ÉCOLE POLYTECHNIQUE
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Representer theorems for ill-posed problems with sparsity constraints

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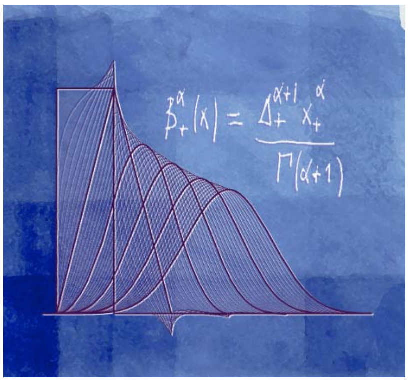

BLISS Seminar, June 28, 2017, University of California, Berkeley.

OUTLINE

- **Linear inverse problems and regularization**
  - Tikhonov regularization
  - The sparsity (r)evolution
  - Compressed sensing and l_1 minimization
- **Part I: Discrete-domain regularization (l_2 vs. l_1)**
- **Part II: Continuous-domain regularization (L_2 vs. gTV)**
  - Classical L_2 regularization: theory of RKHS
  - Splines and operators
  - Minimization of gTV: the optimality of splines
  - Enabling components for the proof
  - Special case TV in 1D

FNSNF Swiss National Science Foundation
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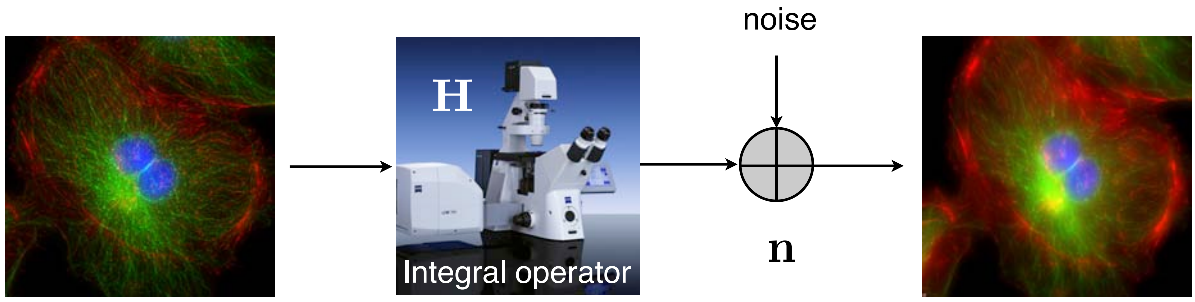
Inverse problems in bio-imaging

- Linear forward model $\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}$



$\mathbf{s}$

Problem: recover $\mathbf{s}$ from noisy measurements $\mathbf{y}$

- The easy scenario

  Inverse problem is **well**

  $\Rightarrow \quad \mathbf{s} \approx \mathbf{H}^{-1}\mathbf{y}$

- Backprojection (po

Basic limitations

1) Inherent noise amplification
2) Difficulty to invert **H** (too large or non-square)
3) All interesting inverse problems are **ill-posed**

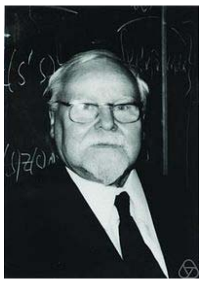
Linear inverse problems (20th century theory)

- Dealing with **ill-posed problems**: Tikhonov **regularization**

$\mathcal{R}(\mathbf{s}) = \|\mathbf{L}\mathbf{s}\|_2^2$: regularization (or smoothness) functional

$\mathbf{L}$: regularization operator (i.e., Gradient)

$$\min_{\mathbf{s}} \mathcal{R}(\mathbf{s}) \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{H}\mathbf{s}\|_2^2 \leq \sigma^2$$



Andrey N. Tikhonov (1906-1993)

- Equivalent variational problem

$$\mathbf{s}^\star = \arg\min \underbrace{\|\mathbf{y} - \mathbf{H}\mathbf{s}\|_2^2}_{\text{data consistency}} + \underbrace{\lambda\|\mathbf{L}\mathbf{s}\|_2^2}_{\text{regularization}}$$

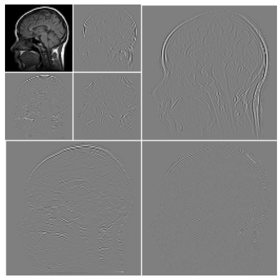
Formal linear solution: $\mathbf{s} = (\mathbf{H}^T\mathbf{H} + \lambda\mathbf{L}^T\mathbf{L})^{-1}\mathbf{H}^T\mathbf{y} = \mathbf{R}_\lambda \cdot \mathbf{y}$

Interpretation: "filtered" backprojection

Linear inverse problems: The sparsity (r)evolution

(20th Century) $p = 2 \longrightarrow 1$ (21st Century)

$$\mathbf{s}_{\text{rec}} = \arg\min_{\mathbf{s}} \left(\|\mathbf{y} - \mathbf{H}\mathbf{s}\|_2^2 + \lambda \mathcal{R}(\mathbf{s}) \right)$$



- Non-quadratic regularization regularization

  $\mathcal{R}(\mathbf{s}) = \|\mathbf{L}\mathbf{s}\|_{\ell_2}^2 \longrightarrow \|\mathbf{L}\mathbf{s}\|_{\ell_p}^p \longrightarrow \|\mathbf{L}\mathbf{s}\|_{\ell_1}$

  - Total variation (Rudin-Osher, 1992)

    $\mathcal{R}(\mathbf{s}) = \|\mathbf{L}\mathbf{s}\|_{\ell_1}$ with $\mathbf{L}$: gradient

  - Wavelet-domain regularization (Figuereido et al., Daubechies et al. 2004)

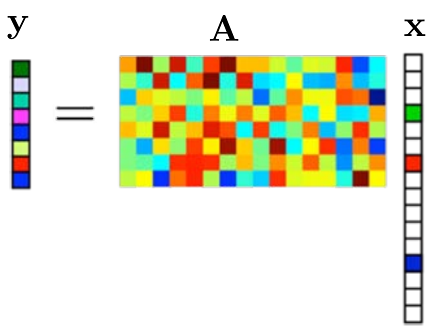
    $\mathbf{v} = \mathbf{W}^{-1}\mathbf{s}$: wavelet expansion of s (typically, sparse)

    $\mathcal{R}(\mathbf{s}) = \|\mathbf{v}\|_{\ell_1}$

- Compressed sensing/sampling (Candes-Romberg-Tao; Donoho, 2006)

Compressive sensing (CS) and l_1 minimization

[Donoho et al., 2005
Candès-Tao, 2006, ...]



\+ “noise”

Sparse representation of signal: $\mathbf{s} = \mathbf{W}\mathbf{x}$ with $\|\mathbf{x}\|_0 = K \ll N_x$

Equivalent $N_y \times N_x$ **sensing matrix** : $\mathbf{A} = \mathbf{H}\mathbf{W}$

- Constrained (synthesis) formulation of recovery problem

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \leq \sigma^2$$

CS: Three fundamental ingredients

(Donoho, *IEEE T. Inf. Theo.* 2006) (Candès-Romberg, *Inv. Prob.* 2007)

1. Existence of sparsifying transform (**W** or **L**)
   - Wavelet basis
   - Dictionary
   - Differential operator (Gradient)

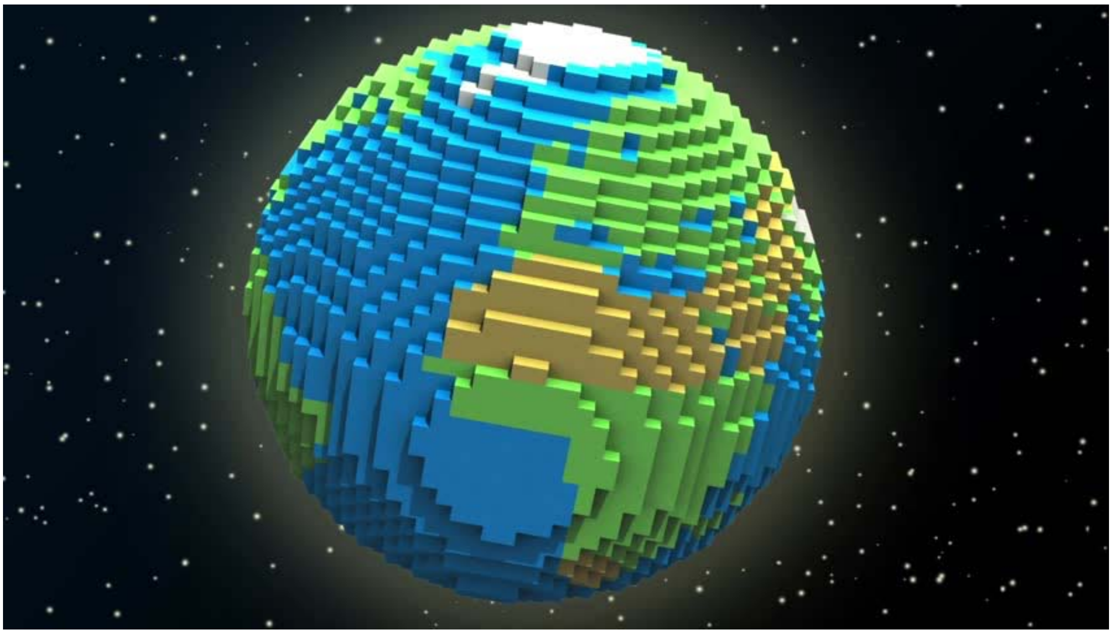
2. Incoherence of sensing matrix **A**
   - Restricted isometry; few linearly dependent columns (spark)
   - Quasi-random and delocalized structure:
     Gaussian matrix with i.i.d. entries,
     random sampling in Fourier domain

3. Non-linear signal recovery (l_1 minimization)

CS: Examples of applications in imaging

- Magnetic resonance imaging (MRI) (Lustig, *Mag. Res. Im.* 2007)
- Radio Interferometry (Wiaux, *Notic. R. Astro.* 2007)
- Teraherz Imaging (Chan, *Appl. Phys.* 2008)
- Digital holography (Brady, *Opt. Express* 2009; Marim 2010)
- Spectral-domain OCT (Liu, *Opt. Express* 2010)
- Coded-aperture spectral imaging (Arce, *IEEE Sig. Proc.* 2014)
- Localization microscopy (Zhu, *Nat. Meth.* 2012)
- Ultrafast photography (Gao, *Nature* 2014)

Part I: Discrete-domain regularization



Classical regularized least-squares estimator

- Linear measurement model:

  $y_m = \langle \mathbf{h}_m, \mathbf{x} \rangle + n[m], \quad m = 1, \ldots, M$

- System matrix of size $M \times N$: $\mathbf{H} = [\mathbf{h}_1 \cdots \mathbf{h}_M]^T$

$$\mathbf{x}_{\mathrm{LS}} = \arg\min_{\mathbf{x} \in \mathbb{R}^N} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_2^2$$

$$\Rightarrow \quad \mathbf{x}_{\mathrm{LS}} = (\mathbf{H}^T\mathbf{H} + \lambda \mathbf{I}_N)^{-1}\mathbf{H}^T\mathbf{y}$$

$$= \mathbf{H}^T\mathbf{a} = \sum_{m=1}^{M} a_m \mathbf{h}_m$$ where $\mathbf{a} = (\mathbf{H}\mathbf{H}^T + \lambda \mathbf{I}_M)^{-1}\mathbf{y}$

Interpretation: $\mathbf{x}_{\mathrm{LS}} \in \mathrm{span}\{\mathbf{h}_m\}_{m=1}^{M}$

Lemma

$(\mathbf{H}^T\mathbf{H} + \lambda \mathbf{I}_N)^{-1}\mathbf{H}^T = \mathbf{H}^T(\mathbf{H}\mathbf{H}^T + \lambda \mathbf{I}_M)^{-1}$

Generalization: constrained l_2 minimization

- Discrete signal to reconstruct: $x = (x[n])_{n\in\mathbb{Z}}$
- Sensing operator $\mathrm{H}: \ell_2(\mathbb{Z}) \to \mathbb{R}^M$
  $x \mapsto \mathbf{z} = \mathrm{H}\{x\} = (\langle x, h_1\rangle, \dots, \langle x, h_M\rangle)$ with $h_m \in \ell_2(\mathbb{Z})$
- Closed convex set in measurement space: $\mathcal{C} \subset \mathbb{R}^M$

Example: $\mathcal{C}_{\mathbf{y}} = \{\mathbf{z} \in \mathbb{R}^M : \|\mathbf{y} - \mathbf{z}\|_2^2 \le \sigma^2\}$

Representer theorem for constrained ℓ_2 minimization

(P2) $\min_{x\in\ell_2(\mathbb{Z})} \|x\|_{\ell_2}^2$ s.t. $\mathrm{H}\{x\} \in \mathcal{C}$

The problem (P2) has a unique solution of the form

$$x_{\mathrm{LS}} = \sum_{m=1}^{M} a_m h_m = \mathrm{H}^*\{\mathbf{a}\}$$

with expansion coefficients $\mathbf{a} = (a_1, \cdots, a_M) \in \mathbb{R}^M$.

(U.-Fageot-Gupta *IEEE Trans. Info. Theory*, Sept. 2016)

Constrained l_1 minimization $\Rightarrow$ sparsifying effect

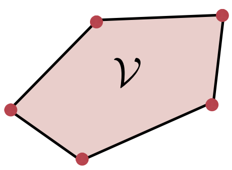
- Discrete signal to reconstruct: $x = (x[n])_{n\in\mathbb{Z}}$
- Sensing operator $\mathrm{H}: \ell_1(\mathbb{Z}) \to \mathbb{R}^M$
  $x \mapsto \mathbf{z} = \mathrm{H}\{x\} = (\langle x, h_1\rangle, \dots, \langle x, h_M\rangle)$ with $h_m \in \ell_\infty(\mathbb{Z})$
- Closed convex set in measurement space: $\mathcal{C} \subset \mathbb{R}^M$

Representer theorem for constrained ℓ_1 minimization

(P1) $\mathcal{V} = \arg\min_{x\in\ell_1(\mathbb{Z})} \|x\|_{\ell_1}$ s.t. $\mathrm{H}\{x\} \in \mathcal{C}$

is convex, weak*-compact with extreme points of the form

$$x_{\mathrm{sparse}}[\cdot] = \sum_{k=1}^{K} a_k \delta[\cdot - n_k] \quad \text{with} \quad K = \|x_{\mathrm{sparse}}\|_0 \le M.$$



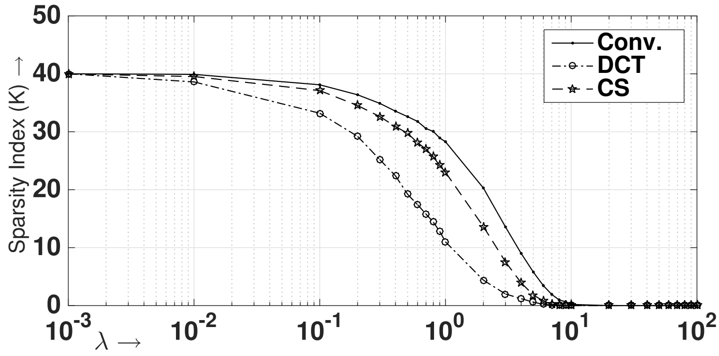
If CS condition is satisfied, then solution is unique

(U.-Fageot-Gupta *IEEE Trans. Info. Theory*, Sept. 2016)

Controlling sparsity

Measurement model: $y_m = \langle h_m, x \rangle + n[m], \quad m = 1, \ldots, M$

$$x_{\text{sparse}} = \arg \min_{x \in \ell_1(\mathbb{Z})} \left(\sum_{m=1}^{M} \left| y_m - \langle h_m, x \rangle \right|^2 + \lambda \|x\|_{\ell_1} \right)$$

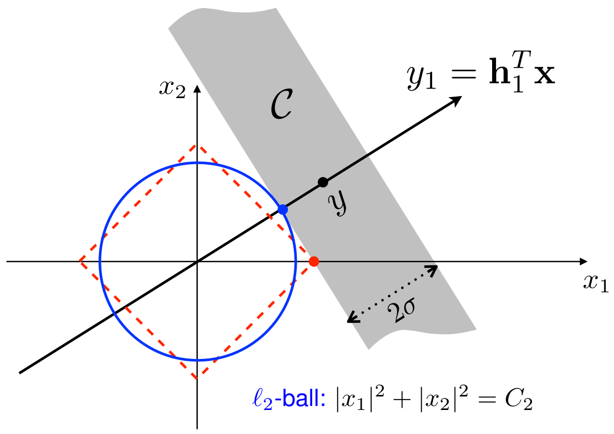


Geometry of l_2 vs. l_1 minimization

- Prototypical inverse problem

$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 + \lambda \|\mathbf{x}\|_{\ell_2}^2 \right\} \quad \Leftrightarrow \quad \min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_2} \text{ subject to } \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 \leq \sigma^2$$

$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 + \lambda \|\mathbf{x}\|_{\ell_1} \right\} \quad \Leftrightarrow \quad \min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_1} \text{ subject to } \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 \leq \sigma^2$$



ℓ_2-ball: $|x_1|^2 + |x_2|^2 = C_2$

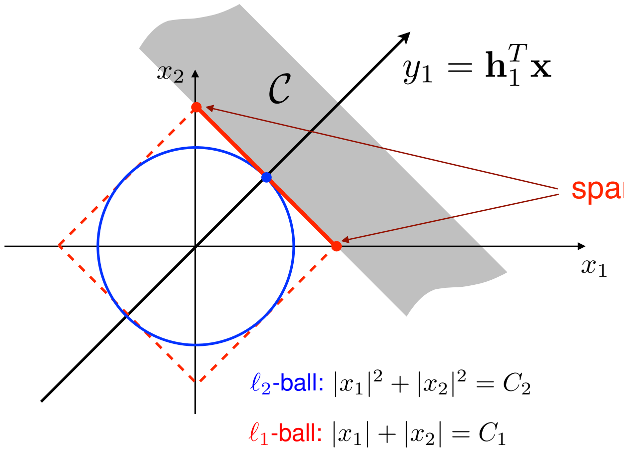
ℓ_1-ball: $|x_1| + |x_2| = C_1$

Geometry of l_2 vs. l_1 minimization

- Prototypical inverse problem

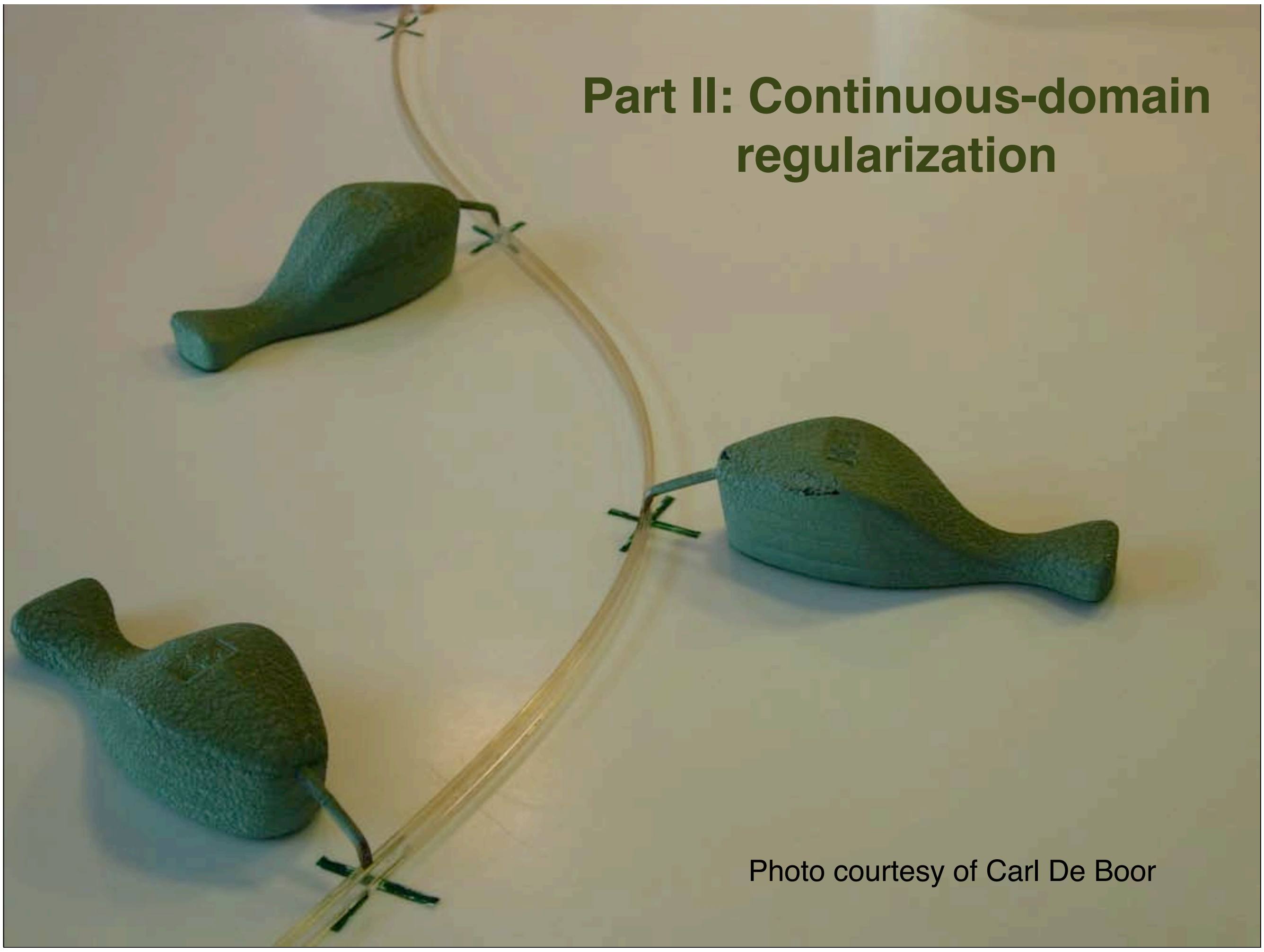
$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 + \lambda \|\mathbf{x}\|_{\ell_2}^2 \right\} \quad \Leftrightarrow \quad \min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_2} \text{ subject to } \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 \leq \sigma^2$$

$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 + \lambda \|\mathbf{x}\|_{\ell_1} \right\} \quad \Leftrightarrow \quad \min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_1} \text{ subject to } \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 \leq \sigma^2$$



Configuration for **non-unique** ℓ_1 solution

Part II: Continuous-domain regularization



Continuous-domain regularization (L_2 scenario)

Regularization functional: $\|\mathrm{L}f\|_{L_2}^2 = \int_{\mathbb{R}^d} |\mathrm{L}f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}$

L: suitable differential operator

- Theory of reproducing kernel Hilbert spaces (Aronszajn 1950)

  $\langle f, g \rangle_{\mathcal{H}} = \langle \mathrm{L}f, \mathrm{L}g \rangle$

- Interpolation and approximation theory
  - Smoothing splines (Schoenberg 1964, Kimeldorf-Wahba 1971)
  - Thin-plate splines, radial basis functions (Duchon 1977)

- Machine learning
  - Radial basis functions, kernel methods (Poggio-Girosi 1990)
  - Representer theorem(s) (Schölkopf-Smola 2001)

Representer theorem for L_2 regularization

(P2) $$\arg\min_{f\in\mathcal{H}} \left(\sum_{m=1}^{M} |y_m - f(\boldsymbol{x}_m)|^2 + \lambda \|f\|_{\mathcal{H}}^2 \right)$$

$h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the (unique) **reproducing kernel** for the Hilbert space $\mathcal{H}(\mathbb{R}^d)$ if

(i) $h(\boldsymbol{x}_0, \cdot) \in \mathcal{H}$ for all $\boldsymbol{x}_0 \in \mathbb{R}^d$

(ii) $f(\boldsymbol{x}_0) = \langle h(\boldsymbol{x}_0, \cdot), f\rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$

Convex loss function: $F : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$

Sample values: $\boldsymbol{f} = \big(f(\boldsymbol{x}_1), \dots, f(\boldsymbol{x}_M)\big)$

(P2') $$\arg\min_{f\in\mathcal{H}} \left(F(\boldsymbol{y}, \boldsymbol{f}) + \lambda \|f\|_{\mathcal{H}}^2 \right)$$

(Schölkopf-Smola 2001)

Representer theorem for L_2-regularization

The generic parametric form of the solution of (P2') is

$$f(\boldsymbol{x}) = \sum_{m=1}^{M} a_m h(\boldsymbol{x}, \boldsymbol{x}_m)$$

Supports the theory of SVM, kernel methods, variational splines, etc.

Sparsity and continuous-domain modeling

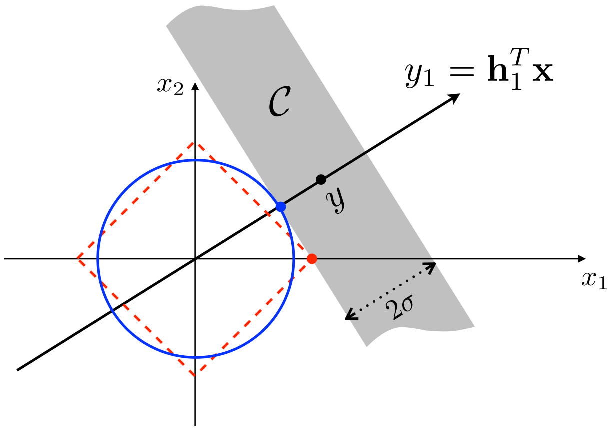
- Compressed sensing (CS)
  - Generalized sampling and infinite-dimensional CS (Adcock-Hansen, 2011)
  - Xampling: CS of analog signals (Eldar, 2011)
- Splines and approximation theory
  - L_1 splines (Fisher-Jerome, 1975)
  - Locally-adaptive regression splines (Mammen-van de Geer, 1997)
  - Generalized TV (Steidl et al. 2005; Bredies et al. 2010)
- Statistical modeling
  - Sparse stochastic processes (Unser et al. 2011-2014)

Geometry of l_2 vs. l_1 minimization

- Prototypical inverse problem

$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 + \lambda \|\mathbf{x}\|_{\ell_2}^2 \right\} \quad \Leftrightarrow \quad \min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_2} \ \text{ subject to } \ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 \le \sigma^2$$

$$\min_{\mathbf{x}} \left\{ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 + \lambda \|\mathbf{x}\|_{\ell_1} \right\} \quad \Leftrightarrow \quad \min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_1} \ \text{ subject to } \ \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_{\ell_2}^2 \le \sigma^2$$

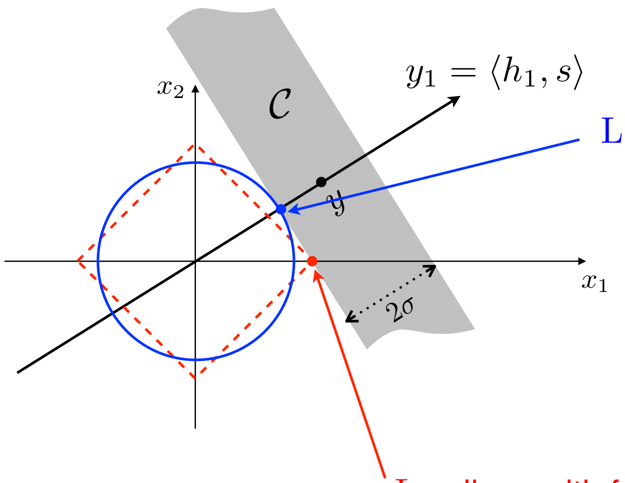


"Geometry" of L_2 vs. TV minimization

- Prototypical inverse problem

$$\min_{s} \left\{ \|\mathbf{y} - \mathrm{H}\{s\}\|_{\ell_2}^2 + \lambda \|\mathrm{L}\{s\}\|_{L_2}^2 \right\} \quad \Leftrightarrow \quad \min_{s} \|\mathrm{L}\{s\}\|_{L_2}^2 \ \text{ subject to } \ \|\mathbf{y} - \mathrm{H}\{s\}\|_{\ell_2}^2 \le \sigma^2$$

$$\min_{s} \left\{ \|\mathbf{y} - \mathrm{H}\{s\}\|_{\ell_2}^2 + \lambda \|\mathrm{L}\{s\}\|_{\mathcal{M}} \right\} \quad \Leftrightarrow \quad \min_{s} \|\mathrm{L}\{s\}\|_{\mathcal{M}} \ \text{ subject to } \ \|\mathbf{y} - \mathrm{H}\{s\}\|_{\ell_2}^2 \le \sigma^2$$



$\mathrm{L}^*\mathrm{L}$-splines with M **fixed** knots
(H: pure sampling operator)

L-splines with few **adaptive** knots
(H: can be arbitrary)

Splines are analog and intrinsically sparse

$\mathrm{L}\{\cdot\}$: admissible differential operator

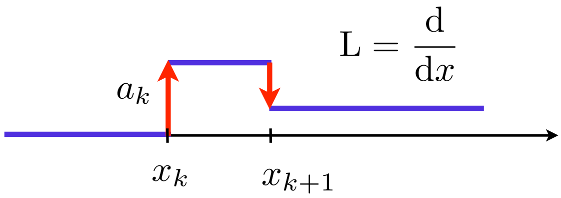
$\delta(\cdot - \boldsymbol{x}_0)$: Dirac impulse shifted by $\boldsymbol{x}_0 \in \mathbb{R}^d$

Definition

The function $s : \mathbb{R}^d \to \mathbb{R}$ is a (non-uniform) L-spline with knots $(\boldsymbol{x}_k)_{k=1}^K$ if

$$\mathrm{L}\{s\} = \sum_{k=1}^{K} a_k \delta(\cdot - \boldsymbol{x}_k) = w_\delta \quad : \text{ spline's innovation}$$

Spline theory: (Schultz-Varga, 1967)



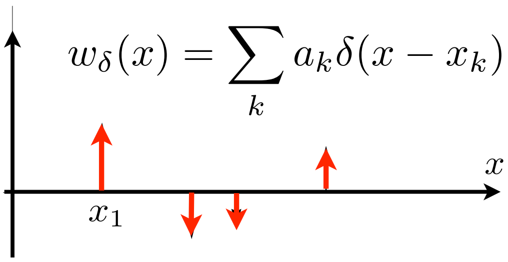
- FRI signal processing: Innovation variables $(2K)$ (Vetterli et al., 2002)
  - Location of singularities (knots) : $\{x_k\}_{k=1}^K$
  - Strength of singularities (linear weights): $\{a_k\}_{k=1}^K$

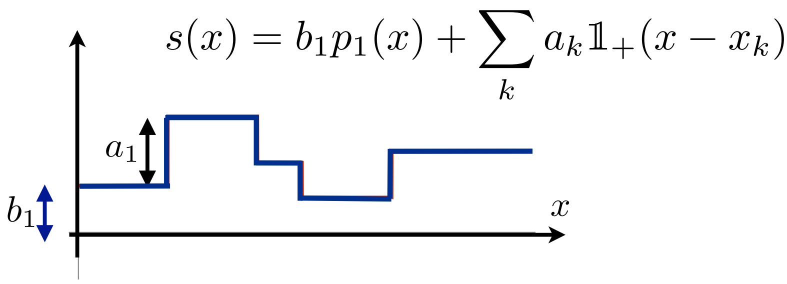
Spline synthesis: example

$$\mathrm{L} = \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}$$

Null space: $\mathcal{N}_\mathrm{D} = \mathrm{span}\{p_1\}, \quad p_1(x) = 1$

$\rho_\mathrm{D}(x) = \mathrm{D}^{-1}\{\delta\}(x) = \mathbb{1}_+(x)$: Heaviside function





Spline synthesis: generalization

L: spline admissible operator (LSI)

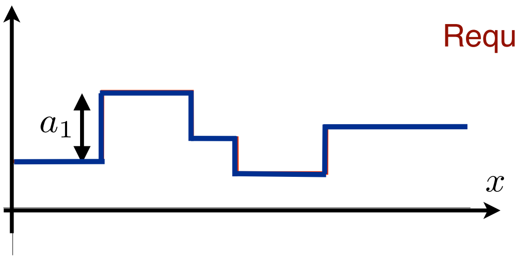
$\rho_{\mathrm{L}}(\boldsymbol{x}) = \mathrm{L}^{-1}\{\delta\}(\boldsymbol{x})$: Green's function of L

Finite-dimensional null space: $\mathcal{N}_{\mathrm{L}} = \mathrm{span}\{p_n\}_{n=1}^{N_0}$

Spline's innovation: $w_\delta(\boldsymbol{x}) = \sum_k a_k \delta(\boldsymbol{x} - \boldsymbol{x}_k)$

$$\Rightarrow \quad s(\boldsymbol{x}) = \sum_k a_k \rho_{\mathrm{L}}(\boldsymbol{x} - \boldsymbol{x}_k) + \sum_{n=1}^{N_0} b_n p_n(\boldsymbol{x})$$

Requires specification of boundary conditions



Principled operator-based approach

- Biorthogonal basis of $\mathcal{N}_{\mathrm{L}} = \mathrm{span}\{p_n\}_{n=1}^{N_0}$
  - $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_{N_0})$ such that $\langle \phi_m, p_n \rangle = \delta_{m,n}$
  - Projection operator: $p = \sum_{n=1}^{N_0} \langle \phi_n, p \rangle p_n$ for all $p \in \mathcal{N}_{\mathrm{L}}$
- Operator-based spline synthesis
  - Boundary conditions: $\langle s, \phi_n \rangle = b_n, \ n = 1, \cdots, N_0$
  - Spline's innovation: $\mathrm{L}\{s\} = w_\delta = \sum_k a_k \delta(\cdot - \boldsymbol{x}_k)$

$$s(\boldsymbol{x}) = \mathrm{L}_{\boldsymbol{\phi}}^{-1}\{w_\delta\}(\boldsymbol{x}) + \sum_{n=1}^{N_0} b_n p_n(\boldsymbol{x})$$

- Existence of $\mathrm{L}_{\boldsymbol{\phi}}^{-1}$ as a stable right-inverse of L ? (see **Theorem 1**)
  - $\mathrm{L}\mathrm{L}_{\boldsymbol{\phi}}^{-1} w = w$
  - $\boldsymbol{\phi}(\mathrm{L}_{\boldsymbol{\phi}}^{-1} w) = \mathbf{0}$

From Dirac impulses to Borel measures

$\mathcal{S}(\mathbb{R}^d)$: Schwartz's space of smooth and rapidly decaying test functions on $\mathbb{R}^d$

$\mathcal{S}'(\mathbb{R}^d)$: Schwartz's space of tempered distributions

- Space of real-valued, countably additive Borel measures on $\mathbb{R}^d$

$$\mathcal{M}(\mathbb{R}^d) = \big(C_0(\mathbb{R}^d)\big)' = \big\{w \in \mathcal{S}'(\mathbb{R}^d) : \|w\|_{\mathcal{M}} = \sup_{\varphi \in \mathcal{S}(\mathbb{R}^d) : \|\varphi\|_\infty = 1} \langle w, \varphi \rangle < \infty \big\},$$

where $w : \varphi \mapsto \langle w, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(\boldsymbol{r}) w(\boldsymbol{r}) \mathrm{d}\boldsymbol{r}$

- Equivalent definition of "total variation" norm

$$\|w\|_{\mathcal{M}} = \sup_{\varphi \in C_0(\mathbb{R}^d) : \|\varphi\|_\infty = 1} \langle w, \varphi \rangle$$

- Basic inclusions
  - $\delta(\cdot - \boldsymbol{x}_0) \in \mathcal{M}(\mathbb{R}^d)$ with $\|\delta(\cdot - \boldsymbol{x}_0)\|_{\mathcal{M}} = 1$ for any $\boldsymbol{x}_0 \in \mathbb{R}^d$
  - $\|f\|_{\mathcal{M}} = \|f\|_{L_1(\mathbb{R}^d)}$ for all $f \in L_1(\mathbb{R}^d) \quad \Rightarrow \quad L_1(\mathbb{R}^d) \subseteq \mathcal{M}(\mathbb{R}^d)$

Optimality result for Dirac measures

- $\mathbf{H}$: linear continuous map $\mathcal{M}(\mathbb{R}^d) \to \mathbb{R}^M$
- $\mathcal{C}$: convex compact subset of $\mathbb{R}^M$
- Generic constrained TV minimization problem

$$\mathcal{V} = \arg \min_{w \in \mathcal{M}(\mathbb{R}^d)\,:\,\mathbf{H}(w) \in \mathcal{C}} \|w\|_{\mathcal{M}}$$

Generalized Fisher-Jerome theorem

The solution set $\mathcal{V}$ is a **convex, weak$*$-compact** subset of $\mathcal{M}(\mathbb{R}^d)$ with **extremal points** of the form

$$w_\delta = \sum_{k=1}^{K} a_k \delta(\cdot - \boldsymbol{x}_k)$$

with $K \leq M$ and $\boldsymbol{x}_k \in \mathbb{R}^d$.

(U.-Fageot-Ward, ArXiv 2016)

Jerome-Fisher, 1975: Compact domain & scalar intervals

General convex problems with gTV regularization

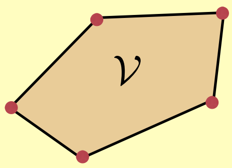
$$\mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) = \left\{ s : \mathrm{gTV}(s) = \|\mathrm{L}\{s\}\|_{\mathcal{M}} = \sup_{\|\varphi\|_\infty \leq 1} \langle \mathrm{L}\{s\}, \varphi \rangle < \infty \right\}$$

- **Linear** measurement operator $\mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) \to \mathbb{R}^M : f \mapsto \mathbf{z} = \mathrm{H}\{f\}$
- $\mathcal{C}$: **convex** compact subset of $\mathbb{R}^M$
- Finite-dimensional **null space** $\mathcal{N}_{\mathrm{L}} = \{q \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) : \mathrm{L}\{q\} = 0\}$ with basis $\{p_n\}_{n=1}^{N_0}$

Admissibility of regularization: $\mathrm{H}\{q_1\} = \mathrm{H}\{q_2\} \Leftrightarrow q_1 = q_2$ for all $q_1, q_2 \in \mathcal{N}_{\mathrm{L}}$

Theorem (gTV optimality of spline for linear inverse problems)

The extremal points of the constrained minimization problem



$$\mathcal{V} = \arg \min_{f \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d)} \|\mathrm{L}\{f\}\|_{\mathcal{M}} \quad \text{s.t.} \quad \mathrm{H}\{f\} \in \mathcal{C}$$

are necessarily of the form $f(\boldsymbol{x}) = \sum_{k=1}^{K} a_k \rho_{\mathrm{L}}(\boldsymbol{x} - \boldsymbol{x}_k) + \sum_{n=1}^{N_0} b_n p_n(\boldsymbol{x})$ with $K \leq M - N_0$; that is, **non-uniform** L**-splines** with knots at the $\boldsymbol{x}_k$ and $\|\mathrm{L}\{f\}\|_{\mathcal{M}} = \sum_{k=1} |a_k|$. The full solution set is the **convex hull** of those extremal points.

(U.-Fageot-Ward, ArXiv 2016)

Representer theorem for gTV regularization

$$\text{(P1)} \quad \arg \min_{f \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d)} \left(\sum_{m=1}^{M} |y_m - \langle h_m, f \rangle|^2 + \lambda \|\mathrm{L}f\|_{\mathcal{M}} \right)$$

- L: spline-admissible operator with null space $\mathcal{N}_{\mathrm{L}} = \mathrm{span}\{p_n\}_{n=1}^{N_0}$
- gTV semi-norm: $\|\mathrm{L}\{s\}\|_{\mathcal{M}} = \sup_{\|\varphi\|_\infty \le 1} \langle \mathrm{L}\{s\}, \varphi \rangle$
- Measurement functionals $h_m : \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) \to \mathbb{R}$ (weak$*$-continuous)

Convex loss function: $F : \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$

Measurements: $\boldsymbol{z} = \big(\langle h_1, f \rangle, \dots, \langle h_M, f \rangle\big)$

$$\text{(P1')} \quad \arg \min_{f \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d)} \left(F(\boldsymbol{y}, \boldsymbol{z}) + \lambda \|\mathrm{L}f\|_{\mathcal{M}} \right)$$

Representer theorem for gTV-regularization

The extreme points of (P1') are **non-uniform** L**-spline** of the form

$$f_{\mathrm{spline}}(\boldsymbol{x}) = \sum_{k=1}^{K_{\mathrm{knots}}} a_k \rho_{\mathrm{L}}(\boldsymbol{x} - \boldsymbol{x}_k) + \sum_{n=1}^{N_0} b_n p_n(\boldsymbol{x})$$

with ρ_{L} such that $\mathrm{L}\{\rho_{\mathrm{L}}\} = \delta$, $K_{\mathrm{knots}} \le M - N_0$, and $\|\mathrm{L}f_{\mathrm{spline}}\|_{\mathcal{M}} = \|\mathbf{a}\|_{\ell_1}$.

All solutions have the same measurements $\boldsymbol{z}_0 = (\langle h_1, f_{\mathrm{spline}} \rangle, \dots, \langle h_M, f_{\mathrm{spline}} \rangle) \in \mathbb{R}^M$

Enabling components for proof of the theorem

Existence of stable right-inverse operator

$$L_{\infty,n_0}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} : \sup_{\boldsymbol{x}\in\mathbb{R}^d} \left(|f(\boldsymbol{x})|(1+\|\boldsymbol{x}\|)^{-n_0}\right) < +\infty\}$$

Theorem 1 (U.-Fageot-Ward, ArXiv 2016)
Let L be a spline-admissible operator with a N_0-dimensional null space $\mathcal{N}_{\mathrm{L}} \subseteq L_{\infty,n_0}(\mathbb{R}^d)$ such that $p = \sum_{n=1}^{N_0}\langle p, \phi_n\rangle p_n$ for all $p \in \mathcal{N}_{\mathrm{L}}$. Then, there exists a **unique and stable operator** $\mathrm{L}_{\boldsymbol{\phi}}^{-1} : \mathcal{M}(\mathbb{R}^d) \to L_{\infty,n_0}(\mathbb{R}^d)$ such that, for all $w \in \mathcal{M}(\mathbb{R}^d)$,

- Right-inverse property: $\mathrm{L}\mathrm{L}_{\boldsymbol{\phi}}^{-1}w = w$,
- Boundary conditions: $\boldsymbol{\phi}(\mathrm{L}_{\boldsymbol{\phi}}^{-1}w) = \mathbf{0}$ with $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_{N_0})$.

Its generalized impulse response $g_{\boldsymbol{\phi}}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{L}_{\boldsymbol{\phi}}^{-1}\{\delta(\cdot - \boldsymbol{y})\}(\boldsymbol{x})$ is given by

$$g_{\boldsymbol{\phi}}(\boldsymbol{x},\boldsymbol{y}) = \rho_{\mathrm{L}}(\boldsymbol{x}-\boldsymbol{y}) - \sum_{n=1}^{N_0} p_n(\boldsymbol{x})q_n(\boldsymbol{y})$$

with ρ_{L} such that $\mathrm{L}\{\rho_{\mathrm{L}}\} = \delta$ and $q_n(\boldsymbol{y}) = \langle \phi_n, \rho_{\mathrm{L}}(\cdot - \boldsymbol{y})\rangle$.

Characterization of generalized Beppo-Levi spaces

- Regularization operator $\mathrm{L} : \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$

$$f \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) \quad \Leftrightarrow \quad \mathrm{gTV}(f) = \|\mathrm{L}\{f\}\|_{\mathcal{M}} < \infty$$

Theorem 2 (U.-Fageot-Ward, ArXiv 2016)
Let L be a spline-admissible operator that admits a stable right-inverse $\mathrm{L}_{\boldsymbol{\phi}}^{-1}$ of the form specified by Theorem 1. Then, any $f \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d)$ has a unique representation as

$$f = \mathrm{L}_{\boldsymbol{\phi}}^{-1}w + p,$$

where $w = \mathrm{L}\{f\} \in \mathcal{M}(\mathbb{R}^d)$ and $p = \sum_{n=1}^{N_0}\langle \phi_n, f\rangle p_n \in \mathcal{N}_{\mathrm{L}}$ with $\phi_n \in \left(\mathcal{M}_{\mathrm{L}}(\mathbb{R}^d)\right)'$.
Moreover, $\mathcal{M}_{\mathrm{L}}(\mathbb{R}^d)$ is a Banach space equipped with the norm

$$\|f\|_{\mathrm{L},\boldsymbol{\phi}} = \|\mathrm{L}f\|_{\mathcal{M}} + \|\boldsymbol{\phi}(f)\|_2.$$

- Generalized Beppo-Levi space: $\mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) = \mathcal{M}_{\mathrm{L},\boldsymbol{\phi}}(\mathbb{R}^d) \oplus \mathcal{N}_{\mathrm{L}}$

$$\mathcal{M}_{\mathrm{L},\boldsymbol{\phi}}(\mathbb{R}^d) = \left\{f \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) : \boldsymbol{\phi}(f) = \mathbf{0}\right\}$$

$$\mathcal{N}_{\mathrm{L}} = \left\{p \in \mathcal{M}_{\mathrm{L}}(\mathbb{R}^d) : \mathrm{L}\{p\} = 0\right\}$$

Example: Convex problem with TV regularization

$$\mathrm{L} = \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}$$

$\mathcal{N}_{\mathrm{D}} = \mathrm{span}\{p_1\}, \quad p_1(x) = 1$

$\rho_{\mathrm{D}}(x) = \mathbb{1}_+(x)$: Heaviside function

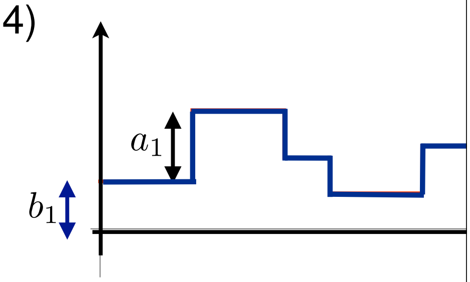
- General linear-inverse problem with TV regularization

$$\min_{s \in \mathcal{M}_{\mathrm{D}}(\mathbb{R})} \|\mathrm{D}\{s\}\|_{\mathcal{M}} \quad \text{s.t.} \quad \mathrm{H}\{s\} = (\langle h_1, s\rangle, \cdots, \langle h_M, s\rangle) \in \mathcal{C}(\mathbf{y})$$

- Generic form of the solution (by Theorem 4)

$$s(x) = b_1 + \sum_{k=1}^{K} a_k \mathbb{1}_+(x - x_k)$$

no penalty



with $K < M$ and free parameters b_1 and $(a_k, x_k)_{k=1}^{K}$

SUMMARY: Sparsity in infinite dimensions

- Discrete-domain formulation
  - Contrasting behavior of l_1 vs. l_2 regularization
  - Minimization of l_1 favors sparse solutions (independently of sensing matrix)
- Continuous-domain formulation
  - Linear measurement model
  - Linear signal model: PDE

$s \in \mathcal{X}$

$s \mapsto \mathbf{z} = \mathrm{H}\{s\}$

$\mathrm{L}s = w \quad \Rightarrow \quad s = \mathrm{L}^{-1}w$

  - L-splines = signals with “sparsest” innovation $\quad \mathrm{gTV}(s) = \|\mathrm{L}s\|_{\mathcal{M}}$
- Deterministic optimality result
  - gTV **regularization**: favors “sparse” innovations
  - Non-uniform L-splines: ***universal*** solutions of linear inverse problems

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- Preprints and demos: http://bigwww.epfl.ch/

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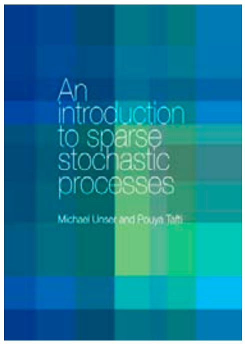
  - **For splines**: see chapter 6



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