ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Vers une théorie unificatrice pour le traitement numérique/analogique des signaux

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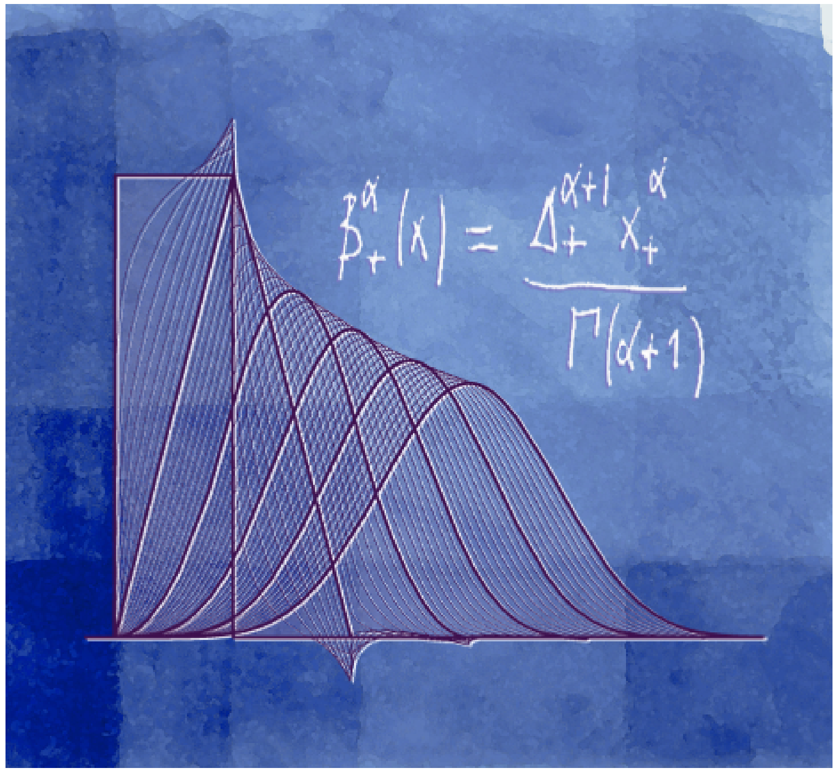


GRETSI'05, 6-9 Septembre, Louvain-la-Neuve, Belgique

Is continuous-time signal processing dead ?

- Arguments in favor of its suppression:
  - The modern world is discrete (CDs, DVDs, WEB, etc…)
  - Modern SP courses concentrate on digital signal processing
  - Most processing is discrete (DSPs, PCs, etc…)
  - Students don't like the Laplace transform...
- However…
  - Real-world signals are continuous
  - Often, the end product is analog: control systems, sound reproduction systems, etc.
  - Don't forget the interface: A-to-D and D-to-A
  - Some discrete algorithms require continuous-time thinking

Revival of continuous-time thinking

- Recent trends in SP
  - Wavelet theory, multiresolution analysis
  - Self-similarity, fractals, analysis of singularities
  - Partial differential equations
  - Spline-based signal processing
- Continuous/discrete formulation
  - “**Think analog, act digital**”
  - Applications:
    - Fractional delays, sampling rate conversion
    - Discretization of differential operators
    - Interpolation
    - ...

OUTLINE

- In search of the missing link
- E-splines
- B-spline calculus
- Application: hybrid signal processing

IN SEARCH OF THE MISSING LINK

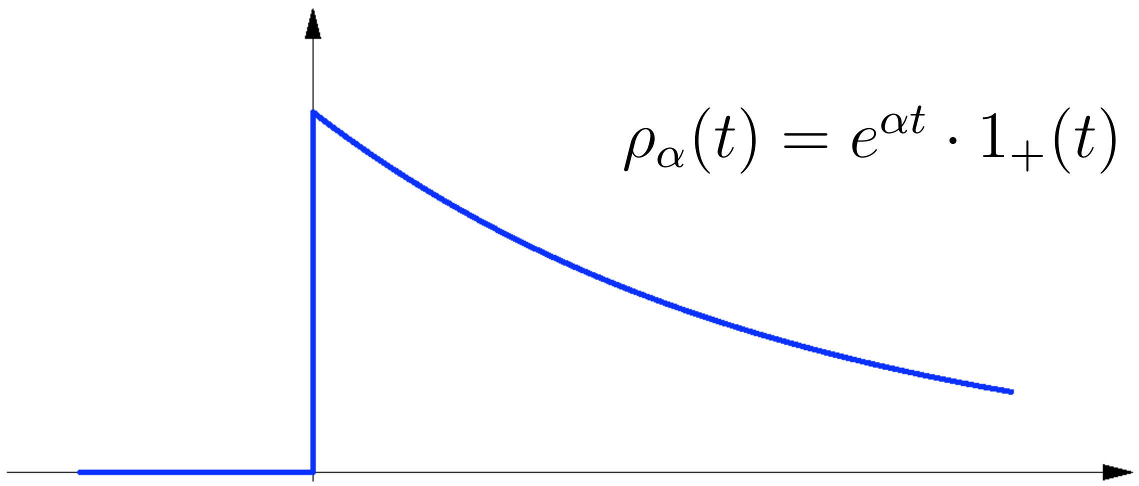
Knowing splines is an advantage

[Schoenberg, 1946]

Teach “Signals and Systems” …

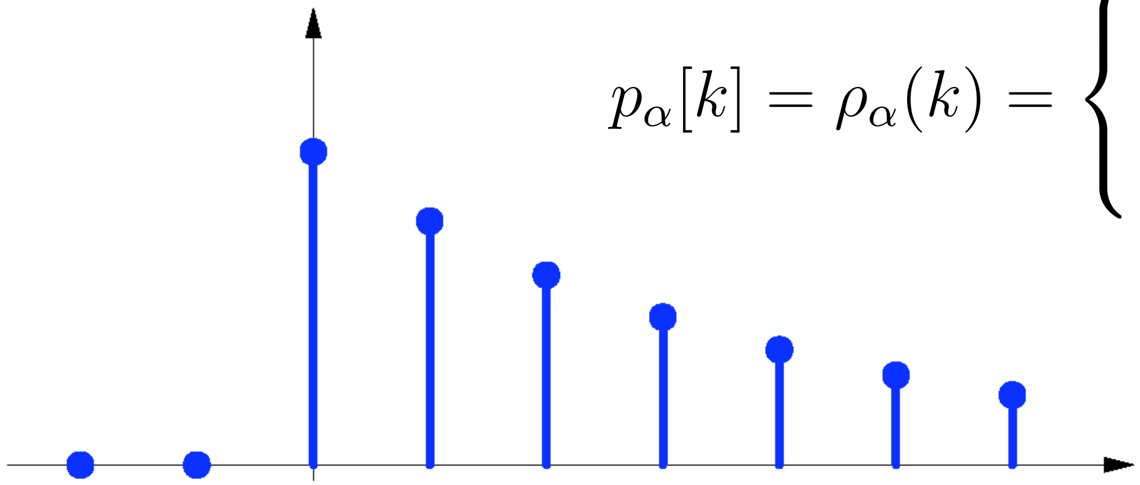
Continuous vs discrete: example

- Causal exponential
  - Continuous-time version



$$\rho_\alpha(t) = e^{\alpha t} \cdot 1_+(t) = \begin{cases} e^{\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \hat{\rho}_\alpha(\omega) = \frac{1}{j\omega - \alpha}$$

  - Discrete-time version



$$p_\alpha[k] = \rho_\alpha(k) = \begin{cases} e^{\alpha k}, & k \geq 0 \\ 0, & k < 0 \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad P_\alpha(z) = \frac{1}{1 - e^{\alpha} z^{-1}}$$

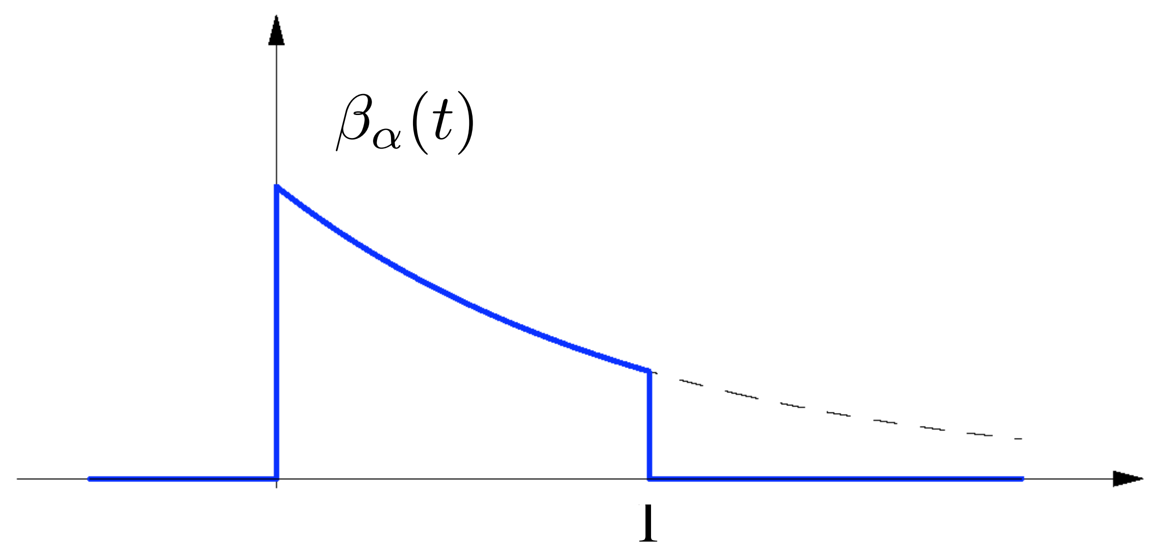
What is the link ?

- Answer: ratio of Fourier transforms

$$\hat{\beta}_\alpha(\omega) = \frac{\hat{\rho}_\alpha(\omega)}{P_\alpha(e^{j\omega})} = \frac{1 - e^{\alpha} e^{-j\omega}}{j\omega - \alpha}$$

$\mathcal{F}^{-1}$

$$\beta_\alpha(t) = \rho_\alpha(t) - e^{\alpha} \cdot \rho_\alpha(t-1)$$



- Reproduction formula

Discrete signal

$$\rho_\alpha(t) = 1_+(t) \cdot e^{\alpha t} = \sum_{k=0}^{+\infty} e^{\alpha k} \beta_\alpha(t-k) = \sum_{k \in \mathbb{Z}} p_\alpha[k] \, \beta_\alpha(t-k)$$

Continuous-time signal

Compactly-supported basis functions

Basic continuous-time convolution operators

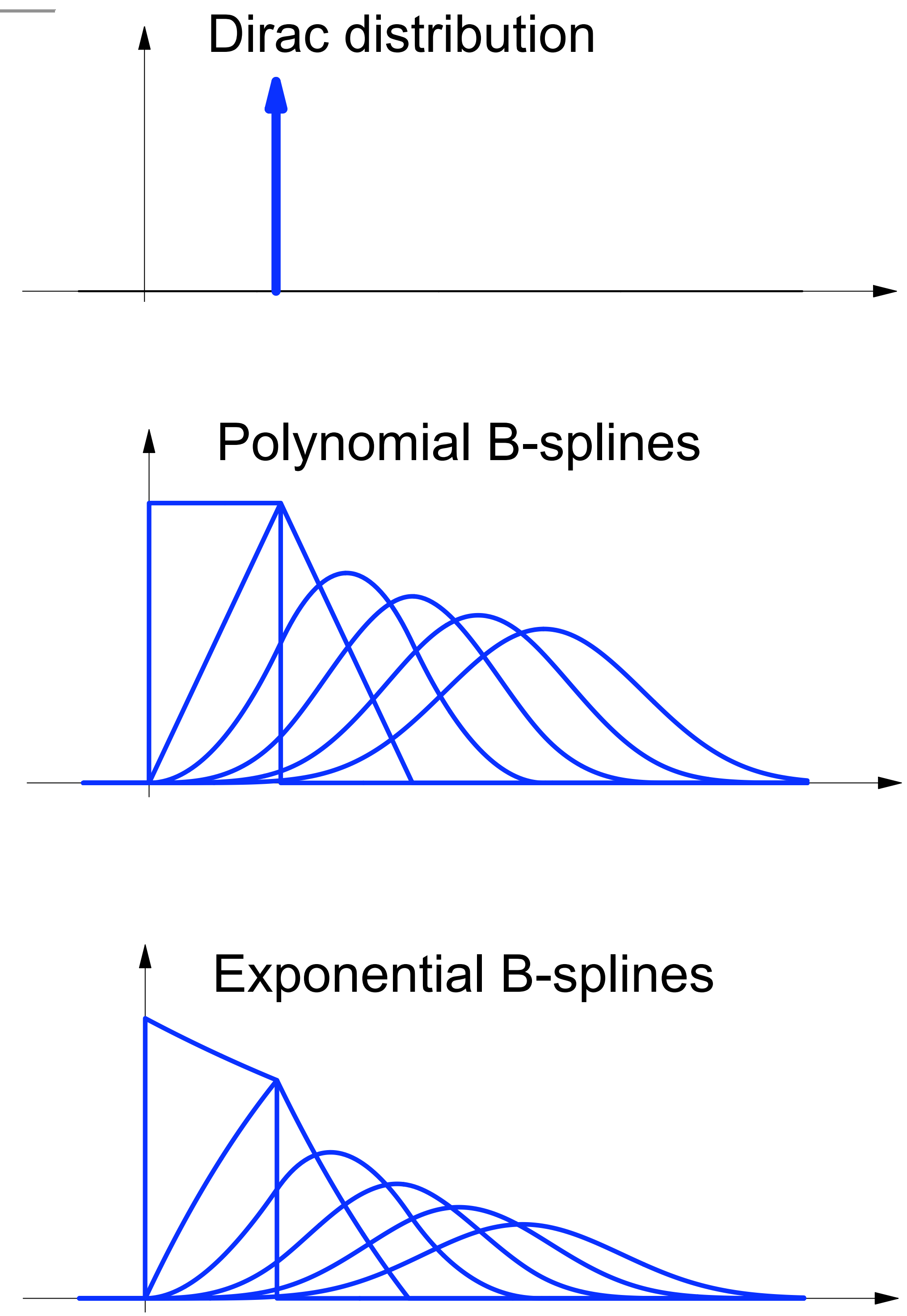
Operator	Notation	Impulse response	Frequency response
Identity	$\mathrm{I}\{\ \}$	$\delta(t)$	1
Shift	$S_\tau\{f\} = f(t-\tau)$	$\delta(t-\tau)$	$e^{-j\omega\tau}$
Integral	$\mathrm{D}^{-1}\{\ \} = \int_{-\infty}^{t} dt$	$1_+(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
Multiple integral	$\mathrm{D}^{-n}\{\ \}$	$\frac{t_+^{n-1}}{(n-1)!}$	$\frac{j^{n-1}\pi\delta^{(n-1)}(\omega)}{(n-1)!} + \frac{1}{(j\omega)^n}$
Simple differential system	$(\mathrm{D}-\alpha\mathrm{I})^{-1}\{\ \}$	$1_+(t)\cdot e^{\alpha t}$	$\frac{1}{j\omega-\alpha}$ $\quad \mathrm{Re}\{\alpha\} < 0$
Iterated differential system	$(\mathrm{D}-\alpha\mathrm{I})^{-n}\{\ \}$	$\frac{t_+^{n-1}e^{\alpha t}}{(n-1)!}$	$\frac{1}{(j\omega-\alpha)^n}$ $\quad \mathrm{Re}\{\alpha\} < 0$

... and their discrete counterparts

Name	Discrete time specification	z-transform
Unit impulse	$\delta[k]$	1
Shift	$\delta[k-k_0]$	z^{-k_0}
Unit step	$p_0[k]=\begin{cases}0, & k<0\\ 1, & k\geq 0\end{cases}$	$\frac{1}{1-z^{-1}}$
Discrete mononial	$p_0^{[n-1]}[k]=\begin{cases}0, & k<0\\ \prod_{m=1}^{n-1}(k+m), & k\geq 0\end{cases}$	$\frac{1}{\left(1-z^{-1}\right)^n}$
Causal exponential	$p_\alpha[k]=\begin{cases}0, & k<0\\ e^{\alpha k}, & k\geq 0\end{cases}$	$\frac{1}{1-e^{\alpha}z^{-1}}$
Discrete exponential monomial	$p_\alpha^{[n-1]}[k]=\begin{cases}0, & k<0\\ e^{\alpha k}\prod_{m=1}^{n-1}(k+m), & k\geq 0\end{cases}$	$\frac{1}{\left(1-e^{\alpha}z^{-1}\right)^n}$

D-to-A translating B-splines

B-spline	Operator L	Order N	Frequency response
$\delta(t)$	$I\{\ \}$	0	1
$\delta(t-\tau)$	$S_\tau\{\ \}$	0	$e^{-j\omega\tau}$
$\beta_{(0)}(t)$	$D\{\ \}=\frac{d}{dt}$	1	$\frac{1-e^{-j\omega}}{j\omega}$
$\beta_{(0,\cdots,0)}(t)$	$D^n\{\ \}$	n	$\left(\frac{1-e^{-j\omega}}{j\omega}\right)^n$
$\beta_\alpha(t)$	$(D-\alpha I)\{\ \}$	1	$\frac{1-e^{\alpha-j\omega}}{j\omega-\alpha}$
$\beta_{(\alpha,\cdots,\alpha)}(t)$	$(D-\alpha I)^n\{\ \}$	n	$\left(\frac{1-e^{\alpha-j\omega}}{j\omega-\alpha}\right)^n$



E-SPLINES

- Generalized splines
- Exponential B-splines
- B-spline properties
- B-spline representation

General concept of an *L*-spline

$\mathrm{L}\{\cdot\}$: differential operator (shift-invariant)

$\delta(t)$: Dirac distribution

Definition A: The continuous-time function $s(t)$ is an ***L-spline*** with knots $\{t_k\}_{k\in\mathbb{Z}}$ iff:

$$\mathrm{L}\{s(t)\} = \sum_{k\in\mathbb{Z}} a_k \delta(t - t_k)$$

Definition B: The continuous-time function $s(t)$ is a ***cardinal L-spline*** iff:

$$\mathrm{L}\{s(t)\} = \sum_{k\in\mathbb{Z}} a[k] \delta(t - k)$$

Exponential spline defining operator

- General differential system

$$\left(\mathrm{D}^N + a_1\mathrm{D}^{N-1} + \cdots a_N\mathrm{I}\right)\{y(t)\} = \left(\mathrm{D}^M + \cdots b_M\mathrm{I}\right)\{x(t)\}$$

$$\Longleftrightarrow \quad \mathrm{L}_{\vec{\alpha}}\{y(t)\} = x(t)$$

- Rational transfer function

$$L_{\vec{\alpha}}(\omega) = \frac{\prod_{n=1}^{N}(j\omega - \alpha_n)}{\prod_{m=1}^{M}(j\omega - \gamma_m)}$$

- Exponential spline parameters

$$\vec{\alpha} = (\underbrace{\alpha_1, \cdots, \alpha_N}_{\text{Poles}}; \underbrace{\gamma_1, \cdots, \gamma_M}_{\text{Zeros (optional)}})$$ with $M < N$

Example: piecewise-constant splines

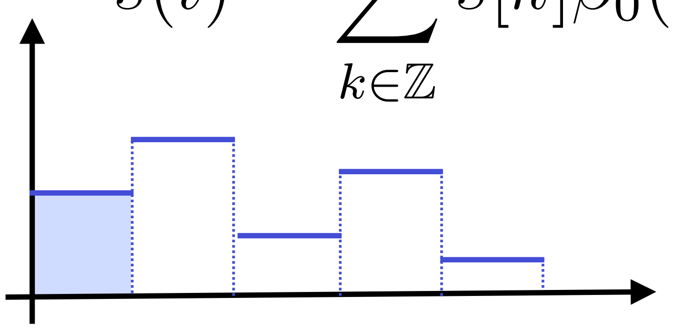
- Spline-defining operators

Continuous-time derivative: $\mathrm{D} = \mathrm{L}_0\{\cdot\} \quad \longleftrightarrow \quad j\omega$

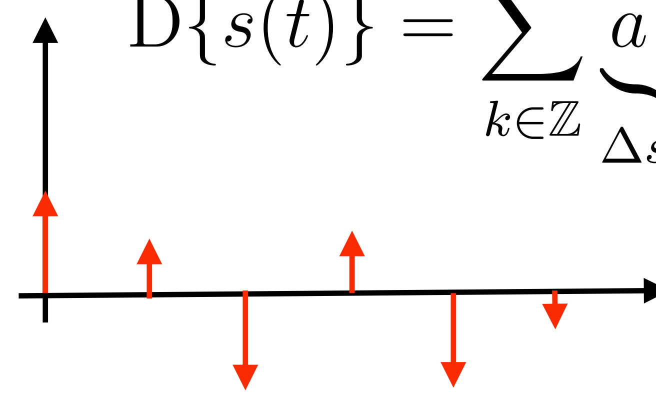
Discrete-time derivative: $\Delta\{\cdot\} \quad \longleftrightarrow \quad 1 - e^{-j\omega}$

- Piecewise constant or D-spline

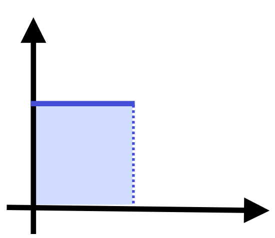
$$s(t) = \sum_{k \in \mathbb{Z}} s[k] \beta_0(t-k)$$



$$\mathrm{D}\{s(t)\} = \sum_{k \in \mathbb{Z}} \underbrace{a[k]}_{\Delta s(k)} \delta(t-k)$$



- B-spline function:



$$\beta_0(t) = \Delta\{1_+(t)\} \quad \longleftrightarrow \quad \frac{1 - e^{-j\omega}}{j\omega}$$

Exponential B-splines

- Localization operator (weighted finite differences)

$$\Delta_{\vec{\alpha}}(z) = \prod_{n=1}^{N} (1 - e^{\alpha_n} z^{-1})$$

Mapping: $z = e^s$

- Fourier domain formula

$$\hat{\beta}_{\vec{\alpha}}(\omega) = \frac{\Delta_{\vec{\alpha}}(e^{j\omega})}{L_{\vec{\alpha}}(\omega)}$$

Discrete-time approximation

Continuous-time operator

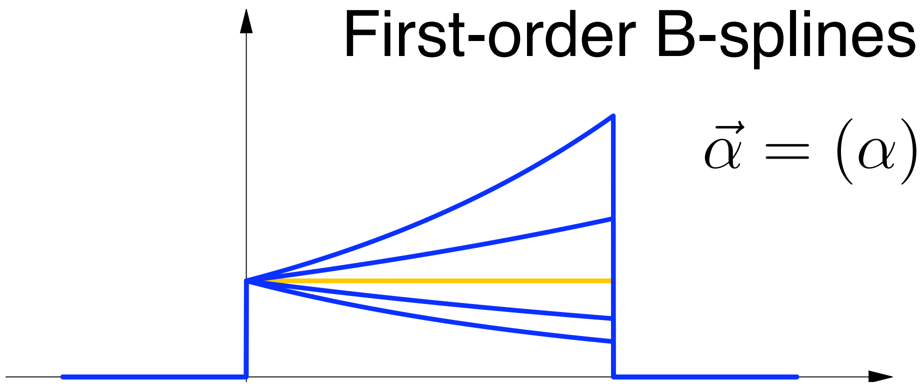
- Time-domain formula (inverse Laplace transform)

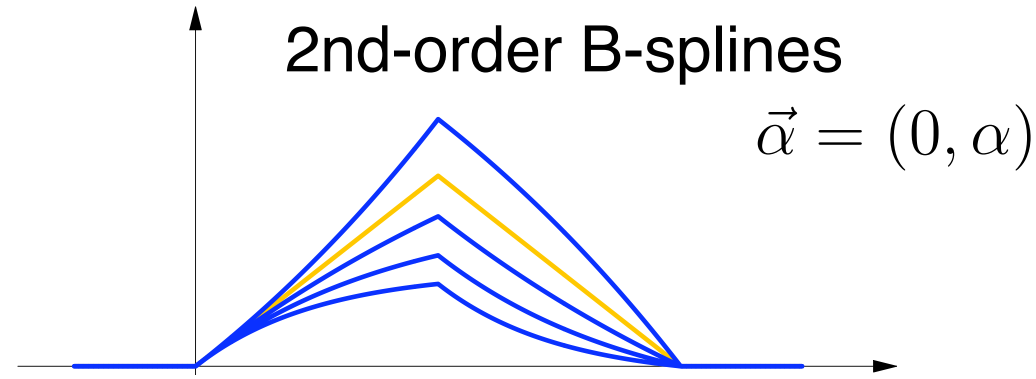
$$\beta_{\vec{\alpha}}(t) = \mathcal{L}^{-1}\left\{ \left(\prod_{n=1}^{N} \frac{1 - e^{\alpha_n - s}}{s - \alpha_n} \right) \cdot \prod_{m=1}^{M} (s - \gamma_m) \right\}$$

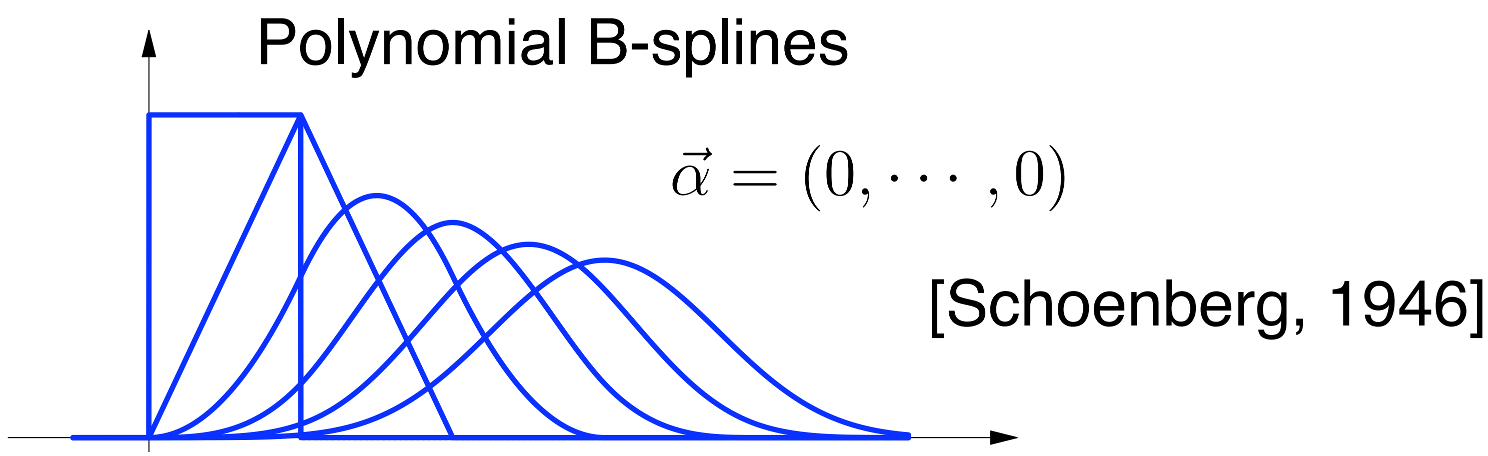
poles

zeros

Exponential B-splines (Cont'd)







- Properties
  - Piecewise exponential/polynomial (E-spline)
  - Compact support: size N
  - Continuity: Hölder of order $n = N - M - 1$

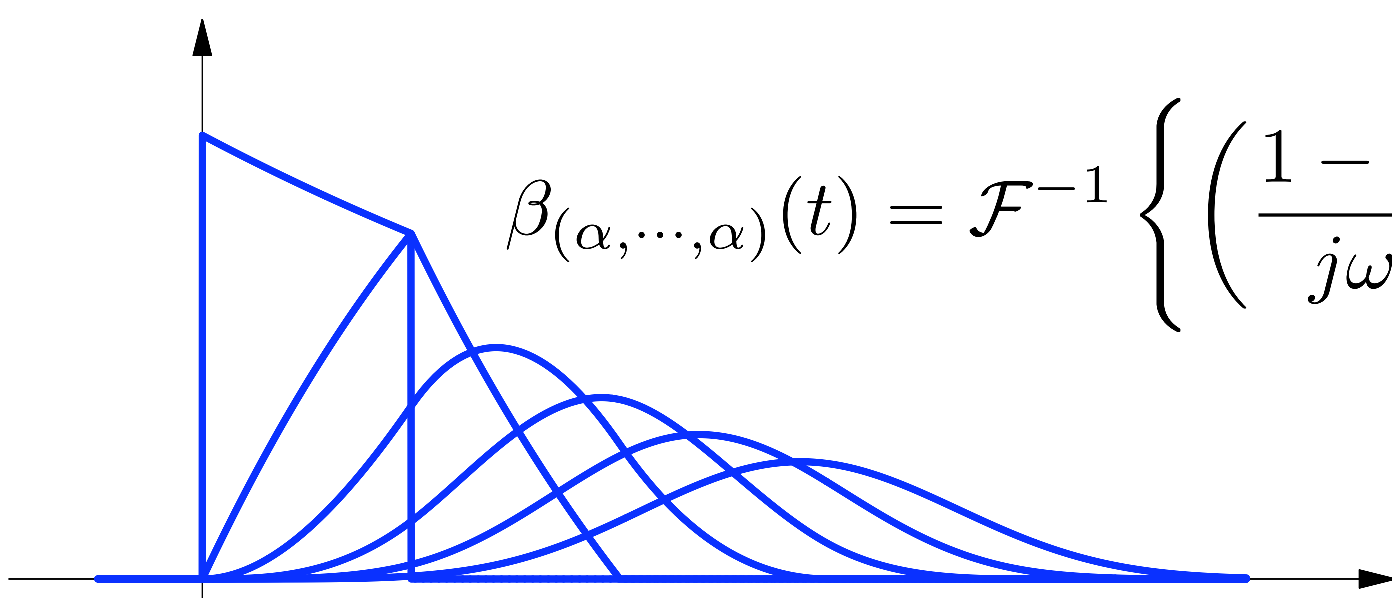
B-spline convolution property

- Convolution property

$$(\beta_{\vec{\alpha}_1} * \beta_{\vec{\alpha}_2})(t) = \beta_{(\vec{\alpha}_1 : \vec{\alpha}_2)}(t)$$

$$(\vec{\alpha}_1 : \vec{\alpha}_2) = \left(\underbrace{\alpha_{1,1}, \ldots, \alpha_{1,N_1}, \alpha_{2,1}, \ldots, \alpha_{2,N_2}}_{\text{concatenation of poles}};\ \underbrace{\gamma_{1,1}, \ldots, \gamma_{1,M_1}, \gamma_{2,1}, \ldots, \gamma_{2,M_2}}_{\text{concatenation of zeros}}\right)$$

- Example: g-splines [Panda et al., 1996]



$$\beta_{(\alpha,\cdots,\alpha)}(t) = \mathcal{F}^{-1}\left\{\left(\frac{1 - e^{\alpha - j\omega}}{j\omega - \alpha}\right)^N\right\}$$

E-splines: B-spline representation

- Space of cardinal E-splines

$$V_{\vec{\alpha}} = \left\{ s(t) : \mathrm{L}_{\vec{\alpha}}\{s(t)\} = \sum_{k\in\mathbb{Z}} a[k]\delta(t-k) \right\} \cap L_2$$

- B-spline representation

Theorem: The set of functions $\{\beta_{\vec{\alpha}}(t-k)\}_{k\in\mathbb{Z}}$ provides a Riesz basis of $V_{\vec{\alpha}}$ if and only if $\alpha_n - \alpha_m \neq j2\pi k, k \in \mathbb{Z}$ for all pairs of distinct, purely imaginary poles.

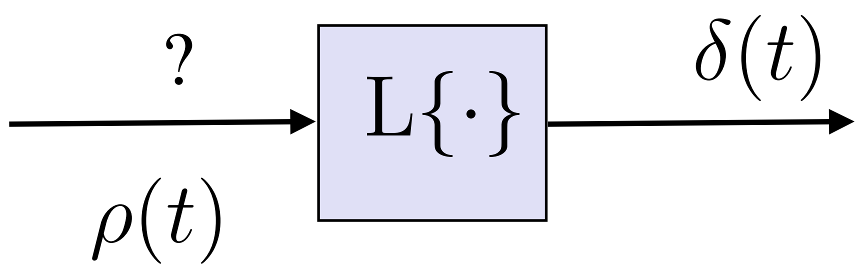
$$V_{\vec{\alpha}} = \left\{ s(t) = \sum_{k\in\mathbb{Z}} c[k]\beta_{\vec{\alpha}}(t-k) : c \in \ell_2 \right\}$$

continuous-time signal ($s(t)$)

discrete-time signal (B-spline coefficients) ($c[k]$)

Green function reproduction

- Green function



$\rho(t)$: Green function of $\mathrm{L}\{\cdot\}$

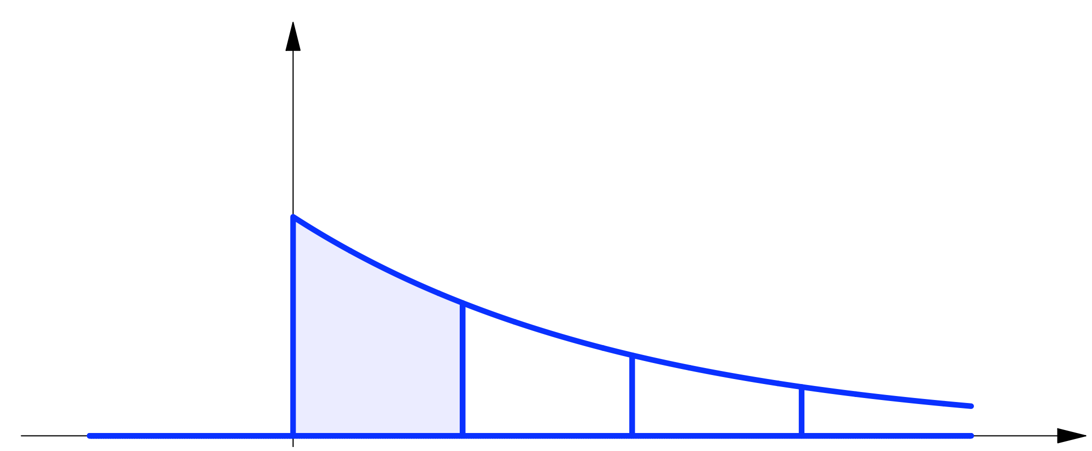
$$\Updownarrow$$

$$\mathrm{L}\{\rho(t)\} = \delta(t)$$

- Green function reproduction = A-to-D translation

$$\rho_{\vec{\alpha}}(t) = \sum_{k \in Z} p_{\vec{\alpha}}[k] \beta_{\vec{\alpha}}(t-k)$$

with $$P_{\vec{\alpha}}(z) = \prod_{n=1}^{N} \frac{1}{1 - e^{\alpha_n} z^{-1}}$$



B-SPLINE CALCULUS

- Interpolation
- Convolution
- Modulation
- Differential operators
- ...

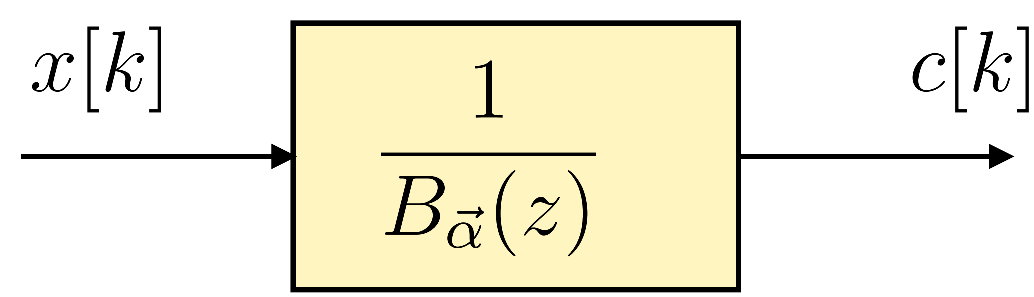
Interpolation

- Interpolation condition

$$x[k] = \sum_{n \in \mathbb{Z}} c[n] \beta_{\vec{\alpha}}(t-n) \Big|_{t=k} = (b_{\vec{\alpha}} * c)\,[k]$$

- B-spline kernel: $B_{\vec{\alpha}}(z) = \sum_{k=0}^{N-1} \beta_{\vec{\alpha}}(k) z^{-k}$

- Digital filtering algorithm



Recursive IIR filter

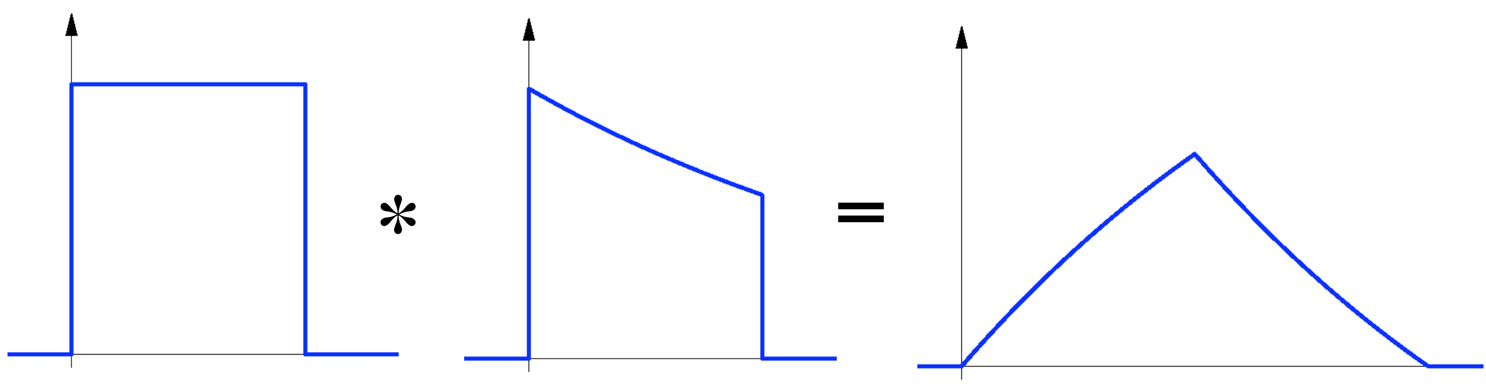
Convolution

- Input signals

$$s_1(t) = \sum_{k\in\mathbb{Z}} c_1[k]\beta_{\vec{\alpha}_1}(t-k) \qquad s_2(t) = \sum_{k\in\mathbb{Z}} c_2[k]\beta_{\vec{\alpha}_2}(t-k)$$

- B-spline convolution property

$$\left(\beta_{\vec{\alpha}_1} * \beta_{\vec{\alpha}_2}\right)(t) = \beta_{(\vec{\alpha}_1:\vec{\alpha}_2)}(t)$$



- Continuous-time convolution

$$\left(s_1 * s_2\right)(t) = \sum_{k\in\mathbb{Z}} \left(c_1 * c_2\right)[k]\, \beta_{(\vec{\alpha}_1:\vec{\alpha}_2)}(t-k)$$

Discrete-time convolution

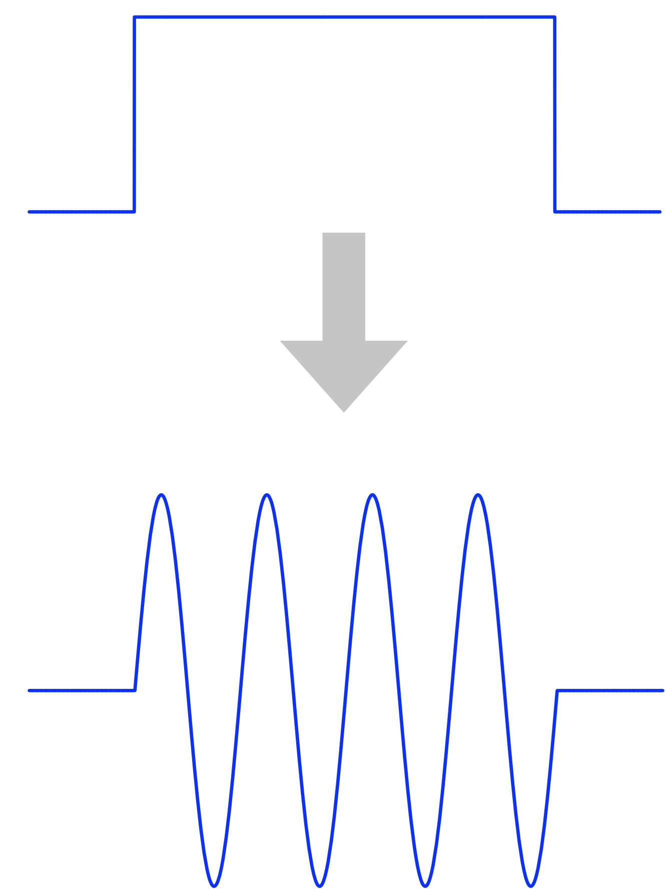
Augmented order B-spline

Modulation

- Input signal $s(t) = \sum_{k \in \mathbb{Z}} c[k] \beta_{\vec{\alpha}}(t-k)$

- B-spline modulation property

$$\beta_{\vec{\alpha}}(t) \cdot e^{j\omega_0 t} = \beta_{\vec{\alpha}+\vec{j}\omega_0}(t)$$



- Continuous-time modulation

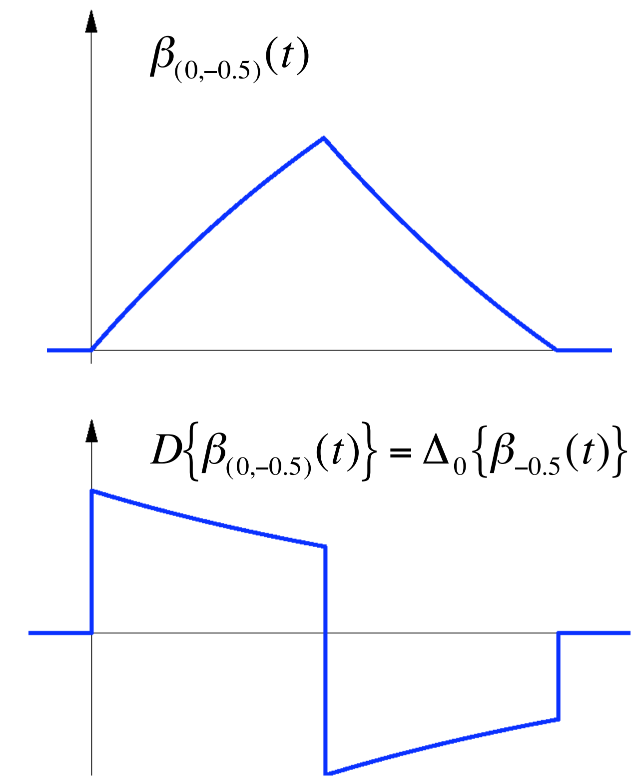
$$s(t) \cdot e^{j\omega_0 t} = \sum_{k \in \mathbb{Z}} \left(c[k] \cdot e^{j\omega_0 k} \right) \beta_{\vec{\alpha}+\vec{j}\omega_0}(t-k)$$

Discrete-time modulation

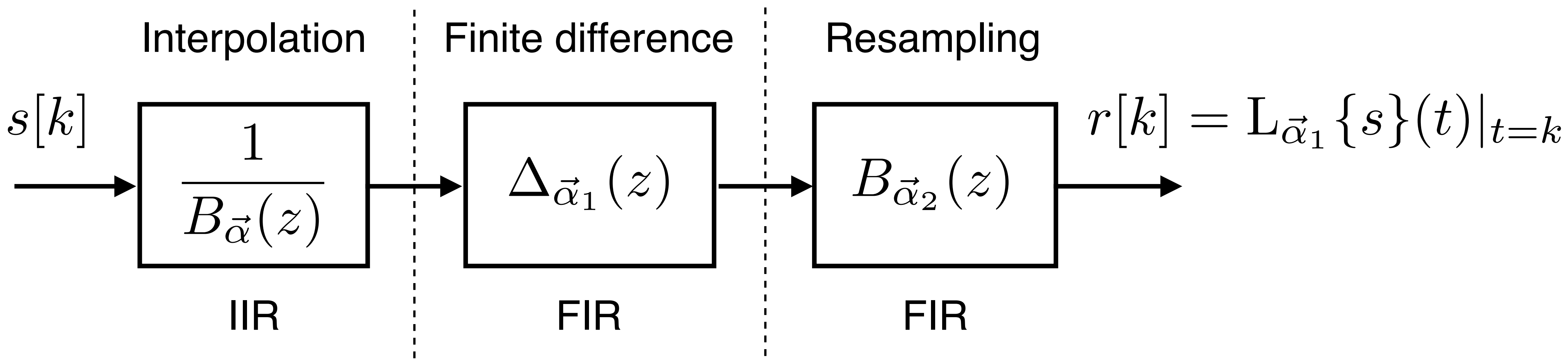
Differential operators

- B-spline differentials

$$\mathrm{L}_{\vec{\alpha}_1}\left\{\beta_{(\vec{\alpha}_1:\vec{\alpha}_2)}(t)\right\} = \Delta_{\vec{\alpha}_1}\left\{\beta_{\vec{\alpha}_2}(t)\right\}$$



- Implementation of differential operator



APPLICATION: HYBRID SIGNAL PROCESSING

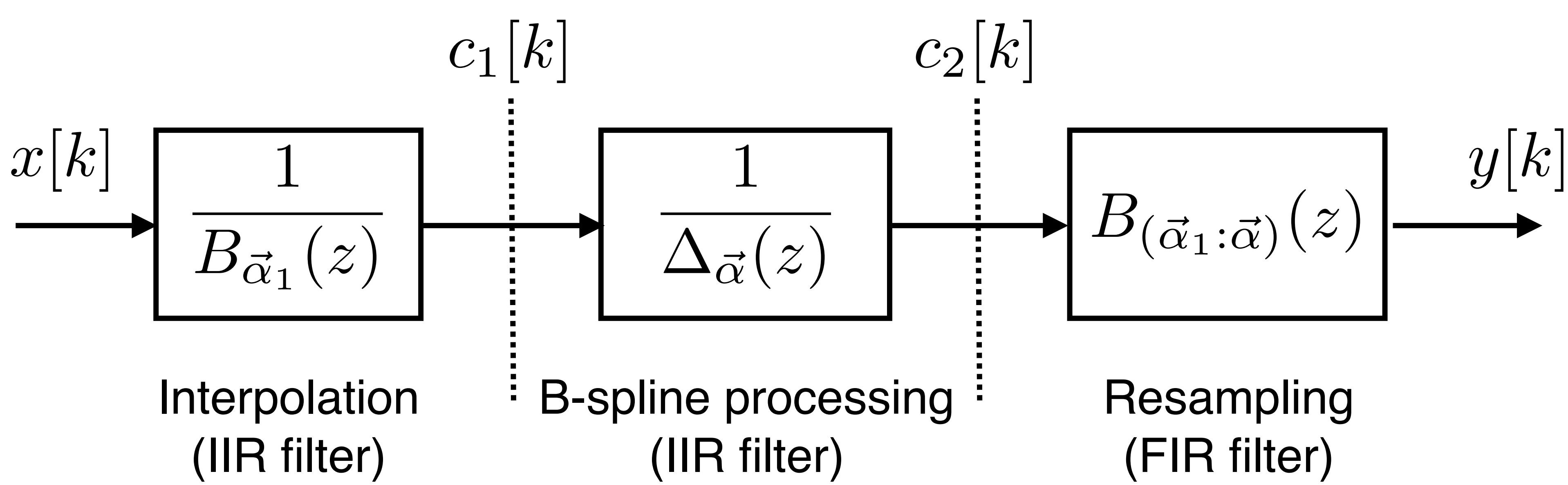
- Analog filtering in the B-spline domain
- Consistent sampling
- Digitally-compensated D-to-A conversion

Analog filtering in the B-spline domain

Analog filter: $h(t) = \sum_{k\in\mathbb{Z}} p[k]\beta_{\vec{\alpha}}(t-k)$

Input signal: $x(t) = \sum_{k\in\mathbb{Z}} c_1[k]\beta_{\vec{\alpha}_1}(t-k)$

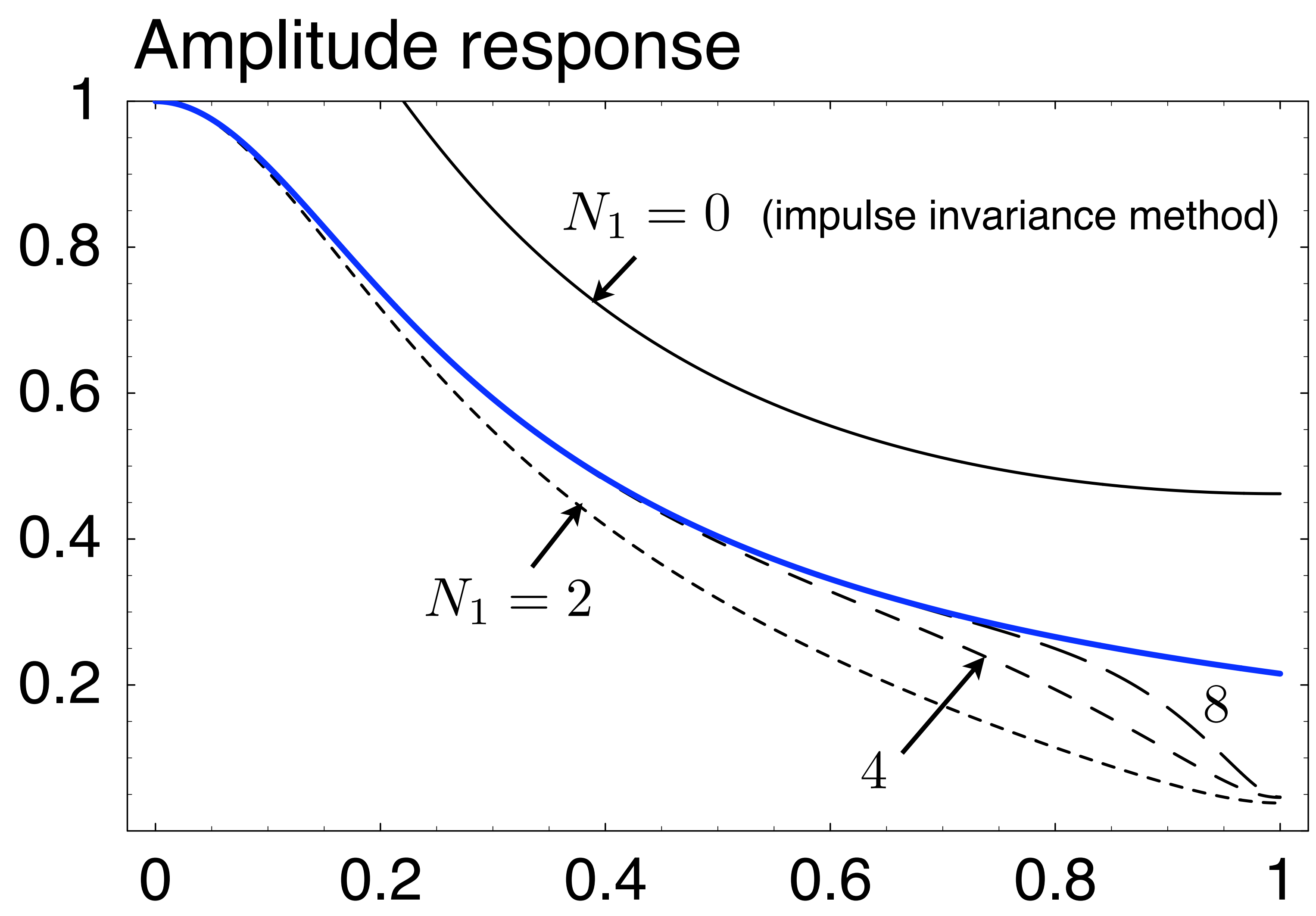
Output signal: $y(t) = \sum_{k\in\mathbb{Z}} (p * c_1)[k]\beta_{(\vec{\alpha}_1:\vec{\alpha})}(t-k)$



$$R_2(z) = \frac{B_{(\vec{\alpha}_1:\vec{\alpha})}(z)}{\Delta_{\vec{\alpha}}(z)} = P(z)\cdot B_{(\vec{\alpha}_1:\vec{\alpha})}(z)$$

Example: first order butterworth

Filter to design: $H(s) = \frac{-\alpha}{s - \alpha}$



Input model: polynomial spline of order N_1

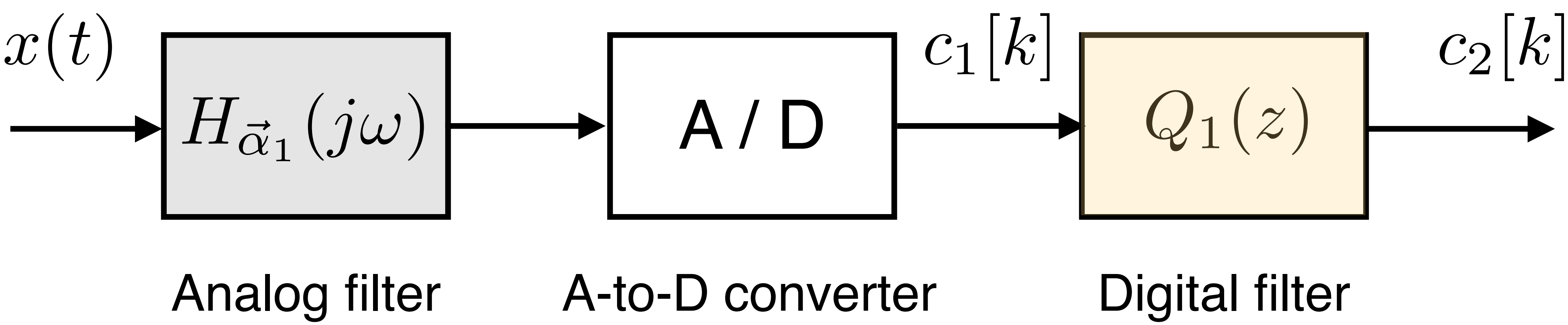
Design example: $\vec{\alpha}_1 = (0, 0) \implies R_{12}(z) = \frac{0.2786+0.2213z^{-1}}{1-0.5z^{-1}}$

Consistent sampling system

Reconstructed signal: $y(t) = \sum_{k\in\mathbb{Z}} c_2[k]\varphi_2(t-k)$

Consistency requirement:

$$\forall k \in \mathbb{Z},\ \langle x(t), \varphi_1(t-k)\rangle = \langle y(t), \varphi_1(t-k)\rangle$$



Digital reconstruction filter: $Q_1(z) = \dfrac{\Delta_{\vec{\alpha}_1}(z)}{\sum_{k=0}^{N_1+N_2} \beta_{(\vec{\alpha}_1:\vec{\alpha}_2)}(k) z^{-k}}$

Digitally-compensated D-to-A conversion

Reconstructed signal:

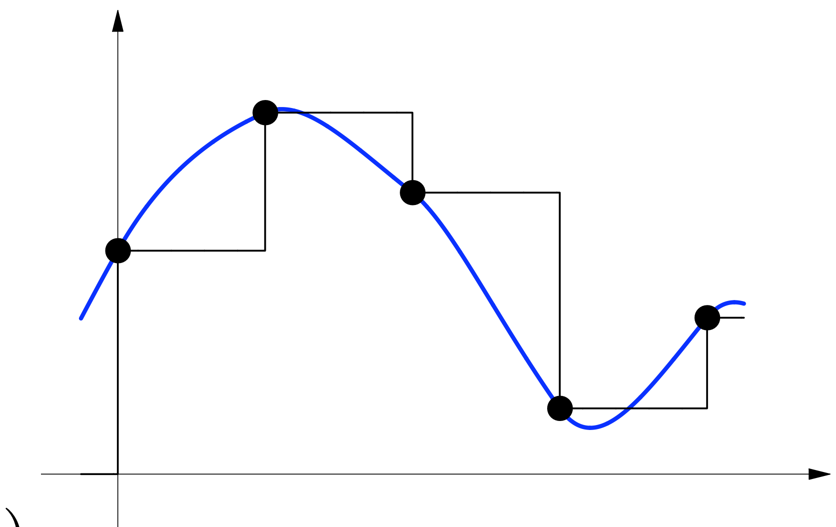
$$y(t) = \sum_{k \in \mathbb{Z}} c_2[k] \varphi_2(t-k)$$

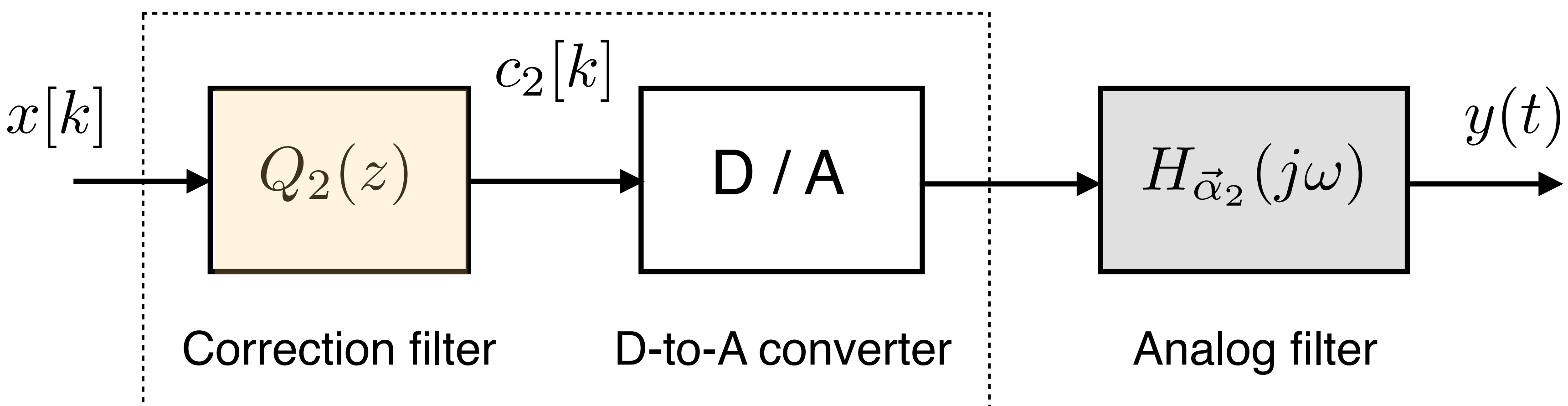
Equivalent synthesis function:

$$\varphi_2(t) = \left(\beta_{(0)} * \rho_{\vec{\alpha}_2}\right)(t)$$

Interpolation condition:

$$y(t)|_{t=k} = x[k]$$





Digital correction filter: $Q_2(z) = \dfrac{\Delta_{\vec{\alpha}_2}(z)}{\sum_{k=0}^{N_2+1} \beta_{(0:\vec{\alpha}_2)}(k) z^{-k}}$

CONCLUSION

- Cardinal E-splines: numerous attractive properties
  - B-spline representation = discrete signal
  - Family closed with respect to primary continuous-time signal processing operators (e.g., convolution, modulation, differential operators)
  - Easy to manipulate (e.g., recursive filtering algorithms, explicit formulas)
  - Generality: include all known brands of splines (polynomial, trigonometric, hyperbolic) and many more

The end: Thank you!

- The key collaborator: **Thierry Blu**
- For more info:
  - M. Unser, "Splines: A Perfect Fit for Signal and Image Processing," *IEEE Signal Processing Magazine*, vol. 16, no. 6, pp. 22-38, November 1999.
  - M. Unser, T. Blu, "Cardinal Exponential Splines: Part I—Theory and Filtering Algorithms," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1425-1438, April 2005.
  - M. Unser, "Cardinal Exponential Splines: Part II—Think Analog, Act Digital," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1425-1438, April 2005.
- Preprints and demos: **http://bigwww.epfl.ch/**

More to come ...

- Variational properties: “Tikhonov” splines
- Unified formulation of stochastic signal processing
  - Hybrid Wiener filter
  - Fractals
- New type of exponential-preserving wavelets and multiresolution analysis
- Multi-dimensional extensions: polyharmonic splines, vector-splines

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FÉDÉRALE DE LAUSANNE

Think analog, act digital

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