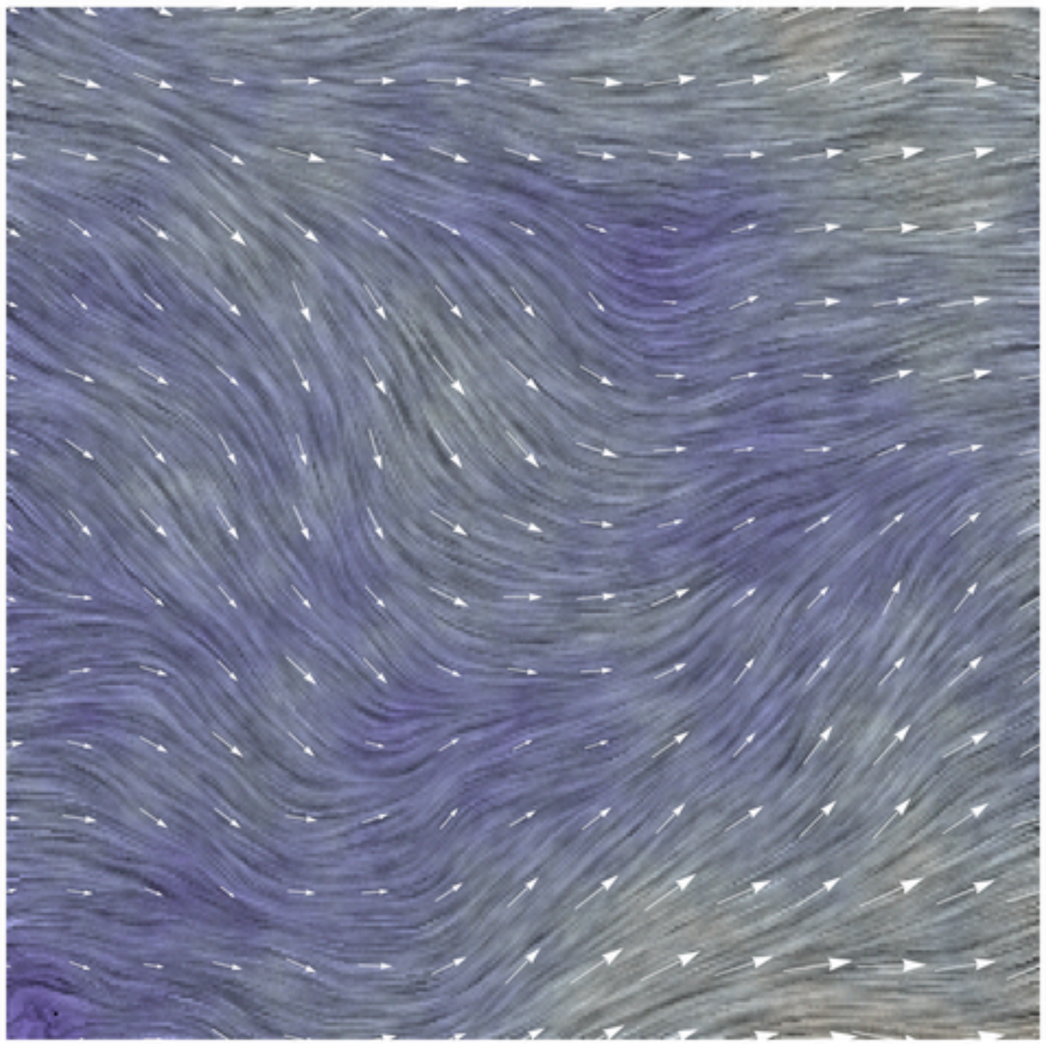
$$\underline{L}_{H,(r_1,r_2)}^{(\alpha_1,\alpha_2)} = \underline{L}_{H,(r_1,0)}^{\alpha_1} + \underline{L}_{H,(0,r_2)}^{\alpha_2}$$

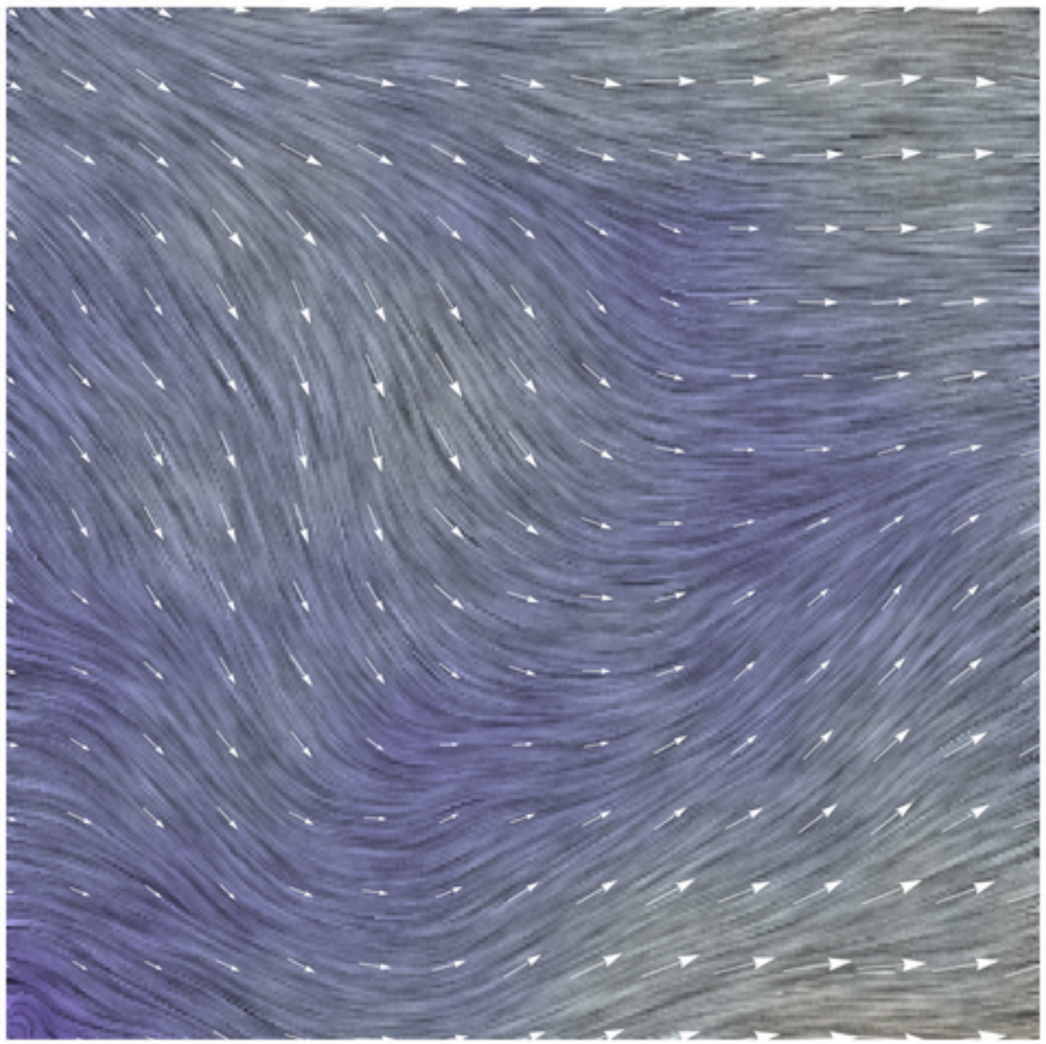
curl-free div.-free

- Variogram (Gaussian, $0 < H < 1$) :

$$\mathbb{E}\left\{\left[\underline{B}_{H,\underline{r}}(x) - \underline{B}_{H,\underline{r}}(y)\right]\left[\underline{B}_{H,\underline{r}}(x) - \underline{B}_{H,\underline{r}}(y)\right]^{\mathrm{H}}\right\} = P_{-2\hat{\underline{r}}'}^{2H}(x-y)$$

# Realizations





Gaussian, $H = 0.6$, $r_1 = r_2$ balanced Gaussian, $H = 0.9$, $r_1 = r_2$

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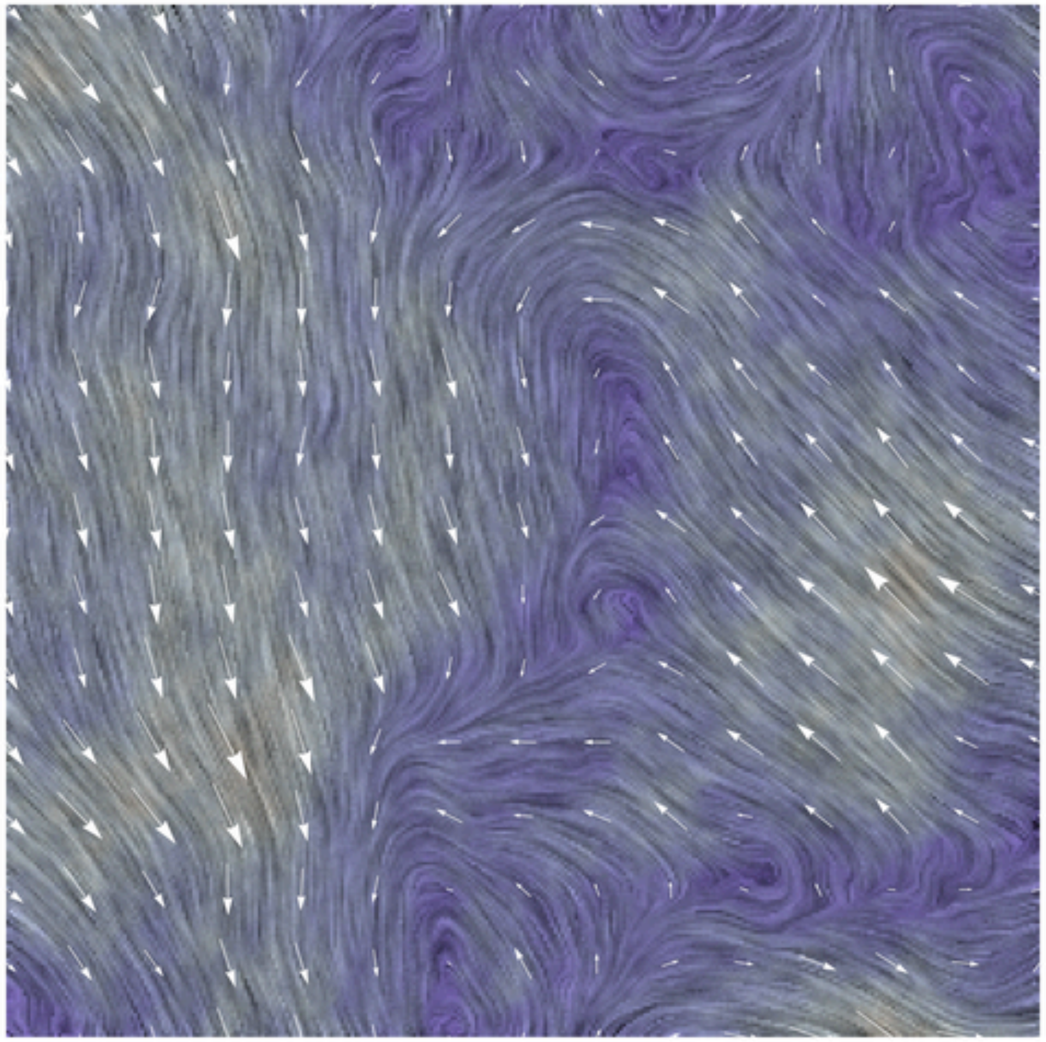
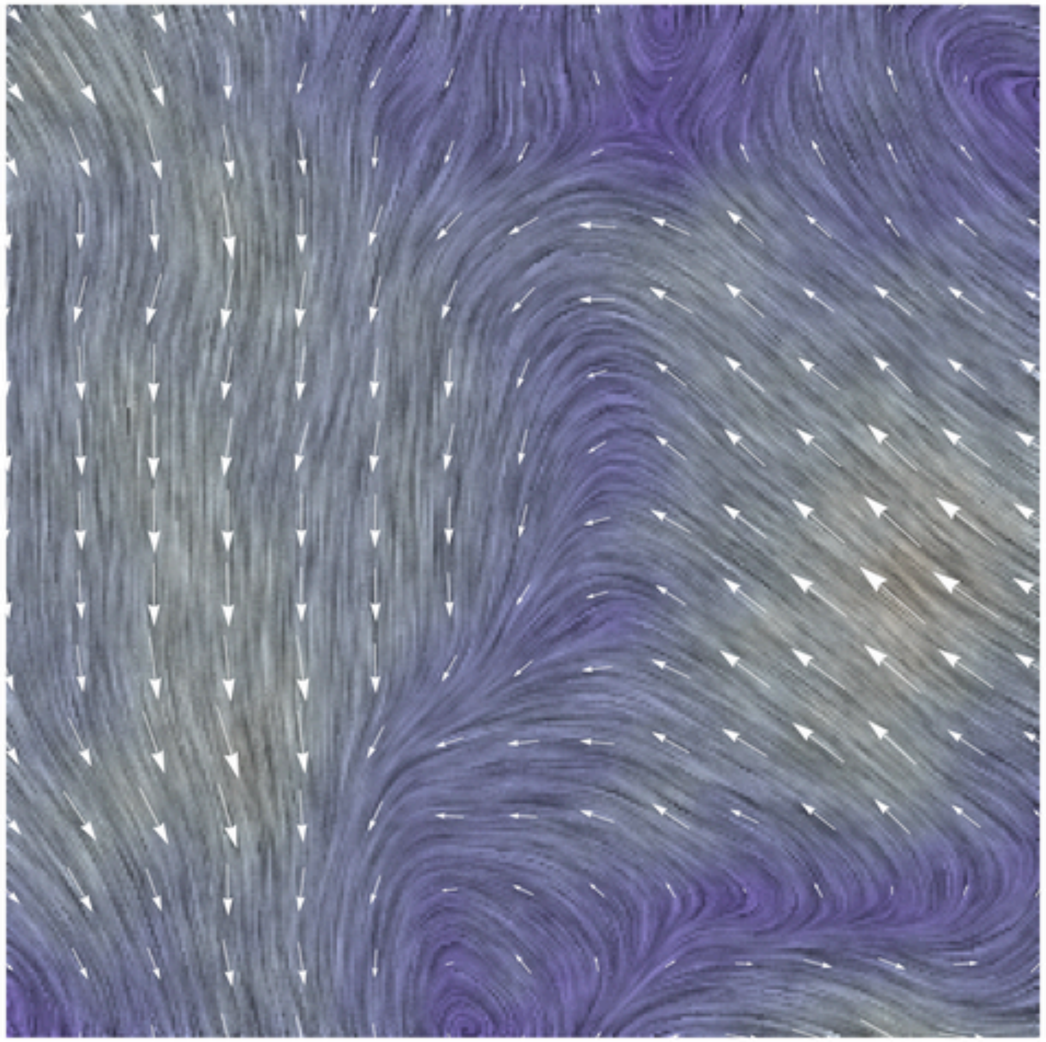


Gaussian, $H = 0.6$, $r_1 = 0$

div.-free



Gaussian, $H = 0.9$, $r_1 = 0$

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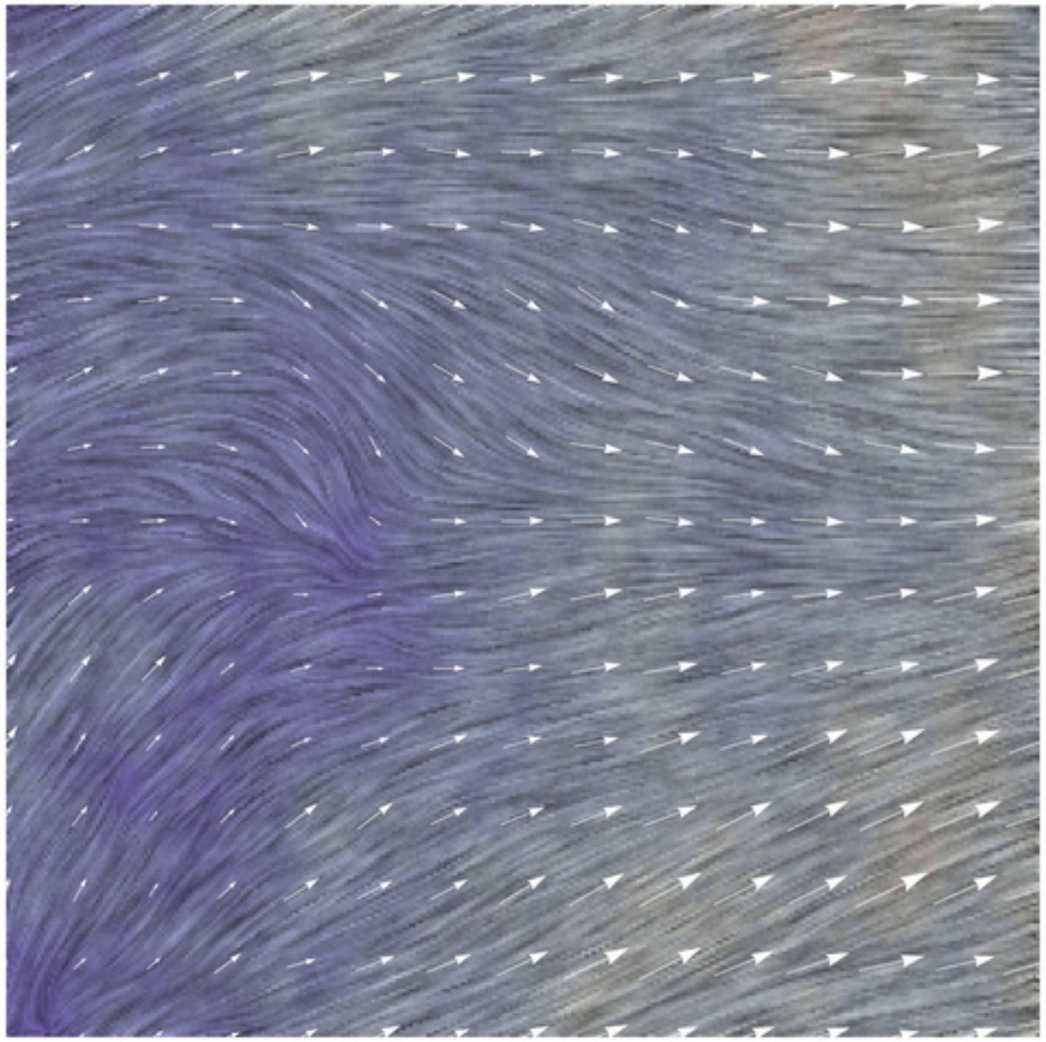
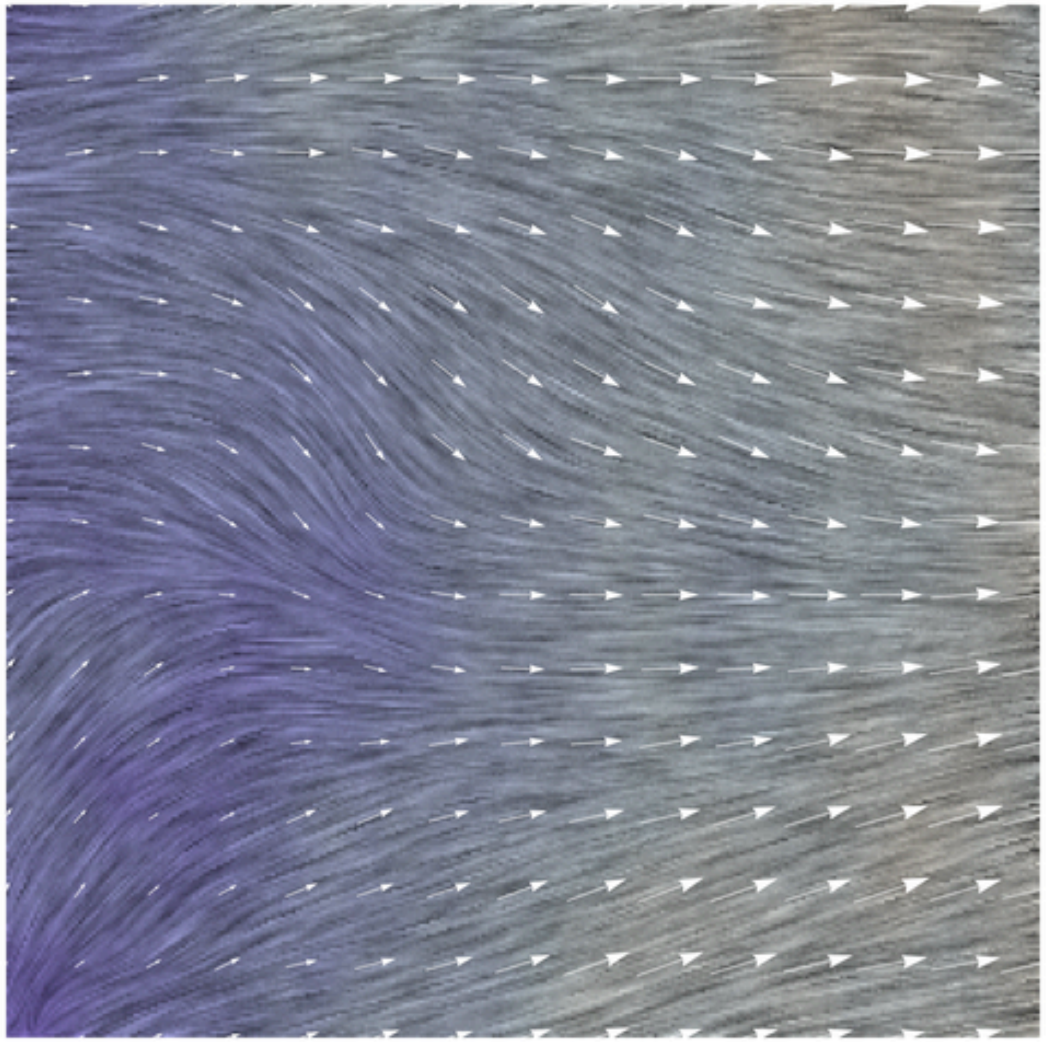




Gaussian, $H = 0.6$, $r_2 = 0$

curl-free

Gaussian, $H = 0.9$, $r_2 = 0$

# Vec. Field Reconstruction (1)

- Problem:
  - Given imperfect, possibly indirect observations:

$$Y = \Phi f_{\text{true}} + \text{noise}$$

  - To reconstruct an approximation of $f_{\text{true}}$

# Vec. Field Reconstruction (2)

- Solution:
  - Initial solution set:

typically quadratic distance (i.e. sample variance)

$$f \quad \textbf{s.t.} \quad \mathrm{dist}(\Phi f; Y) = \mu$$

# Vec. Field Reconstruction (2)

- Solution:
  - Initial solution set:

typically quadratic distance (i.e. sample variance)

parameter for exploration

$$f \quad \text{s.t.} \quad \operatorname{dist}(\Phi f; Y) \leq \mu$$

# Vec. Field Reconstruction (2)

- ## Solution:
  - Initial solution set:

typically quadratic distance (i.e. sample variance)

parameter for exploration

$$f \quad \text{s.t.} \quad \mathrm{dist}(\Phi f\,; Y) \leq \mu$$

  - **Parametric** regularity energy/criterion:

$$\min_f \mathfrak{R}_{\underline{\alpha}}(f)$$

Energy: total potential of independent contributions

decoupling (inverse-mixing) operator

potential functional

$$\mathfrak{R}(f) = \Xi(\mathrm{R}\, f) = \int_\Omega \xi\left(\mathrm{R}\, f(u)\right) \; \mu(\mathrm{d}u)$$

  - Invariance (lack of preference):

$$\exists \underline{\alpha}' \quad \text{s.t.} \quad \mathfrak{R}_{\underline{\alpha}}(\mathrm{T}f) \equiv \mathfrak{R}_{\underline{\alpha}'}(f)$$

# Vec. Field Reconstruction (3)

- Scale-, rot.-invariant regularity criterion (4.r):

$$\mathfrak{R}_{\underline{\alpha}}(f) = \alpha_c \|\mathrm{Curl} f\|_{p_c}^{p_c} + \alpha_d \|\mathrm{Div} f\|_{p_d}^{p_d} + \sum_i \alpha_i \|\underline{\mathrm{U}}_{\underline{r}_i}^{\lambda_i} f\|_{p_i}^{p_i}$$

- Important special cases: Curl-Div. reg.

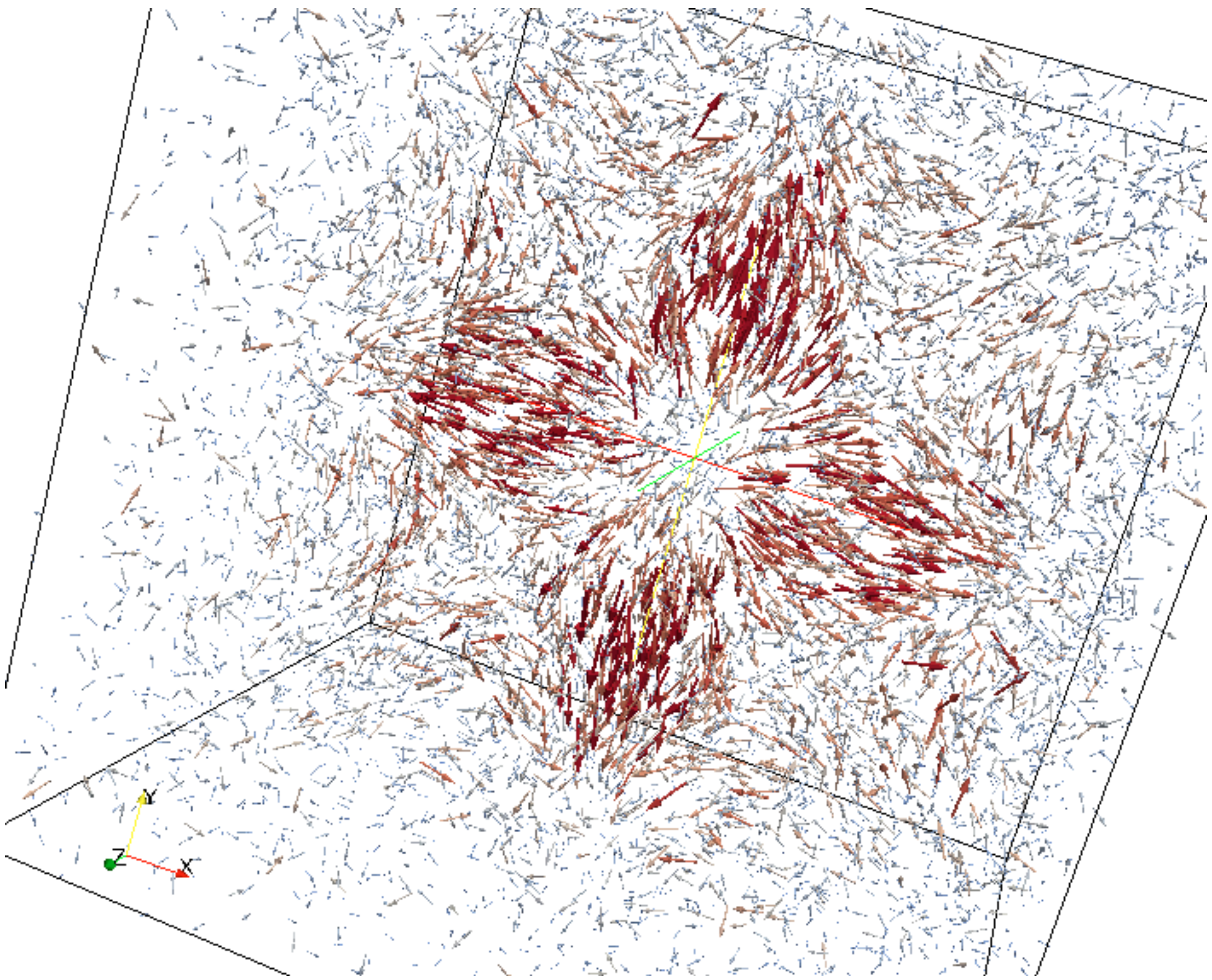
$$\mathfrak{R}_{\underline{\alpha}}(f) = \alpha_c \|\mathrm{Curl} f\|_{p}^{p} + \alpha_d \|\mathrm{Div} f\|_{p}^{p}, \quad p = 1, 2$$

# Algorithm

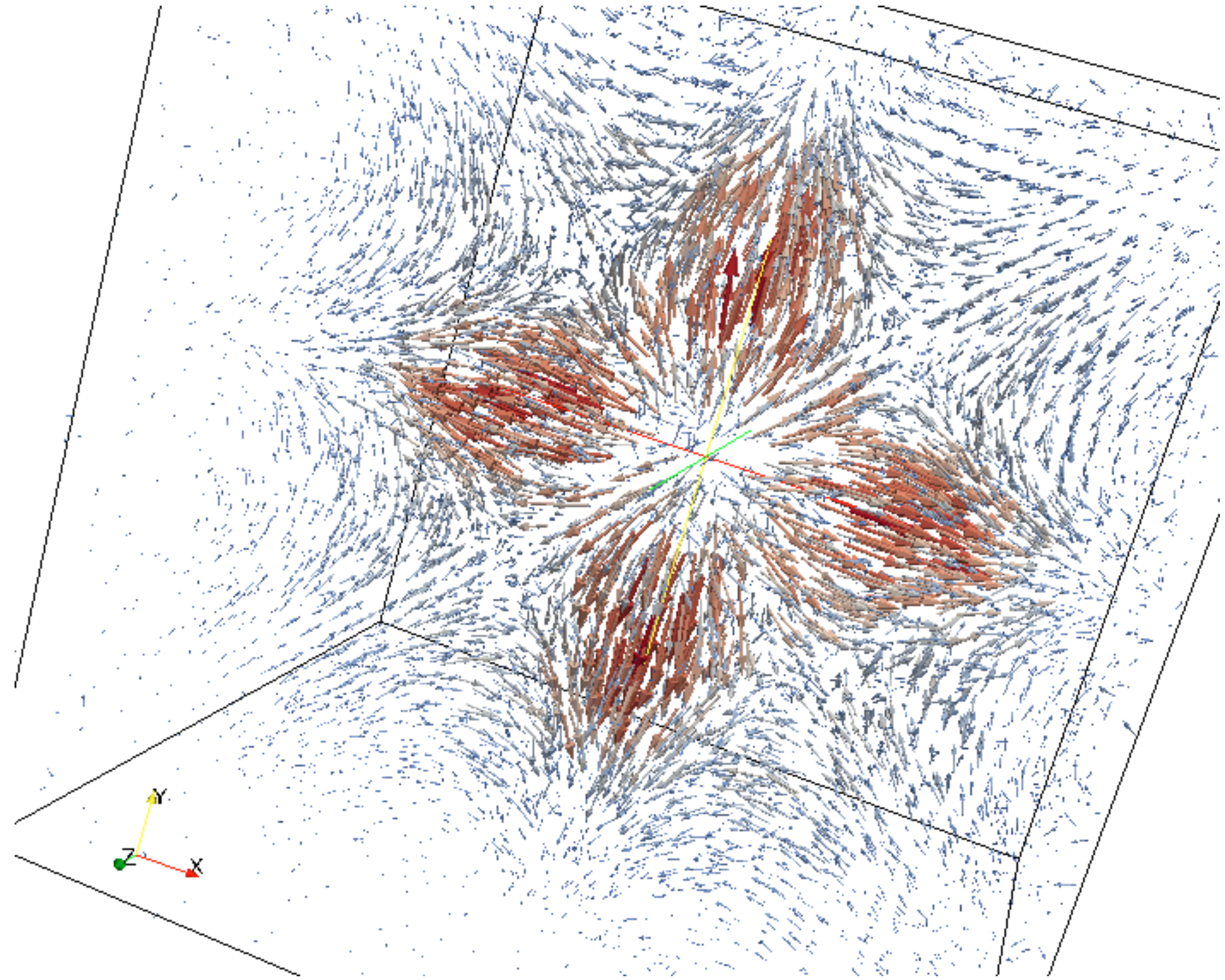
$$\underset{f}{\text{minimize}} \quad \text{dist}(\Phi f\,; Y) + \alpha_c \|\text{Curl} f\|_p^p + \alpha_d \|\text{Div} f\|_p^p, \qquad p = 1, 2$$

- Discretization (finite diffs, more sophisticated)
- Non-quadratic optimization:
  - Sequence of tight quadratic upper bounds (4.z)
  - Each local bound optimized using an iterative linear solver
- Algorithm parameters adjusted for best performance (empirical, theoretical for specific noise models)

# Some Results (1)

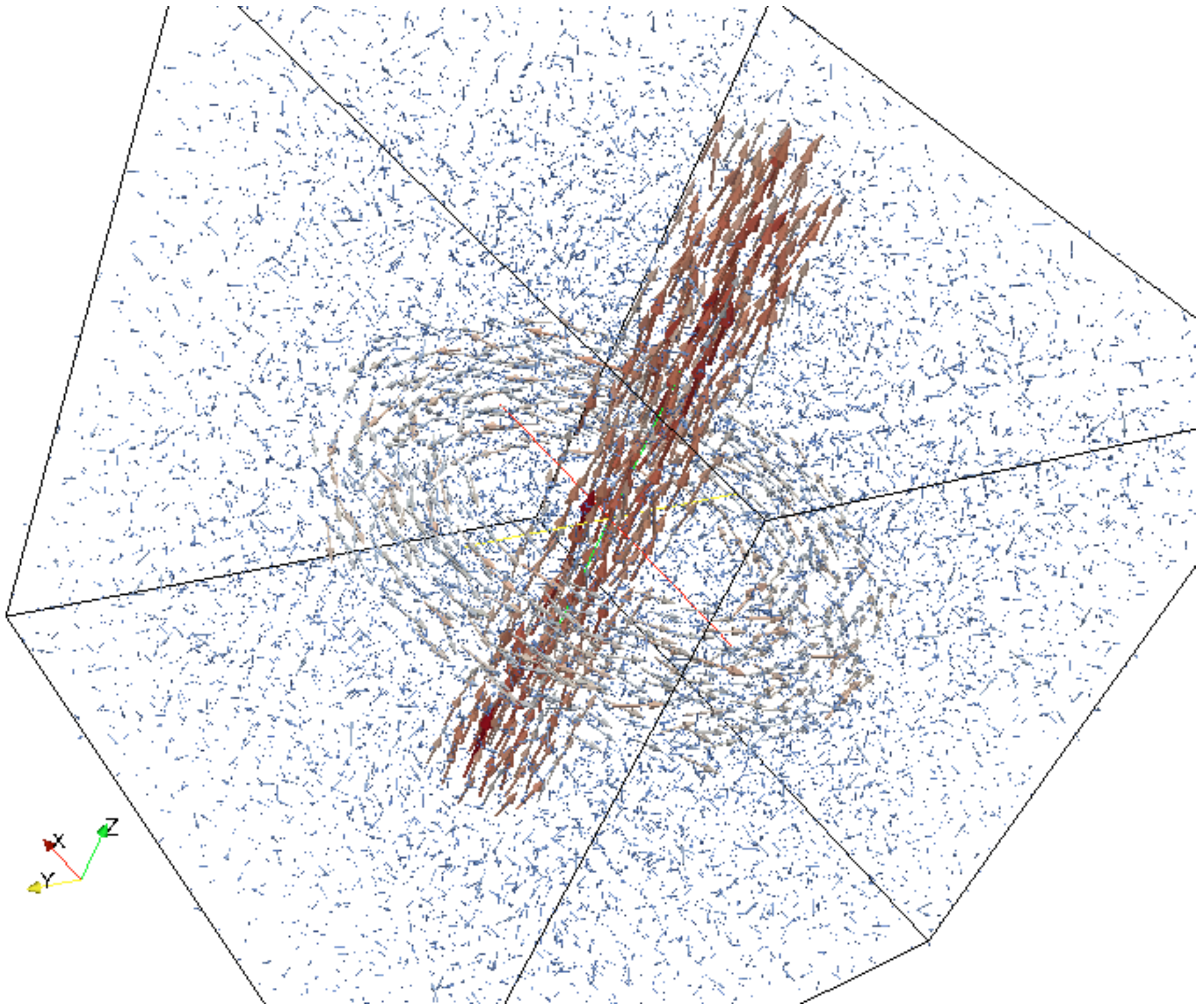



Noisy (0 dB SNR)

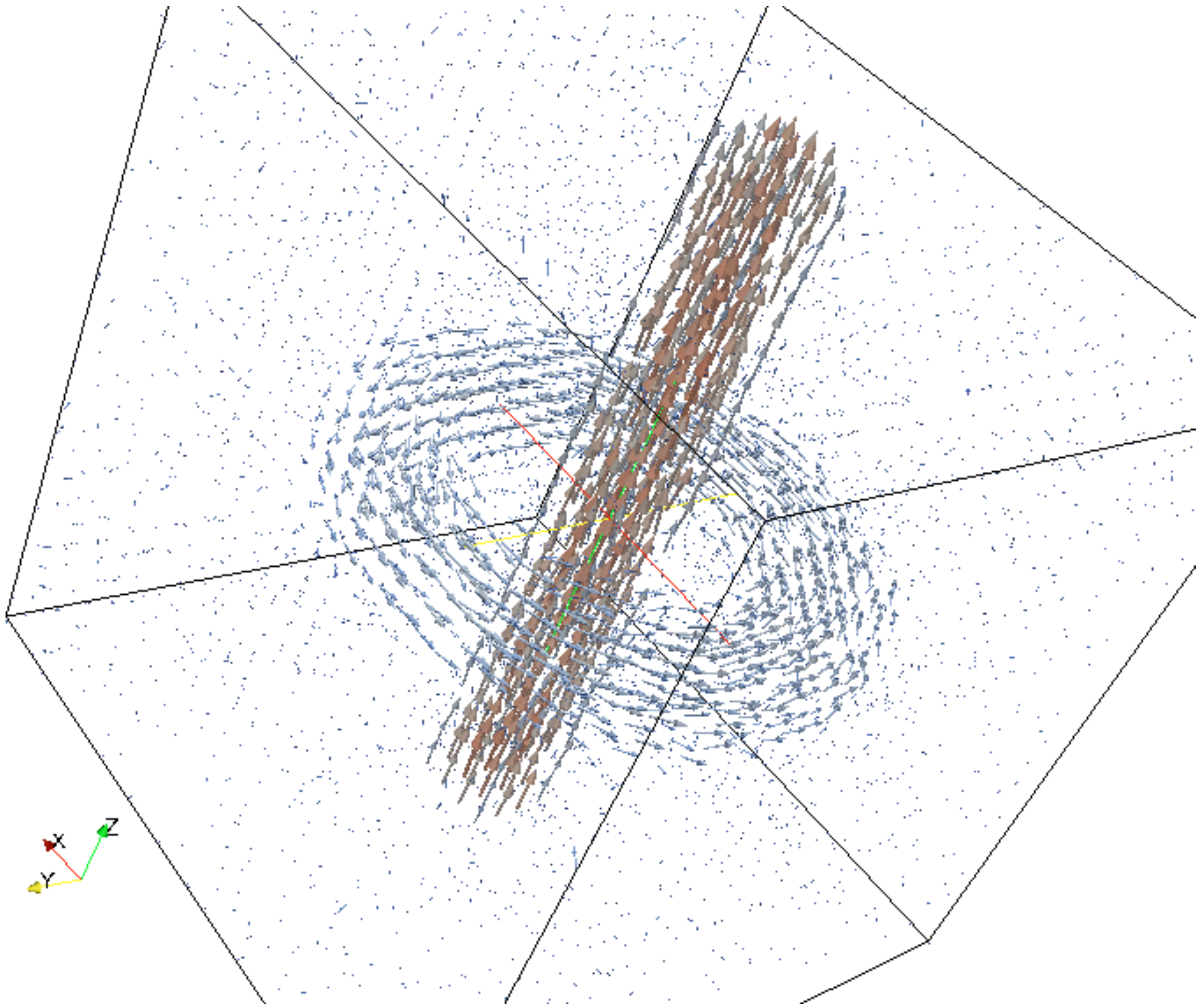



Denoised (11.70 dB SNR)

# Some Results (2)

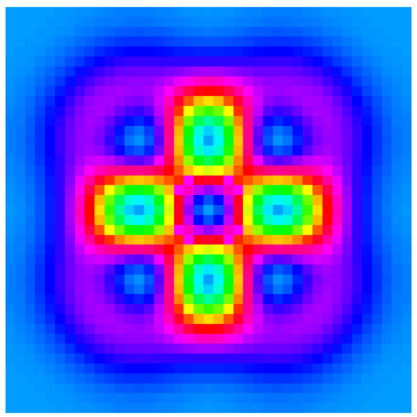



Noisy (0 dB SNR)

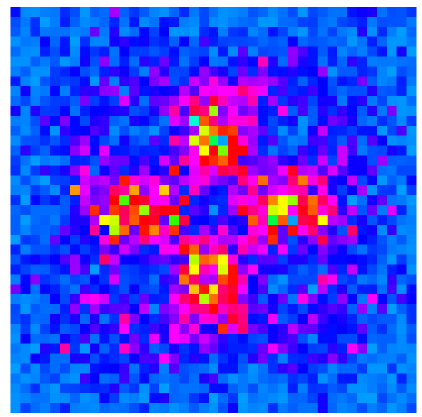



Denoised (9.01 dB SNR)

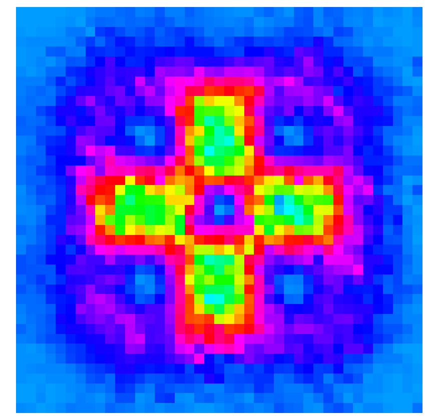
# Some Results (3)



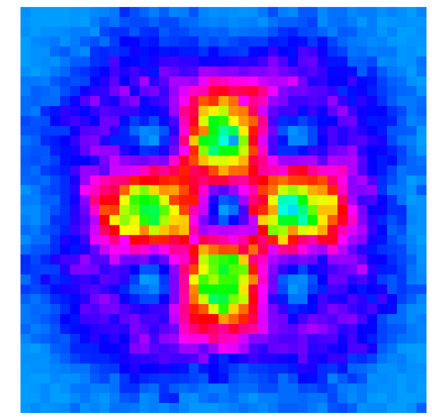
(a) Original



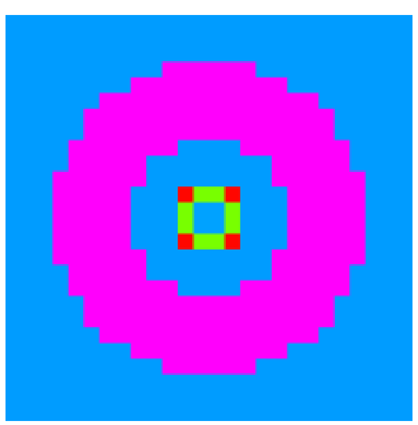
(b) Noisy (0 dB SNR)



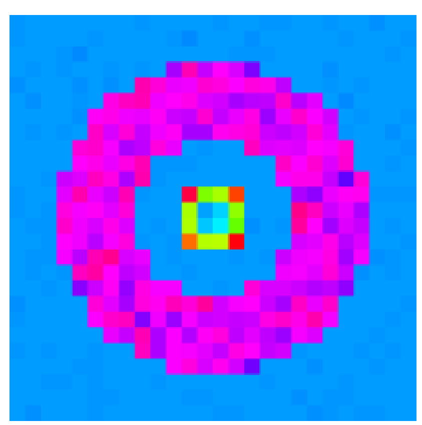
(c) $L_1$ denoised (11.70 dB improvement)



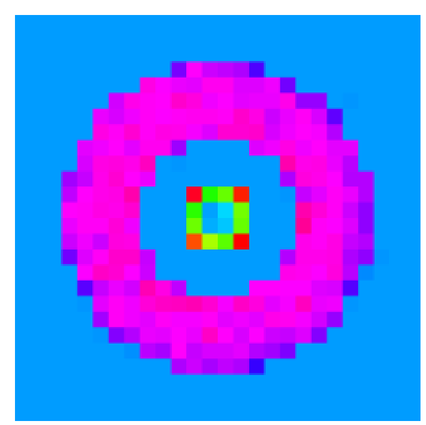
(d) $L_2$ denoised (11.04 dB improvement)



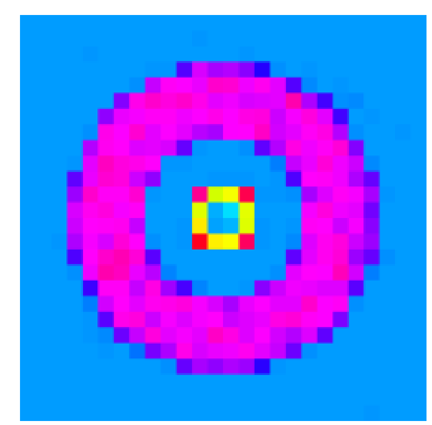
(a) Original



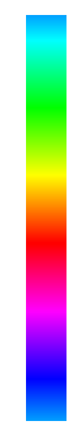
(b) Noisy (10 dB SNR)



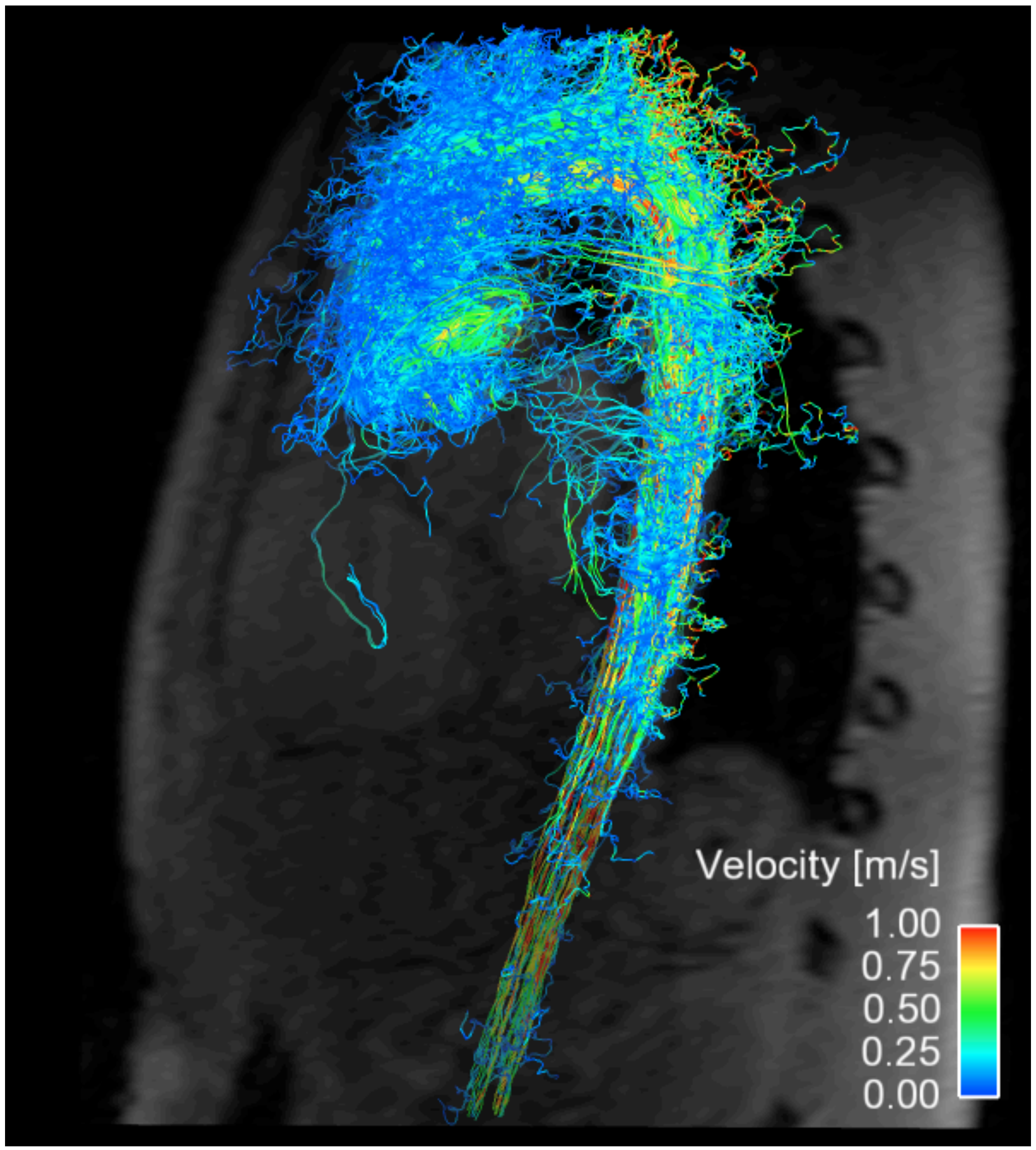
(c) $L_1$ denoised (7.96 dB improvement)



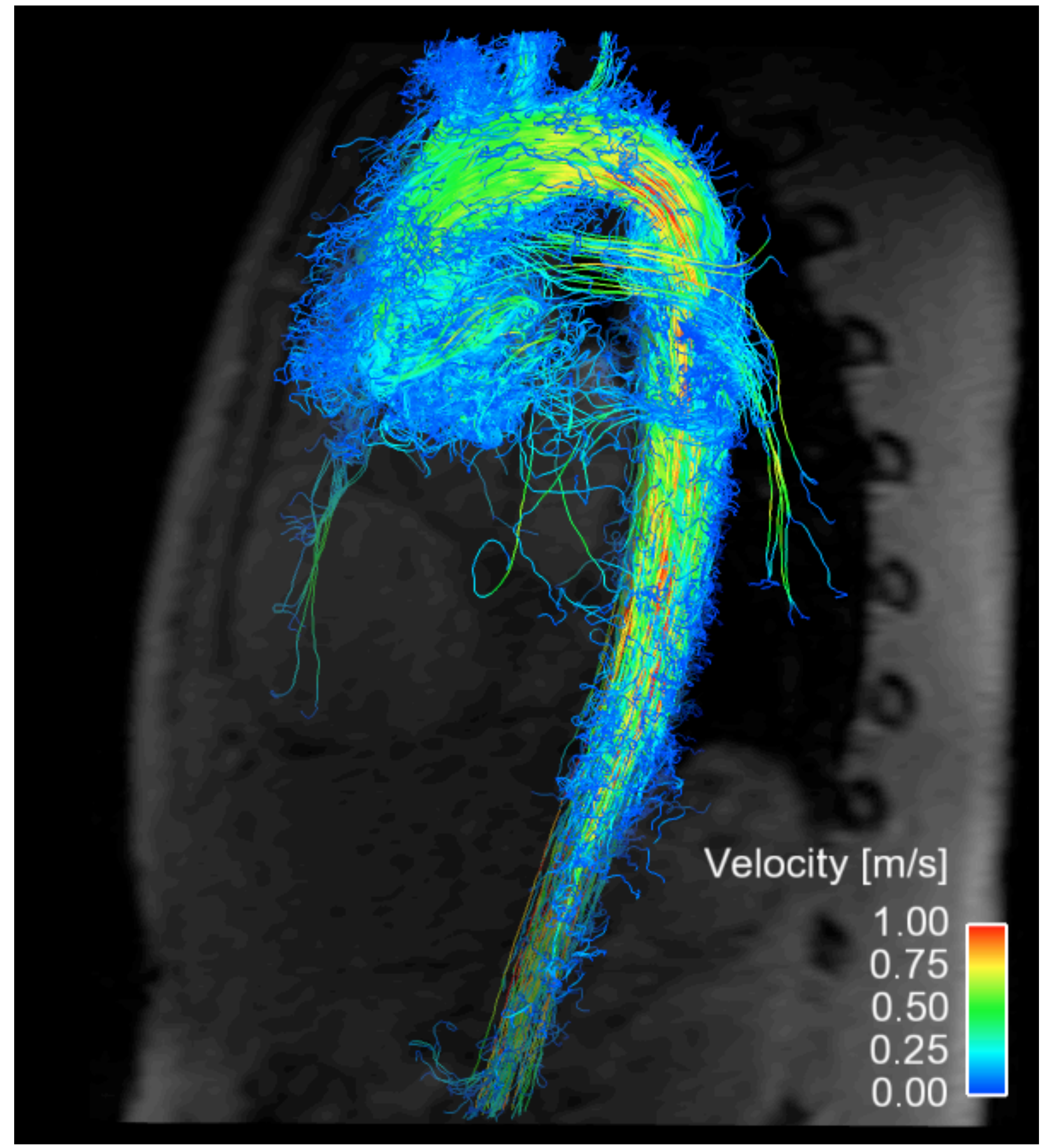
(d) $L_2$ denoised (2.55 dB improvement)



# Some Results (4)




Original pathlines




Enhanced pathlines

# Summary

- Innovation modelling framework
- Self-similar random vector field models
- Invariance-based vector field reconstruction algorithms

# Outlook

- Other innovation models (may involve other spaces beside $L_p$)
- Other operators (e.g. with local parameters)
- General formulation of invariance for tensors of any order
- Statistical interpretation of algorithm
- Other algorithms (primal-dual, etc.)
- Other modelling and reconstruction applications