Fractional Brownian Vector Fields

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Outline

Scalar fractional Brownian motion (fBm)

- Invariances
- Fractional PDE formulation (innovation model)

Fractional Brownian vector fields

- Vector invariances
- Generalized fractional Laplacians
- Characterization of vector fBm
- Some properties
- Parameter estimation with wavelets

Scalar Fractional Brownian Motion

Non-stationary random field on $\mathbb{R}^d$ with

- Gaussian statistics;
- zero mean;
- zero boundary conditions ($B_H(0) = 0$);
- *stationary* increments with variance

$$\boxed{\mathbb{E}\{|B_H(\boldsymbol{x}) - B_H(\boldsymbol{y})|^2\} \propto |\boldsymbol{x} - \boldsymbol{y}|^{2H}}$$

($H \in (0, 1)$: Hurst exponent).

Statistical invariances:

- Scaling:

$$S_\sigma \, B_H = \sigma^H B_H \qquad \text{in law},$$

$(S_\sigma : f \mapsto f(\sigma^{-1}\cdot),\ \sigma \in \mathbb{R}_+)$;

- Scalar rotation (and reflection):

$$R_\Omega^{\text{scalar}} \, B_H = B_H \qquad \text{in law},$$

$(R_\Omega^{\text{scalar}} : f \mapsto f(\Omega^T\cdot),\ \Omega$ orthogonal$)$.

Whitening/innovation modelling

- Characterization/generalization by means of a *whitening equation:*

$$\boxed{\mathrm{U}^* \, \mathrm{B}_\mathrm{H} = W}$$

  where:

  - W is *white Gaussian noise;*
  - U^* is the whitening operator.

$\Rightarrow$ Non-stationary generalization of spectral shaping.

Whitening/innovation modelling: Steps

1. Identify U (using invariances);

2. Find a *continuous* *linear* *left inverse* $\mathrm{L} : \mathcal{S} \to \mathcal{L}^2$:

$$\mathrm{LU} = \text{identity};$$

3. Define $\mathrm{B_H}$ as a particular solution (generalized random field):

$$\langle \mathrm{B_H}, \phi \rangle := \langle W, \mathrm{L}\phi \rangle \qquad (*)$$

Justification:

$$(*) \Longrightarrow \langle \mathrm{B_H}, \mathrm{U}\psi \rangle = \langle W, \mathrm{LU}\psi \rangle = \langle W, \psi \rangle$$
$$\Longrightarrow \boxed{\mathrm{U}^* \mathrm{B_H} = W}.$$

1. The fractional Laplacian $\mathrm{U}^\gamma \overset{\mathcal{F}}{\longleftrightarrow} \kappa_\gamma |\omega|^{2\gamma}$ satisfies

$$\mathrm{U}^\gamma \, \mathrm{S}_\sigma = \sigma^{2\gamma} \, \mathrm{S}_\sigma \, \mathrm{U}^\gamma \,; \qquad \text{(homogeneity)}$$

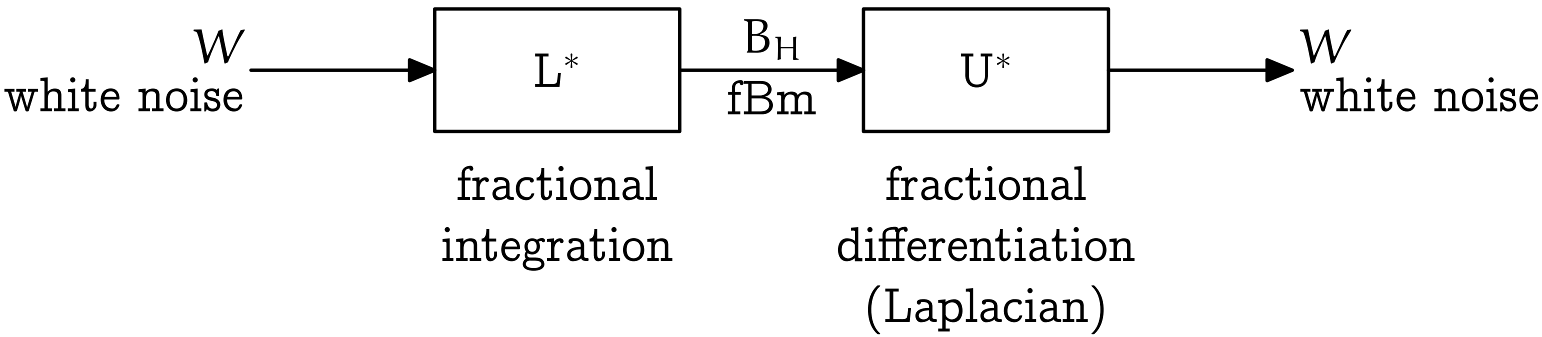
$$\mathrm{U}^\gamma \, \mathrm{R}_\Omega^{\mathrm{scalar}} = \mathrm{R}_\Omega^{\mathrm{scalar}} \, \mathrm{U}^\gamma \,. \qquad \text{(rotation invariance)}$$

2. Continuous linear left inverse ($\mathcal{S} \to \mathcal{L}_2$):

$$\mathrm{L}^\gamma : f \mapsto \frac{1}{\kappa_\gamma (2\pi)^d} \int_{\mathbb{R}^d} e^{j\langle x,\omega\rangle} \frac{1}{|\omega|^{2\gamma}} \left(\hat{f}(\omega) - \sum_{|k| \leqslant \lfloor 2\gamma - \frac{d}{2} \rfloor} \frac{\hat{f}^{(k)}(0)\omega^k}{k!} \right) d\omega .$$

Invariances: Like U, L is *homogeneous* and *rotation-invariant*.

3. Innovation/whitening model:



- Captures the inverse power-law spectrum of B_H;
- Generalizes to $H > 1$;
- Non-Gaussian $W \Rightarrow$ non-Gaussian models à la Lévy motion (may need to redefine L).

Fractional Brownian Vector Fields

Fractional Brownian vector fields

How to define fractional Brownian *vector* fields?

- **Trivial definition:** Vector of independent scalar fBms.

  No constraints on the interdependency of the components;

$\Rightarrow$ Hence no control over directional behaviour.

- **Solution:** More general definition based on *invariances.*

- *Vector rotaion:* Rotate the domain, but keep directions fixed.

  Rotation by $\Omega \in O(n)$:

  $$R_\Omega^{\text{vector}} : \mathbf{f} \mapsto \Omega\, \mathbf{f}(\Omega^T \cdot).$$

- Desired invariances for vector fBm:

  $$S_\sigma\, B_H = \sigma^H\, B_H \qquad \text{in law,}$$

  $$R_\Omega^{\text{vector}}\, B_H = B_H \qquad \text{in law.}$$

Idea: Whitening/innovation model as before:

$$\mathrm{U}^* \mathbf{B}_{\mathrm{H}} = W,$$

$\boldsymbol{W}$: vector of white noises; U is:

- Homogeneous:

$$\mathrm{U}\ \boxed{\mathrm{S}_\sigma} = \sigma^{2\gamma}\ \boxed{\mathrm{S}_\sigma}\ \mathrm{U}\,;$$

- Vector rotation invariant:

$$\mathrm{U}\ \boxed{\mathrm{R}_\Omega^{\mathrm{vector}}} = \boxed{\mathrm{R}_\Omega^{\mathrm{vector}}}\ \mathrm{U} \quad .$$

Fractional vector Laplacians (1)

Theorem (Arigovindan & Unser '05, PDT & Unser '10): A vector convolution operator with the said invariances has a Fourier multiplier of the form

$$\mathrm{U}^{\gamma}_{(\xi_1,\xi_2)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \kappa_\gamma \Phi^{\gamma}_{\xi}(\omega) := \kappa_\gamma |\omega|^{2\gamma} \left[e^{\xi_1} \frac{\omega\omega^{\mathrm{T}}}{|\omega|^2} + e^{\xi_2} \left(\mathrm{I} - \frac{\omega\omega^{\mathrm{T}}}{|\omega|^2} \right) \right].$$

Interpretation:

$$|\omega|^{2\gamma} \quad : \quad \text{fractional Laplacian}$$

$$\frac{\omega\omega^{\mathrm{T}}}{|\omega|^2} \quad : \quad \text{projection onto the curl-free component}$$

$$\mathrm{I} - \frac{\omega\omega^{\mathrm{T}}}{|\omega|^2} \quad : \quad \text{projection onto the div-free component}$$

Fractional vector Laplacians (2)

Properies of Φ_{ξ}^{γ}:

- Homogeneity: $S_\sigma \, \Phi_{\xi}^{\gamma} = \sigma^{2\gamma} \, \Phi_{\xi}^{\gamma}$;
- Rotation contra-variance: $R_{\Omega}^{\text{vector}} \, \Phi_{\xi}^{\gamma} = \Phi_{\xi}^{\gamma}(\cdot) \, \Omega$;
- Inversion: $\Phi_{\xi}^{\gamma}(\omega) \, \Phi_{-\xi}^{-\gamma}(\omega) = 1$, $\omega \neq 0$;
- Fourier transform: $\mathcal{F}\{\Phi_{\xi}^{\gamma}\} = \Phi_{\hat{\xi}}^{-\gamma-d}$;
- Products: $\Phi_{\xi_1}^{\gamma_1} \, \Phi_{\xi_1}^{\gamma_1} = \Phi_{\xi_1+\xi_2}^{\gamma_1+\gamma_2}$.

Fractional vector Laplacians (3)

- Continuous linear left inverse defined same as before:

$$L_{\xi}^{\gamma} : f \mapsto \frac{1}{\kappa_{\gamma}(2\pi)^d} \int_{\mathbb{R}^d} e^{j\langle x, \omega \rangle} \Phi_{-\xi}^{-\gamma}(\omega) \left(\hat{f}(\omega) - \sum_{|k| \leqslant \lfloor 2\gamma - \frac{d}{2} \rfloor} \frac{\hat{f}^{(k)}(0)\omega^k}{k!} \right) d\omega.$$

Key properties:

- Homogeneous;
- Vector rotation invariant;
- Continuous $\mathcal{S}^d \to \mathcal{L}_2^d$.

Self-similar and rotation invariant solution of

$$(\mathrm{U}_{(\xi_1,\xi_2)}^{\frac{H}{2}+\frac{d}{4}})^* \, \mathbf{B}_{H,\xi} \;=\; \boldsymbol{W};$$

($\boldsymbol{W}$ is vector of white noise).

- Coordinates are no longer independent (unless $\xi_1 = \xi_2$).
- $\xi_1 - \xi_2$ controls vectorial behaviour:

  $\xi_1 - \xi_2 \to +\infty$: solenoidal (div-free);

  $\xi_1 - \xi_2 \to -\infty$: irrotational (curl-free).
- Interpreted as a *generalized random field* (Gel'fand *& al.*).

Generalized random fields (1)

- $\langle \mathbf{B}_{H,\xi}, \boldsymbol{\phi} \rangle$, $\boldsymbol{\phi} \in \mathcal{S}^d$, are R.V.s with consistent finite-dimensional prob. measures.

- The *stochastic law* (prob. measure) of $\mathbf{B}_{H,\xi}$ is derived from its *characteristic functional:*

  Theorem (Bochner-Minlos): There is a one-to-one correspondence between positive-definite and continuous *characteristic functionals* $Z_B(\phi)$, $\phi \in \mathcal{E}$ (a nuclear space), and probability measures P_B on $\mathcal{E}'$, via the relation

$$Z_B(\phi) = \mathbb{E}\{e^{j\langle B,\phi\rangle}\} = \int_{\mathcal{E}'} e^{j\langle \chi,\phi\rangle}\, P_B(d\chi).$$

Generalized random fields (2)

Example (white Gaussian noise):

$$Z_W(\boldsymbol{\phi}) = \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{\phi}\|^2}$$

Properties:

- Independent values at every point (whiteness):

$$\langle \boldsymbol{W}, \boldsymbol{\phi} \rangle,\ \langle \boldsymbol{W}, \boldsymbol{\psi} \rangle \text{ independent if } \operatorname{Supp} \boldsymbol{\phi} \cap \operatorname{Supp} \boldsymbol{\psi} = \varnothing;$$

- Jointly Gaussian finite-dim. distributions for all

$$\langle \boldsymbol{W}, \boldsymbol{\phi}_i \rangle, \quad 1 \leqslant i \leqslant N.$$

Reminder: Solution in the sense of distributions

$$\langle \mathbf{B}_{H,\xi}, \boldsymbol{\phi} \rangle := \langle W, \mathrm{L}_{\xi}^{\frac{H}{2}+d4} \boldsymbol{\phi} \rangle \quad \Longrightarrow \quad (\mathrm{U}_{\xi}^{\frac{H}{2}+\frac{d}{4}})^{*} \, \mathbf{B}_{H,\xi} = W.$$

Characteristic functional:

$$\begin{aligned} Z_{\mathbf{B}_{H,\xi}}(\boldsymbol{\phi}) &= \mathbb{E}\{e^{j\langle \mathbf{B}_{H,\xi}, \boldsymbol{\phi}\rangle}\} \\ &= \mathbb{E}\{e^{j\langle W, \mathrm{L}\boldsymbol{\phi}\rangle}\} \\ &= Z_{W}(\mathrm{L}_{-\xi}^{-\frac{H}{2}-\frac{d}{4}} \boldsymbol{\phi}) \end{aligned}$$

(requires continuity $\mathcal{S}^{d} \to \mathcal{L}_{2}^{d}$).

Some properties of vector fBm (1)

Scale and rotation invariance of $L_{\xi}^{\frac{H}{2}+\frac{d}{4}}$ $\Longrightarrow$

- Self-similarity:

$$S_{\sigma}\, \mathbf{B}_{H} = \sigma^{H}\, \mathbf{B}_{H} \qquad \text{in law;}$$

- Rotation invariance:

$$R_{\Omega}^{\text{vector}}\, \mathbf{B}_{H} = \mathbf{B}_{H} \qquad \text{in law.}$$

Some properties of vector fBm (2)

- Generalization to $H > 1$

$$\mathbf{B}_{H,\xi} \;=\; (L_{\xi}^{\frac{H}{2}+\frac{d}{4}})^{*}\,\boldsymbol{W}$$

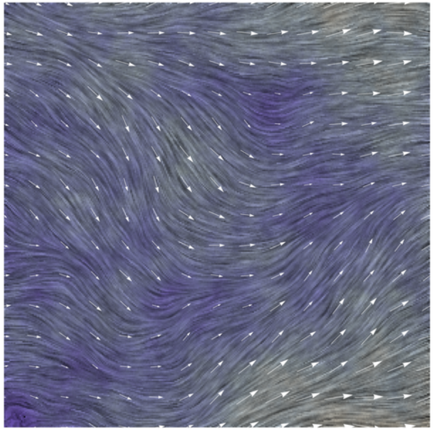
also valid for $H > 1$ (non-integer).

- Stationary nth-order increments for $n \geqslant \lfloor H \rfloor + 1$;

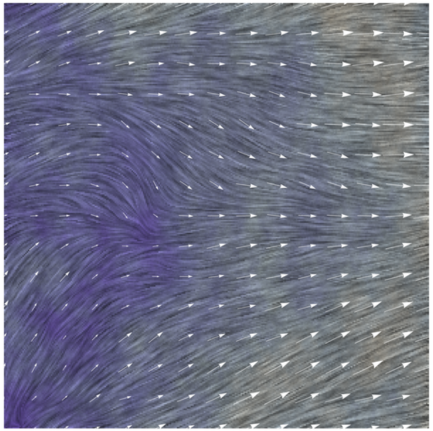
- Covariance structure of increments for $0 < H < 1$:

$$\mathbb{E}\{\left[\mathbf{B}_{H,\xi}(\boldsymbol{x}) - \mathbf{B}_{H,\xi}(\boldsymbol{y})\right]\left[\mathbf{B}_{H,\xi}(\boldsymbol{x}) - \mathbf{B}_{H,\xi}(\boldsymbol{y})\right]^{T}\} \propto \Phi^{H}_{(\eta_1,\eta_2)}(\boldsymbol{x}-\boldsymbol{y})$$

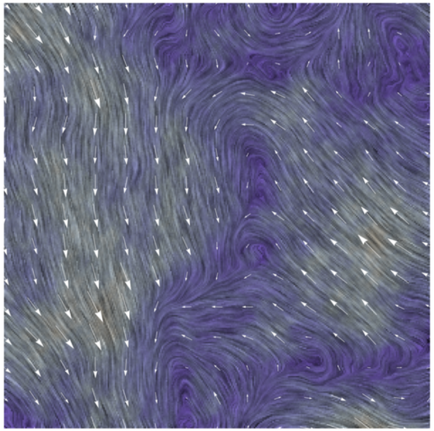
- Vectorial behaviour:
  - $\xi_1 - \xi_2 \to +\infty \quad \Rightarrow$ div-free;
  - $\xi_1 - \xi_2 \to -\infty \quad \Rightarrow$ curl-free;
  - $\xi_1 = \xi_2 \quad \Rightarrow$ independent coordinates.



(a) $H = 0.60,\ \xi_1 = \xi_2 = 0$
(indep. coordinates)



(b) $H = 0.60,\ \xi_1 = 0,\ \xi_2 = 100$
(curl-free)



(c) $H = 0.60,\ \xi_1 = 100,\ \xi_2 = 0$
(div-free)

Wavelet analysis of vector fBm (1)

VECTOR WAVELETS

Let $\mathrm{E} \overset{\mathcal{F}}{\longleftrightarrow} \boldsymbol{\omega}\boldsymbol{\omega}^{\mathrm{T}}/|\boldsymbol{\omega}|^2$ (curl-free projection).

Define vector wavelets (matrix-valued):

- Smoothing kernel Φ (matrix-valued, usu. diagonal);
- Wavelets:

$$\Psi \;=\; \mathrm{U}^{\gamma}\Phi \;=\; \mathrm{U}^{\gamma}\big[\mathrm{E} + (\mathrm{Id} - \mathrm{E})\big]\Phi$$

$$= \underbrace{\mathrm{U}^{\gamma}\mathrm{E}\Phi}_{\Psi_1\text{: captues curl-free comp.}} + \underbrace{\mathrm{U}^{\gamma}(\mathrm{Id} - \mathrm{E})\Phi}_{\Psi_2\text{: captues div-free comp.}} .$$

Wavelet analysis of vector fBm (2)

Parameter Estimation

- $\log$(wavelet energy) varies linearly across scales; slope depends on H.

$\Rightarrow$ Estimates of H.

- Ratio between Ψ_1 and Ψ_2 energy depends on $\xi_1 - \xi_2$.

$\Rightarrow$ Estimates of vectorial character $(\xi_1 - \xi_2)$.

Thank you.