ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Affine invariance, splines, wavelets and fractional Brownian fields

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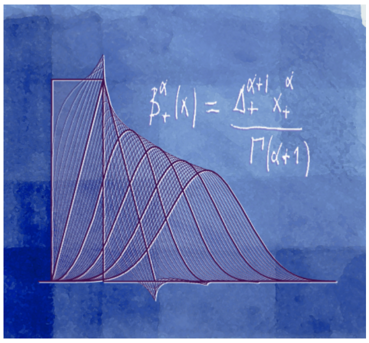


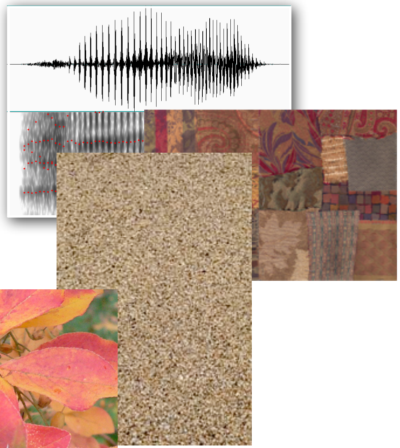
Mathematical Image Processing, Sept. 2007, CIRM, Marseille, France

The quest for invariance

- Invariance to coordinate transformations
  - Primary transformations (X): translation (T), scaling (S), rotation (R), affine (similarity) (A=S+R)
  - A continuous-domain operator L is X-invariant iff. it commutes with X; i.e,
    $\forall f \in L_2(\mathbb{R}^d), \mathrm{XL}f = C_\mathrm{X} \cdot \mathrm{LX}f$ $\quad$ C_X: normalization constant
  - All classical physical laws are TSR-invariant
- Classical signal/image processing operators are invariant (to various extents)
  - Filters (linear or non-linear): T-invariant
  - Differentiators, Hilbert transform, wavelet transform: TS-invariant
  - Contour/ridge detectors (Gradient, Laplacian, Hessian): TSR-invariant
  - Steerable filters: TR-invariant

Invariant signals

- Natural signals/images often exhibit some degree of invariance (at least locally, if not globally)
  - Stationarity, texture: T-invariance
  - Isotropy (no preferred orientation): R-invariance
  - Self-similarity, fractality: S-invariance



OUTLINE

- Splines and T-invariant operators
  - Green functions as elementary building blocks
  - Multiresolution revisited
- Imposing scale (resp., affine) invariance
  - Fractional derivatives
  - Fractional (resp., polyharmonic) B-splines
- Fractional wavelets
  - Exact Hilbert-transform pairs of bases
  - Isotropic wavelets
  - Analysis of fractal processes

General concept of an *L*-spline

$\mathrm{L}\{\cdot\}$: differential operator (translation-invariant)

$\delta(\boldsymbol{x}) = \prod_{i=1}^{d} \delta(x_i)$: multidimensional Dirac distribution

Definition

The continuous-domain function $s(\boldsymbol{x})$ is a ***cardinal L-spline*** iff.

$$\mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

- Cardinality: the knots (or spline singularities) are on the (multi-)integers
- Generalization: includes polynomial splines as particular case ($\mathrm{L} = \frac{\mathrm{d}^N}{\mathrm{d}x^N}$)

Example: piecewise-constant splines

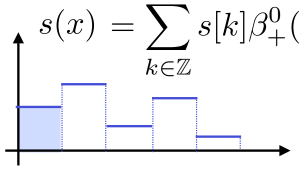
- Spline-defining operators

Continuous-domain derivative: $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}x} \quad \longleftrightarrow \quad j\omega$

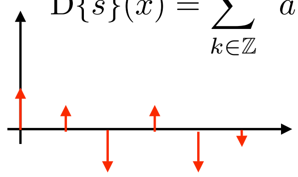
Discrete derivative: $\Delta_+\{\cdot\} \quad \longleftrightarrow \quad 1 - e^{-j\omega}$

- Piecewise-constant or D-spline

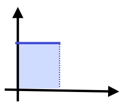
$$s(x) = \sum_{k\in\mathbb{Z}} s[k]\beta_+^0(x-k)$$



$$\mathrm{D}\{s\}(x) = \sum_{k\in\mathbb{Z}} \overbrace{a[k]}^{\Delta_+ s(k)}\,\delta(x-k)$$



- B-spline function



$$\beta_+^0(x) = \Delta_+\mathrm{D}^{-1}\{\delta\}(x) \quad \longleftrightarrow \quad \boxed{\frac{1-e^{-j\omega}}{j\omega}}$$

Existence of a local, shift-invariant basis?

- Space of cardinal L-splines

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) : \mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \right\} \cap L_2(\mathbb{R}^d)$$

- Generalized B-spline representation

A “localized” function $\varphi(\boldsymbol{x}) \in V_{\mathrm{L}}$ is called *generalized B-spline* if it generates a Riesz basis of V_{L}; i.e., iff. there exists $(A>0, B<\infty)$ s.t.

$$A \cdot \|c\|_{\ell_2(\mathbb{Z}^d)} \le \left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \right\|_{L_2(\mathbb{R}^d)} \le B \cdot \|c\|_{\ell_2(\mathbb{Z}^d)}$$

$$\Downarrow$$

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) : \boldsymbol{x}\in\mathbb{R}^d, c \in \ell_2(\mathbb{Z}^d) \right\}$$

continuous-domain signal

discrete signal
(B-spline coefficients)

Splines and Green’s functions

Definition

$\rho(\boldsymbol{x})$ is a Green function of the shift-invariant operator L iff $\mathrm{L}\{\rho\} = \delta$

$$\rho(\boldsymbol{x}) \longrightarrow \boxed{\mathrm{L}\{\cdot\}} \longrightarrow \delta(\boldsymbol{x}) \quad \Rightarrow \quad \delta(\boldsymbol{x}) \longrightarrow \boxed{\mathrm{L}^{-1}\{\cdot\}} \longrightarrow \rho(\boldsymbol{x})$$

(+ null-space component?)

- Cardinal L-spline: $\mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$

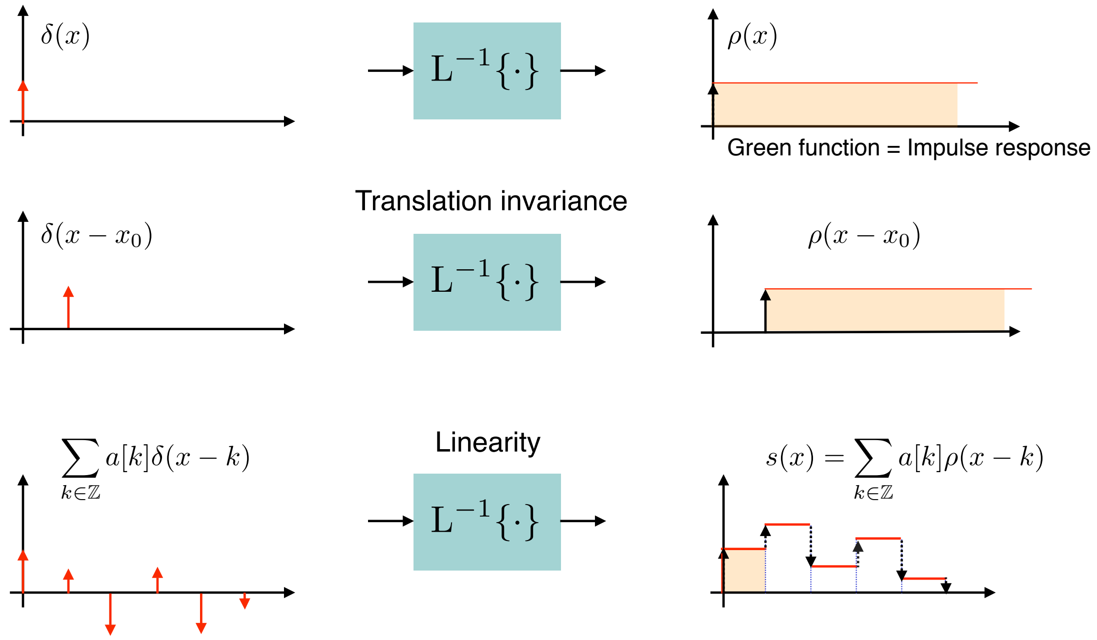
Formal integration

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \longrightarrow \boxed{\mathrm{L}^{-1}\{\cdot\}} \longrightarrow s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\rho(\boldsymbol{x}-\boldsymbol{k})$$

$$\Rightarrow \quad V_{\mathrm{L}} = \mathrm{span}\,\{\rho(\boldsymbol{x}-\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$$

Example of spline synthesis

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \Rightarrow \quad \mathrm{L}^{-1}\text{: integrator}$$

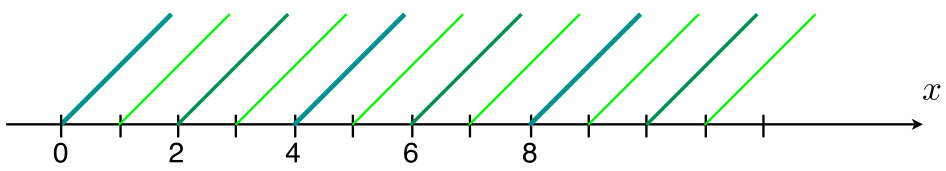


A fresh (Green's) view of multiresolution

Series of embedded spaces: $V_{(i)} \triangleq \mathrm{span}\left\{\rho(\boldsymbol{x} - 2^i\boldsymbol{k})\right\}_{\boldsymbol{k}\in\mathbb{Z}^d}$

Example: $\mathrm{L} = \mathrm{D}^2$ with $\rho(x) = x_+$

$$V_{(2)} \subset V_{(1)} \subset V_{(0)} = V_{\mathrm{L}}$$



Inclusion property: $V_{(j)} \subset V_{(i)}$ for $j \geq i$

- Technical difficulties
  - $\{\rho(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is not always a Riesz basis (e.g., $\rho \notin L_2(\mathbb{R}^d)$)
  - Completeness issue

IMPOSING INVARIANCE

- Scale-invariant operators
  - Fractional B-splines
- Affine-invariant operators
  - Polyharmonic B-splines

Scale-invariant operators

Definition: An operator L is scale-invariant iff it commutes with dilation: i.e., $\forall s(\boldsymbol{x}), \mathrm{L}\{s(\cdot)\}(\boldsymbol{x}/a) = C_a\,\mathrm{L}\{s(\cdot/a)\}(\boldsymbol{x})$.

Theorem
The complete family of real scale-invariant 1D convolution operators is given by the fractional derivatives ∂_τ^γ, whose frequency response is

$$\hat{L}(\omega) = (-j\omega)^{\frac{\gamma}{2}-\tau}(j\omega)^{\frac{\gamma}{2}+\tau}$$

$\gamma \in \mathbb{R}^+$: order of the derivative (i.e., $|\hat{L}(\omega)| = |\omega|^\gamma$)
$\tau \in \mathbb{R}$: phase (or asymmetry)

(Unser & Blu, *IEEE-SP*, 2007)

Construction of causal B-splines

Derivative operator: $\mathrm{D} = \partial^1_{\frac{1}{2}} \quad \xleftrightarrow{\mathcal{F}} \quad j\omega$

Finite difference: $\Delta_+ \quad \xleftrightarrow{\mathcal{F}} \quad 1 - e^{-j\omega}$

Liouville's fractional derivative: $\mathrm{D}^\gamma = \partial^\gamma_{\gamma/2} \quad \xleftrightarrow{\mathcal{F}} \quad (j\omega)^\gamma$

Fractional finite differences: $\Delta^\gamma_+ \quad \xleftrightarrow{\mathcal{F}} \quad (1 - e^{-j\omega})^\gamma$

- Causal fractional B-splines

Discrete operator: localization filter $Q(e^{j\omega})$

Spline degree: $\alpha = \gamma - 1$

$$\frac{(1 - e^{-j\omega})^{\alpha+1}}{(j\omega)^{\alpha+1}} \quad \xrightarrow{\mathcal{F}^{-1}} \quad \beta^\alpha_+(x)$$

Continuous-domain operator: $\hat{L}(\omega)$

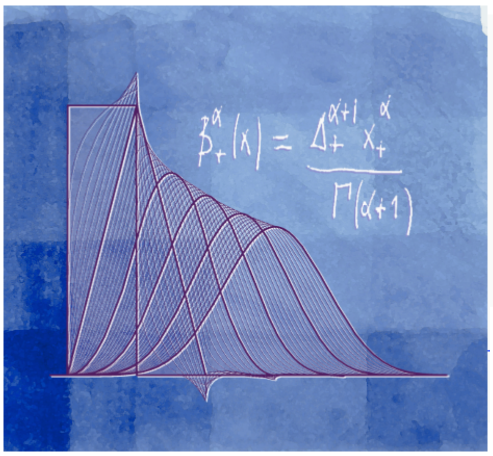
Causal fractional B-splines

$$\beta^0_+(x) = \Delta_+ x^0_+ \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1 - e^{-j\omega}}{j\omega}$$

$$\vdots \qquad\qquad \vdots$$

$$\beta^\alpha_+(x) = \frac{\Delta^{\alpha+1}_+ x^\alpha_+}{\Gamma(\alpha+1)} \quad \xleftrightarrow{\mathcal{F}} \quad \left(\frac{1 - e^{-j\omega}}{j\omega}\right)^{\alpha+1}$$

One-sided power function: $x^\alpha_+ = \begin{cases} x^\alpha, & x \geq 0 \\ 0, & x < 0 \end{cases}$



(Unser & Blu, *SIAM Rev*, 2000)

Fractional B-splines properties

- Stable representation of fractional splines (Riesz basis)

$$V_{\partial_\tau^{\alpha+1}} = \left\{ s(x) = \sum_{k\in\mathbb{Z}} c[k]\beta_\tau^\alpha(x-k) : c[k] \in \ell_2 \right\}$$

Condition: $\alpha > -\frac{1}{2}$

- Degree versus order of approximation
  - Building blocks (Green function of ∂_τ^γ): power functions of degree $\alpha = \gamma - 1$
  - Order of approximation: $\gamma = \alpha + 1$

- Reproduction of polynomials

The fractional B-splines $\{\beta_\tau^\alpha(x-k)\}_{k\in\mathbb{Z}}$ reproduce the polynomials of degree $n = \lceil\alpha\rceil$. In particular,

$$\sum_{k\in\mathbb{Z}} \beta_\tau^\alpha(x-k) = 1 \qquad \text{(partition of unity)}$$

Scale- and rotation-invariant operators

Definition: An operator L is affine-invariant (or SR-invariant) iff.

$$\forall s(\boldsymbol{x}),\ \mathrm{L}\{s(\cdot)\}(\mathbf{R}_\theta \boldsymbol{x}/a) = C_a \cdot \mathrm{L}\{s(\mathbf{R}_\theta \cdot /a)\}(\boldsymbol{x})$$

where $\mathbf{R}_\theta$ is an arbitrary $d \times d$ unitary matrix and C_a a constant

- **Invariance theorem**

  The complete family of real, scale- and rotation-invariant convolution operators is given by the fractional Laplacians

$$\Delta^{\frac{\gamma}{2}} \quad \xleftrightarrow{\mathcal{F}} \quad \|\boldsymbol{\omega}\|^\gamma$$

- Invariant Green functions (RBF) (Duchon, 1979)

$$\rho(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|^{\gamma-d}\log\|\boldsymbol{x}\|, & \text{if } \gamma - d \text{ is even} \\ \|\boldsymbol{x}\|^{\gamma-d}, & \text{otherwise} \end{cases}$$

Construction of polyharmonic B-splines

Laplacian operator: $\Delta \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad -\|\boldsymbol{\omega}\|^2$

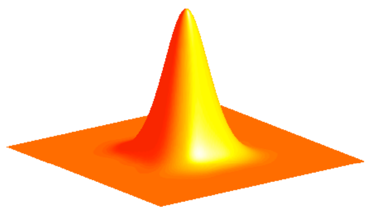
Discrete Laplacian: $\Delta_{\mathrm{d}} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad -\sum_{i=1}^{d} 4\sin^2(\omega_i/2) \triangleq -\|2\sin(\boldsymbol{\omega}/2)\|^2$

0	-1	0
-1	4	-1
0	-1	0

- Polyharmonic B-splines (Rabut, 1992)

Discrete operator: localization filter $Q(e^{j\boldsymbol{\omega}})$

$$\frac{\|2\sin(\boldsymbol{\omega}/2)\|^{\gamma}}{\|\boldsymbol{\omega}\|^{\gamma}} \quad \overset{\mathcal{F}^{-1}}{\longrightarrow} \quad \varphi_{\gamma}(\boldsymbol{x})$$



Continuous-domain operator: $\hat{L}(\boldsymbol{\omega})$

FRACTIONAL WAVELETS

- General scaling relations
- Fractional B-spline wavelets
- Exact Hilbert-transform pairs
- Fractional Mexican-hat wavelets
- Analysis of fractal processes
  (multidimensional generalization of pioneering work of Flandrin and Abry)

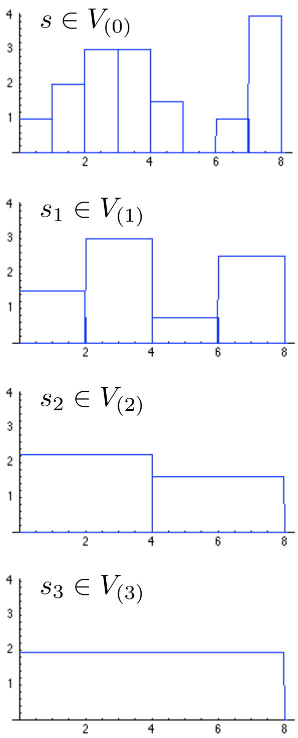
General scaling relations (m integer)

- Generalized splines of order γ
  - Scale-invariant operator of order γ $\Leftrightarrow$ $\hat{L}_\gamma(m\boldsymbol{\omega}) = m^\gamma \hat{L}_\gamma(\boldsymbol{\omega})$
  - A cardinal L_γ-spline dilated by m remains an L_γ-spline:

$$\mathrm{L}_\gamma\{s(\boldsymbol{x})\} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \Rightarrow \mathrm{L}_\gamma\{s(\boldsymbol{x}/m)\} = \frac{1}{m^{\gamma-d}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-m\boldsymbol{k})$$

- Scaling function
  - Generalized B-spline: $\varphi_\gamma(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} q[\boldsymbol{k}]\rho(\boldsymbol{x}-\boldsymbol{k}) \quad \xleftrightarrow{\mathcal{F}} \quad \dfrac{Q(e^{j\boldsymbol{\omega}})}{\hat{L}_\gamma(\boldsymbol{\omega})}$
  - General m-scale relation: $\varphi_\gamma(\boldsymbol{x}/m) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} h_{\gamma,m}[\boldsymbol{k}]\varphi_\gamma(\boldsymbol{x}-\boldsymbol{k})$
  - Refinement filter: $H_{\gamma,m}(e^{j\boldsymbol{\omega}}) = \dfrac{|m|^d \hat{\varphi}_\gamma(m\boldsymbol{\omega})}{\hat{\varphi}_\gamma(\boldsymbol{\omega})} = \dfrac{1}{m^{\gamma-d}} \dfrac{Q(e^{jm\boldsymbol{\omega}})}{Q(e^{j\boldsymbol{\omega}})}$

Multiresolution analysis of L_2



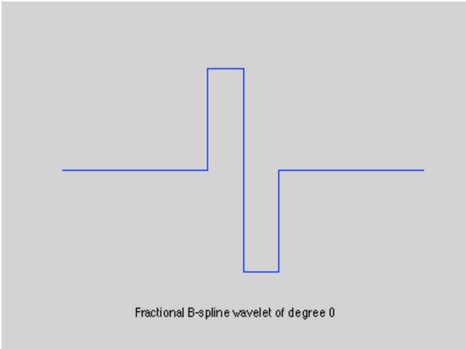
- Multiresolution basis functions: $\varphi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^{-id/2}\varphi\left(\frac{\boldsymbol{x}-2^i\boldsymbol{k}}{2^i}\right)$
- Subspace at resolution i: $V_{(i)} = \mathrm{span}\,\{\varphi_{i,\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$

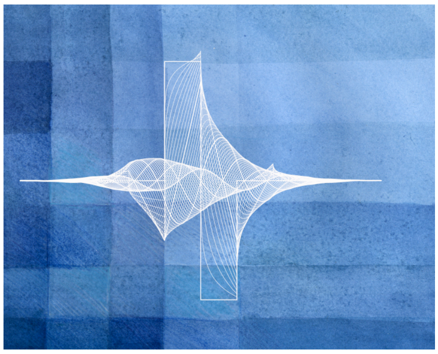
Two-scale relation $\Rightarrow$ $V_{(i)} \subset V_{(j)}$, for $i \geq j$

Partition of unity $\Leftrightarrow$ $\overline{\bigcup_{i\in\mathbb{Z}} V_{(i)}} = L_2(\mathbb{R}^d)$

Fractional B-spline wavelets

$$\psi_\tau^\alpha(x/2) = \sum_{k\in\mathbb{Z}} \underbrace{\sum_{n\in\mathbb{Z}} (-1)^n h_{\tau,2}^\alpha[n] \beta_0^{2\alpha+1}(n+k-1)}_{g_{\mathrm{wave}}[k]} \beta_\tau^\alpha(x-k)$$





Causal B-spline: $\tau = \frac{\alpha+1}{2}$

(Unser and Blu, *SIAM Review*, 2000)

Fractional B-spline wavelet properties

- Semi-orthogonal basis of $L_2(\mathbb{R})$: $\left\{2^{-i/2}\psi_\tau^\alpha(x/2^i - k)\right\}_{(i,k)\in\mathbb{Z}^2}$
- Multiscale fractional derivative behavior:

$$\left\langle \psi_\tau^\alpha\left(\frac{\cdot - x}{a}\right), f\right\rangle = \partial_\tau^{\alpha+1}\left\{\phi\left(\frac{\cdot}{a}\right) * f\right\}(x)$$

$\phi(x)$: smoothing spline kernel of order $2\alpha + 2$

- Spline family closed under fractional differentiation; in particular, $\psi_{\tau-\frac{1}{2}}^\alpha(x)$ is the Hilbert transform of $\psi_\tau^\alpha(x)$
- Optimal time-frequency localization: convergence to Gabor functions as α increases
- Fast filterbank algorithm (using FFT)

Multidimensional, nonseparable wavelets

Search for a **single wavelet** that generates a basis of $L_2(\mathbb{R}^d)$ and that is a multi-scale version of the operator L; i.e., $\psi = \mathrm{L}^*\phi$ where ϕ is a suitable smoothing kernel

- General operator-based construction
  - Basic space V_0 generated by the integer shifts of the Green function ρ of L:
    $V_0 = \mathrm{span}\{\rho(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ with $\mathrm{L}\rho = \delta$
  - Orthogonality between V_0 and $W_0 = \mathrm{span}\{\psi(\boldsymbol{x} - \frac{1}{2}\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d\backslash 2\mathbb{Z}^d}$

$$\begin{aligned}\langle\psi(\cdot - \boldsymbol{x}_0), \rho(\cdot - \boldsymbol{k})\rangle &= \langle\phi, \mathrm{L}\rho(\cdot - \boldsymbol{k} + \boldsymbol{x}_0)\rangle \\ &= \langle\phi, \delta(\cdot - \boldsymbol{k} + \boldsymbol{x}_0)\rangle = \phi(\boldsymbol{k} - \boldsymbol{x}_0) = 0\end{aligned}$$

(can be enforced via a judicious choice of ϕ (interpolator) and $\boldsymbol{x}_0$)

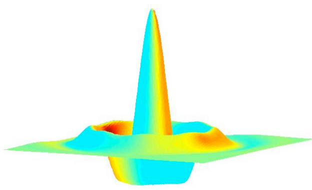
  - Works in arbitrary dimensions and for any dilation matrix $\mathbf{D}$

New isotropic basis: the Mexican-Hat

$$\psi_\gamma(\boldsymbol{x}) = (-\Delta)^{\frac{\gamma}{2}}\phi_{2\gamma}(\boldsymbol{x})$$

Gaussian-like polyharmonic spline kernel:

$$\phi_{2\gamma}(\boldsymbol{x}) \approx \exp\left(-\frac{\|\boldsymbol{x}\|^2}{2\gamma/9}\right)$$



$$\psi_{(i,\boldsymbol{k})} \triangleq |\det(\mathbf{D})|^{-i/2}\psi_\gamma\left(\mathbf{D}^{-i}\boldsymbol{x} - \mathbf{D}^{-1}\boldsymbol{k}\right)$$

- The set $\left\{\psi_{(i,\boldsymbol{k})}\right\}_{(\boldsymbol{k}\in\mathbb{Z}^d\backslash\mathbf{D}\mathbb{Z}^d, i\in\mathbb{Z})}$ is a semi-orthogonal basis of $L_2(\mathbb{R}^d)$ for $\gamma > 1$:

$$\forall f \in L_2(\mathbb{R}^d), \quad f = \sum_{i\in\mathbb{Z}}\sum_{\boldsymbol{k}\in\mathbb{Z}^d\backslash\mathbf{D}\mathbb{Z}^d}\langle f, \psi_{(i,\boldsymbol{k})}\rangle\,\tilde{\psi}_{(i,\boldsymbol{k})} = \sum_{i\in\mathbb{Z}}\sum_{\boldsymbol{k}\in\mathbb{Z}^d\backslash\mathbf{D}\mathbb{Z}^d}\langle f, \tilde{\psi}_{(i,\boldsymbol{k})}\rangle\,\psi_{(i,\boldsymbol{k})}$$

$\{\tilde{\psi}_{(i,\boldsymbol{k})}\}$: dual wavelet basis of $\{\psi_{(i,\boldsymbol{k})}\}$

- The Mexican-Hat wavelet analysis implements a multiscale version of the Laplace operator and is perfectly reversible (one-to-one transform)

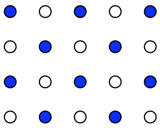
- The wavelet transform has a fast filterbank algorithm (Van De Ville, IEEE-IP, 2005)

Mexican-Hat wavelet basis

- Nonredundant transform

Quincunx sampling pattern

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

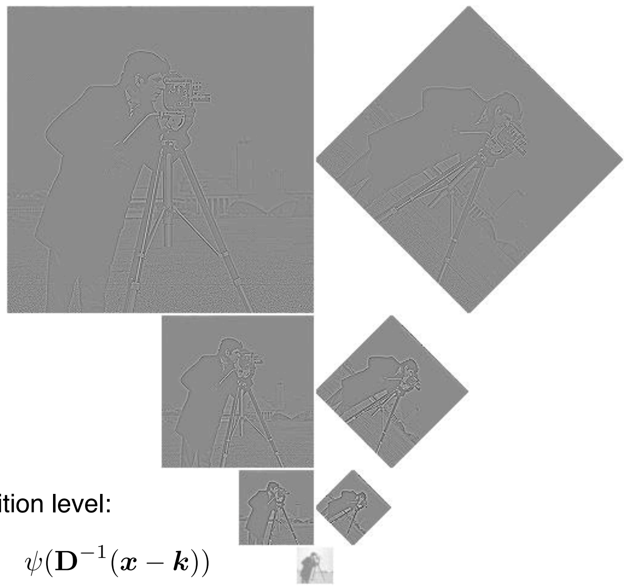


First decomposition level (one-to-one):

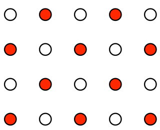
$\varphi(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$	$\psi(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$
Scaling functions (dilated by $\sqrt{2}$)	Wavelets (dilated by $\sqrt{2}$)

Mexican-Hat wavelet analysis

- Pyramid decomposition: redundancy 2



Quincunx sampling pattern



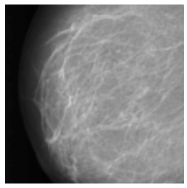
First decomposition level:

$\varphi(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$	$\psi(\mathbf{D}^{-1}(\boldsymbol{x} - \boldsymbol{k}))$
Scaling functions	Wavelets (redundant by 2)

Self-similar image models

- Motivation: most images exhibit some level of self-similarity
  - 2D projection of a 3D scene: objects appear with various degrees of magnification (depth-dependent scaling)
  - Step edges and singularities give rise to $1/\omega$ spectral decay
  - Natural/biological growth processes often generate fractal structures

(Penland, 1984; Mumford 2001)



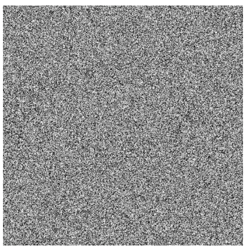
- Stochastic fractals: fractional Brownian fields
  - d-dimensional version of Mandelbrot's fractional Brownian motion: self-similar analogs of stationary processes—loosely referred to as "$1/\omega$-processes"
  - Distributional solution of the **stochastic partial differential equation**:

    $\Delta^{\frac{H}{2}+\frac{d}{4}} B_H = W,$ where W is white Gaussian noise
  - **Key finding**: Polyharmonic splines are the optimal function spaces for the MMSE estimation of such processes from their noisy samples (Blu, 2007; Tirosh, 2006)

fBm: innovation model

Formalism: Gelfand's theory of generalized stochastic processes

White noise



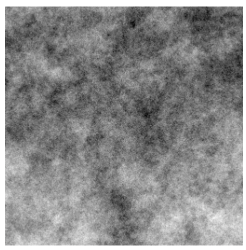
$\Delta^{-\frac{\gamma}{2}}$

fractional integrator
(appropriate boundary conditions)

$\Delta^{\frac{\gamma}{2}}$

Whitening
(fractional Laplacian)

fractional Brownian field



Hurst exponent: $H = \gamma - \frac{d}{2}$

- Deterministic counterpart

Train of Dirac impulses:

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

$\Delta^{\frac{\gamma}{2}}$

Polyharmonic spline

$$s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\rho_\gamma(\boldsymbol{x}-\boldsymbol{k})$$

Wavelet analysis of fBm: whitening revisited

- Operator-like behavior of wavelet
  - Analysis wavelet: $\psi_\gamma = \Delta^{\frac{\gamma}{2}}\phi(\boldsymbol{x}) = \Delta^{\frac{H}{2}+\frac{d}{4}}\psi'_{\gamma'}(\boldsymbol{x})$
  - Reduced-order wavelet: $\psi'_{\gamma'}(\boldsymbol{x}) = \Delta^{\frac{\gamma'}{2}}\phi(\boldsymbol{x})$ with $\gamma' = \gamma - (H + \frac{d}{2}) > 0$
- Stationarizing effect of wavelet analysis
  - Analysis of fractional Brownian field with exponent H:

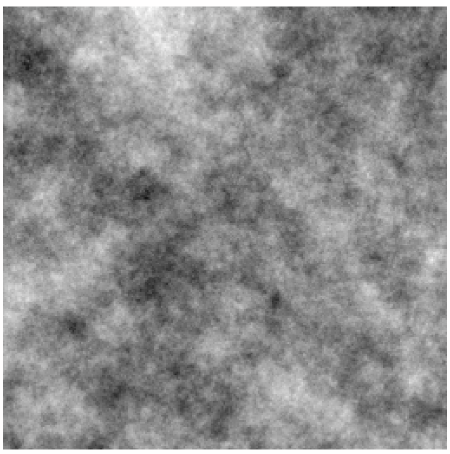
    $\langle B_H, \psi_\gamma\left(\frac{\cdot-\boldsymbol{x}_0}{a}\right)\rangle \propto \langle \Delta^{\frac{H}{2}+\frac{d}{4}} B_H, \psi'_{\gamma'}\left(\frac{\cdot-\boldsymbol{x}_0}{a}\right)\rangle = \langle W, \psi'_{\gamma'}\left(\frac{\cdot-\boldsymbol{x}_0}{a}\right)\rangle$
  - Equivalent spectral noise shaping: $S_{\text{wave}}(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |\hat{\psi}'_\gamma(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2$

    $\Rightarrow$ Extent of wavelet-domain whitening depends on flatness of $S_{\text{wave}}(e^{j\boldsymbol{\omega}})$
  - "Whitening" effect is the same at all scales up to a proportionality factor

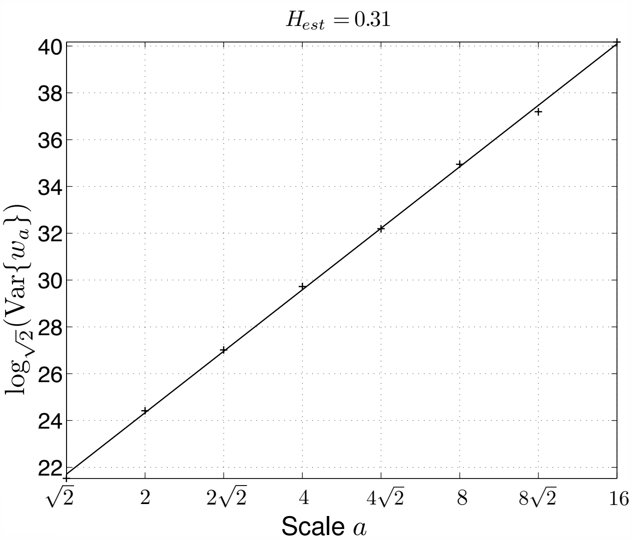
    $\Rightarrow$ fractal exponent can be deduced from the log-log plot of the variance

Wavelet analysis of fractional Brownian fields

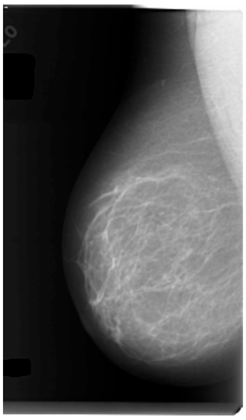
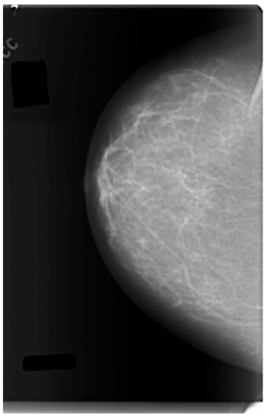
Theoretical scaling law : $\mathrm{Var}\{w_a[k]\} = \sigma_0^2 \cdot a^{(2H+d)}$

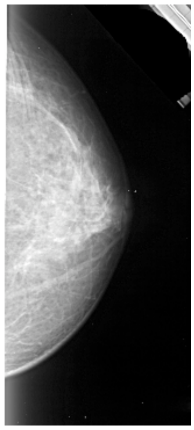


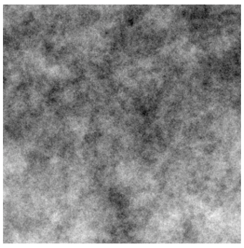
log-log plot of variance



Fractals in bioimaging: fibrous tissue



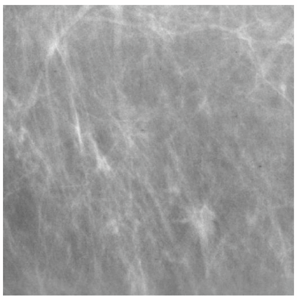


DDSM: University of Florida

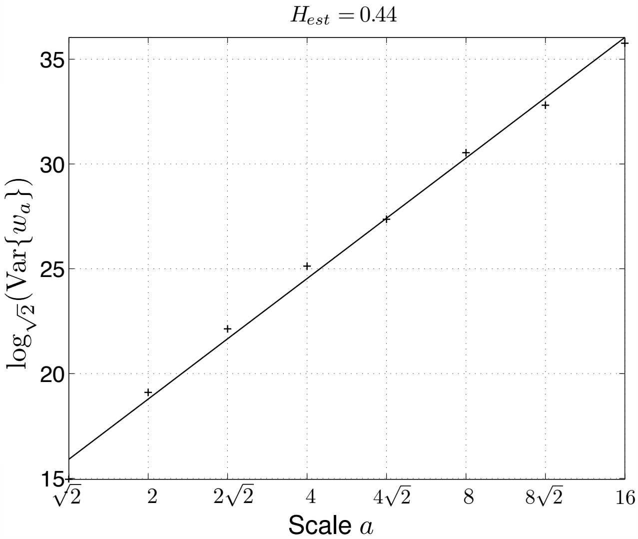
(Digital Database for Screening Mammography)

(Laine, 1993; Li et al., 1997)

Wavelet analysis of mammograms

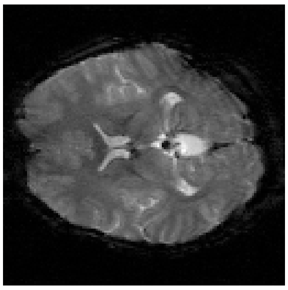


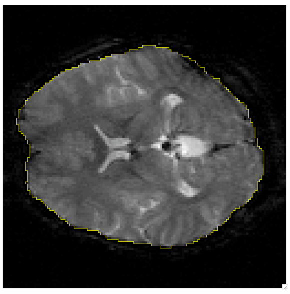
log-log plot of variance

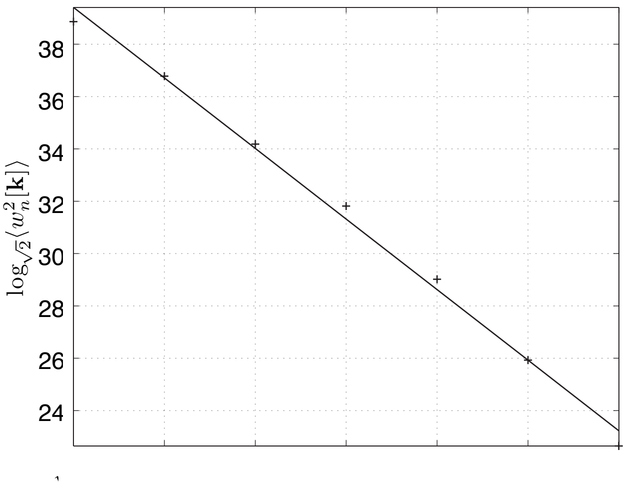


Fractal dimension: $D = 1 + d - H = 2.56$ with $d = 2$ (topological dimension)

Wavelet analysis of fMRI data



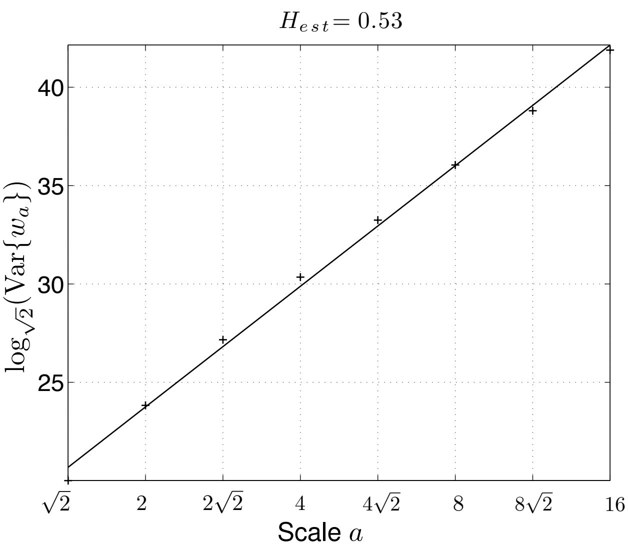




Brain: courtesy of Jan Kybic

Fractal dimension: $D = 1 + d - H = 2.65$ with $d = 2$ (topological dimension)

...and some non-biomedical images...



CONCLUSION

- Affine invariance is universal
  - Natural and biomedical images
  - Operators: fractional derivatives
  - Optimal functions spaces: splines (universal interpolators)
  - Fractal processes
- Invariant operators yield “matched” wavelet bases
  - Existence of “localized” B-spline bases
  - Enforcement of multiresolution property ⇒ wavelets
  - Wavelet = multiscale version of operator
  - “Invariant” wavelets: more operator-like (e.g., better isotropy)
  - Extended family of **fractional** wavelets: tunable, closed under fractional differentiation
  - Promising tool for (multi-D) signal processing and fractal analyses

The end: Thank you!

- **Key collaborators**
  T. Blu, D. Van De Ville, P. Tafti, I. Khalidov, K. Chaudhury, K. Balac
- Selected papers

  - M. Unser, T. Blu, "Fractional Splines and Wavelets," *SIAM Review*, 42(1), pp. 43-67, March 2000.
  - M. Unser, T. Blu, "Self-similarity—Part I: Splines and Operators," *IEEE Trans. Signal Processing*, 55(4), pp. 1352-1363, 2007.
  - T. Blu, M. Unser, "Self-similarity—Part II: Optimal Estimation of Fractal Processes," *IEEE Trans. Signal Processing*, 55(4), pp. 1364-1378, 2007.
  - D. Van De Ville, T. Blu, M. Unser, "Isotropic Polyharmonic B-Splines: Scaling Functions and Wavelets," *IEEE Trans. Image Processing*, 14(11), pp. 1798-1813, 2005.

- Preprints and demos: **http://bigwww.epfl.ch/**