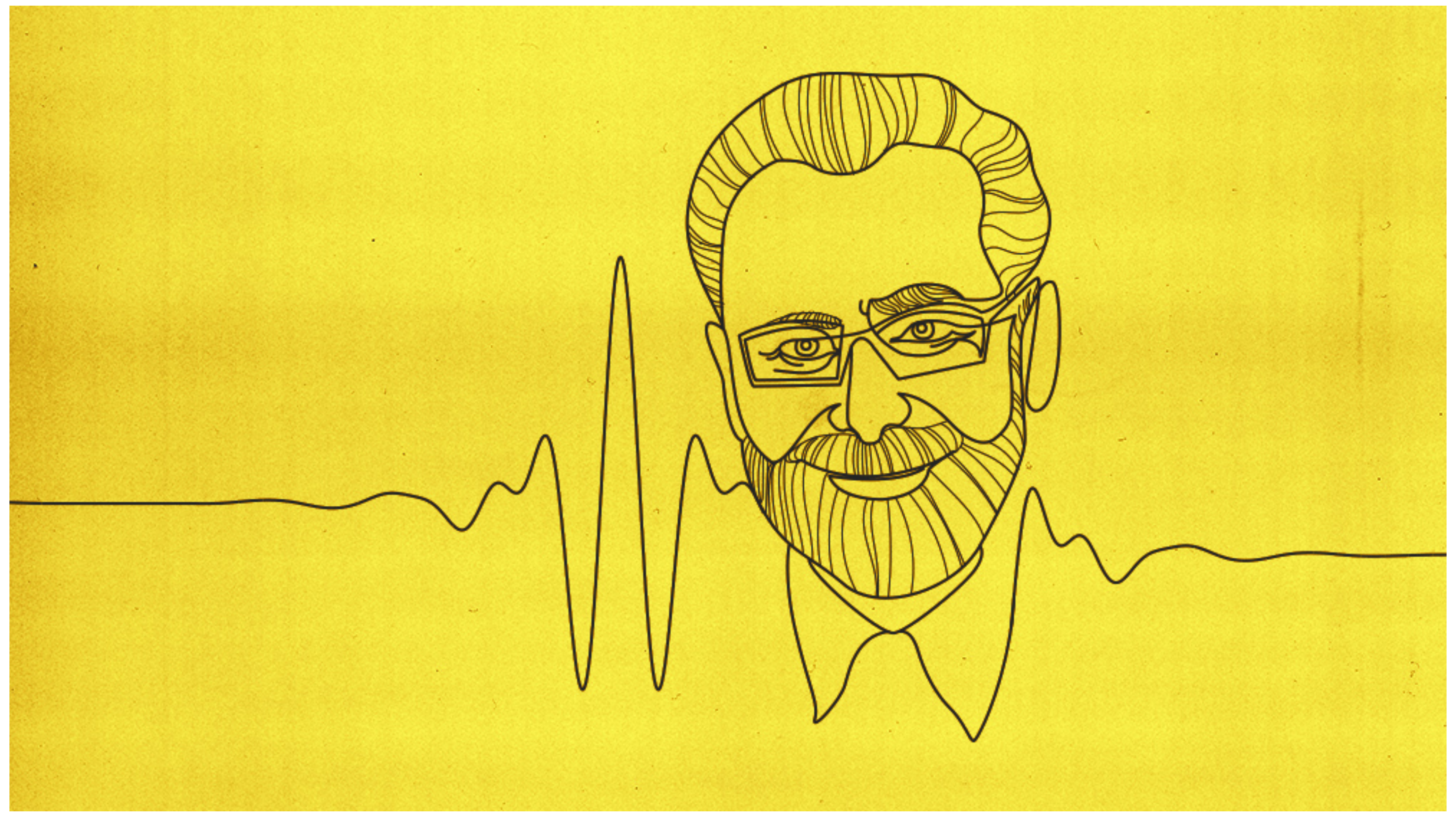
EPFL

The work of Yves Meyer (Abel Prize 2017)
Wavelets in harmonic analysis and signal processing

Michael Unser
Biomedical Imaging Group
EPFL, Lausanne, Switzerland



Olena Shmahalo/Quanta Magazine

Abel Prize Day 2020, Swiss Mathematical Society, September 25, 2020, University Bern.

Yves Meyer

- Born 19 July 1939
- Studied at the Lycée Carnot in Tunis
- Won French General Student Competition (Concours Général) in Mathematics
- First place, entrance exam, ENS Ulm, 1957
- Professorships
  - Université Strasbourg (1980–1986)
  - Université Paris-Sud (1966–1980)
  - Ecole Polytechnique (1980–1986)
  - Université Paris-Dauphine (1985–1995)
  - ENS-Cachan (1999-2003)
- 2010 Gauss Prize
- 2017 Abel Prize

Source Wikipédia

Brief history of wavelets

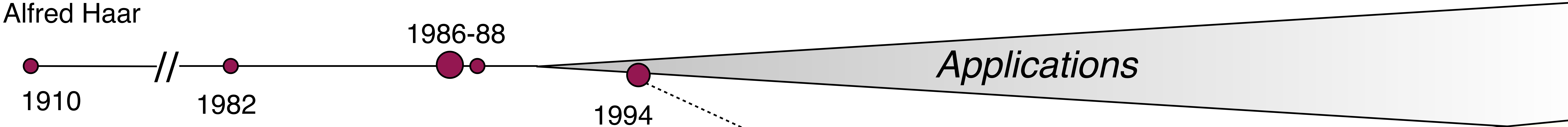
Fundamental tool for functional and numerical analysis

Yves Meyer

Stéphane Mallat

Ingrid Daubechies

Alfred Haar



Sparsity

Martin Vetterli

David Donoho

Emmanuel Candès

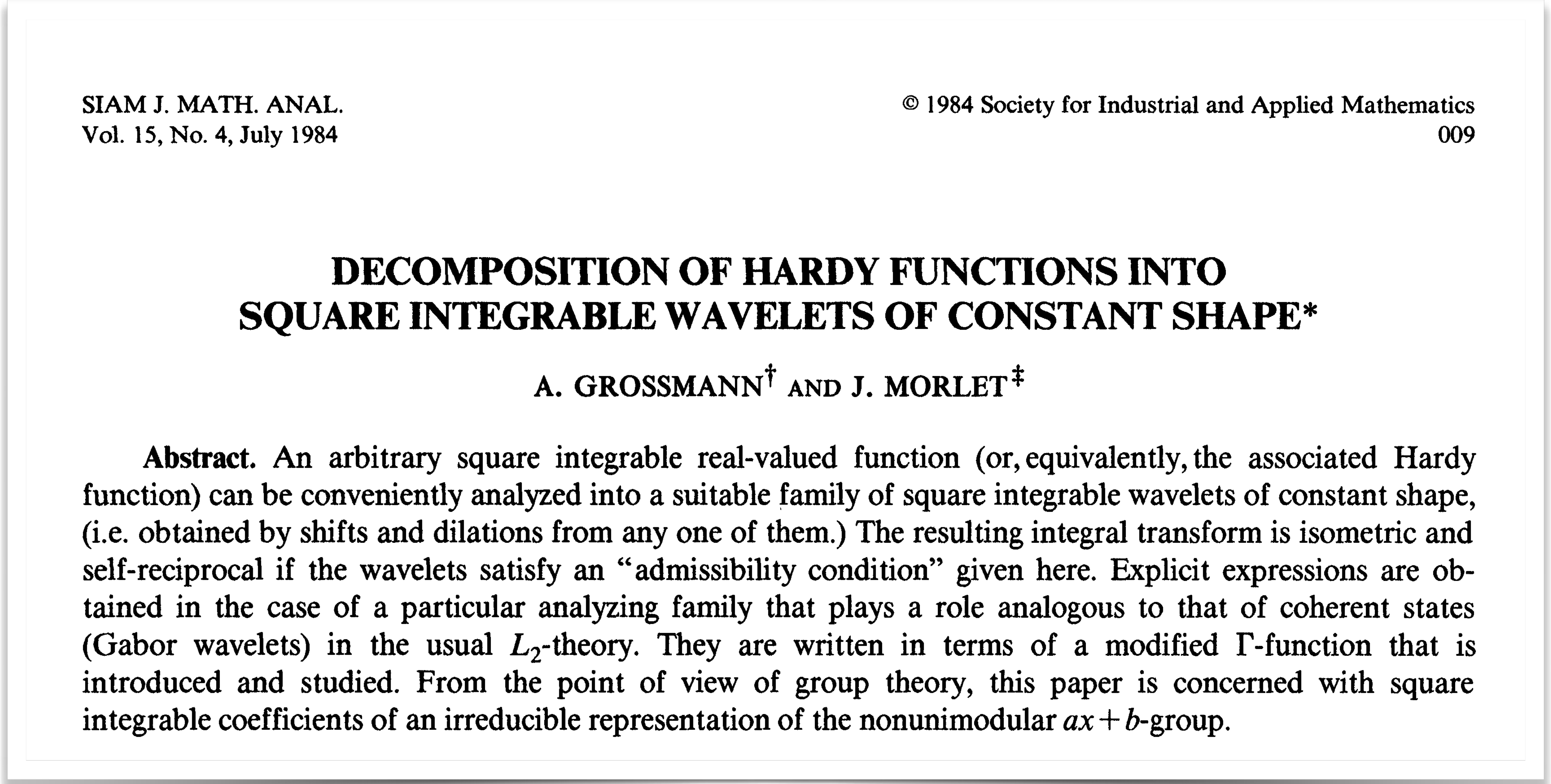

EPFL
Docteur Honoris Causa
Yves Meyer

OUTLINE

- **Construction of wavelet bases**
  - Meyer wavelet
  - The mathematical micoscope
  - Multi-resolution analysis
  - Applications: coding, etc.

- **Wavelets and functional analysis**
  - Wavelets and (fractional) differentiation
  - Wavelets and Besov spaces
  - Best *N*-term approximations
- **Wavelets and sparsity**
  - Denoising
  - Biomedical image reconstruction
  - Compressed sensing
- **Conclusion**

The precursors: the continuous wavelet transform



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Vol. 15, No. 4, July 1984

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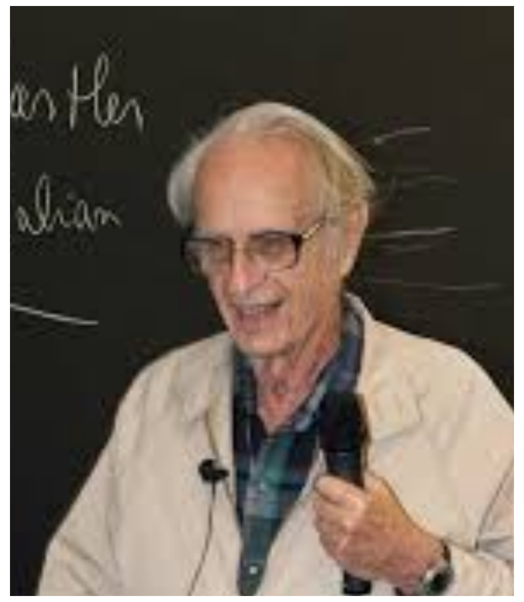
009

DECOMPOSITION OF HARDY FUNCTIONS INTO SQUARE INTEGRABLE WAVELETS OF CONSTANT SHAPE*

A. GROSSMANN† AND J. MORLET‡

Abstract. An arbitrary square integrable real-valued function (or, equivalently, the associated Hardy function) can be conveniently analyzed into a suitable family of square integrable wavelets of constant shape, (i.e. obtained by shifts and dilations from any one of them.) The resulting integral transform is isometric and self-reciprocal if the wavelets satisfy an "admissibility condition" given here. Explicit expressions are obtained in the case of a particular analyzing family that plays a role analogous to that of coherent states (Gabor wavelets) in the usual L_2-theory. They are written in terms of a modified Γ-function that is introduced and studied. From the point of view of group theory, this paper is concerned with square integrable coefficients of an irreducible representation of the nonunimodular $ax+b$-group.

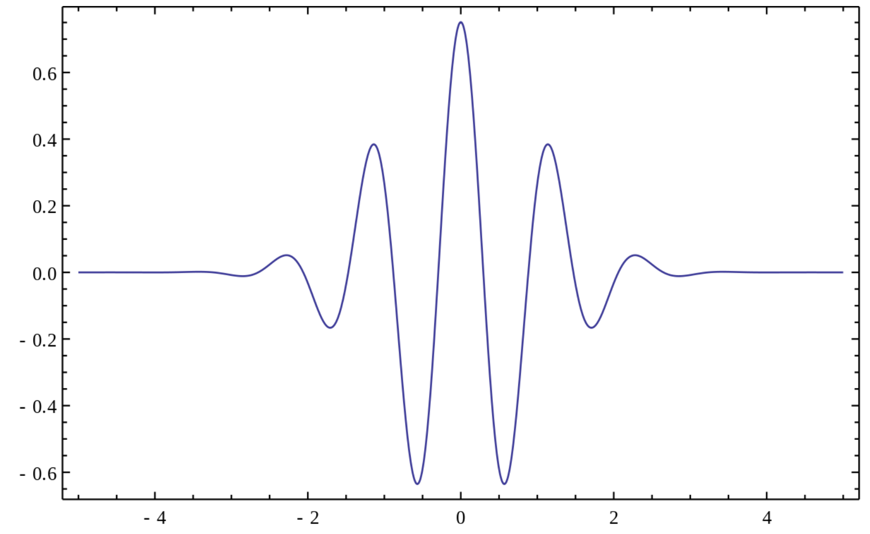
Jean Morlet (1931-2007)



Alex Grossmann (1930-2019)

$$W_\psi\{f\}(\tau,s) = \int_{\mathbb{R}} f(t)\frac{1}{\sqrt{s}}\overline{\psi\left(\frac{t-\tau}{s}\right)}\mathrm{d}t = \langle f, \psi_{\mathbf{s},\tau}\rangle$$

$$\psi(t)$$



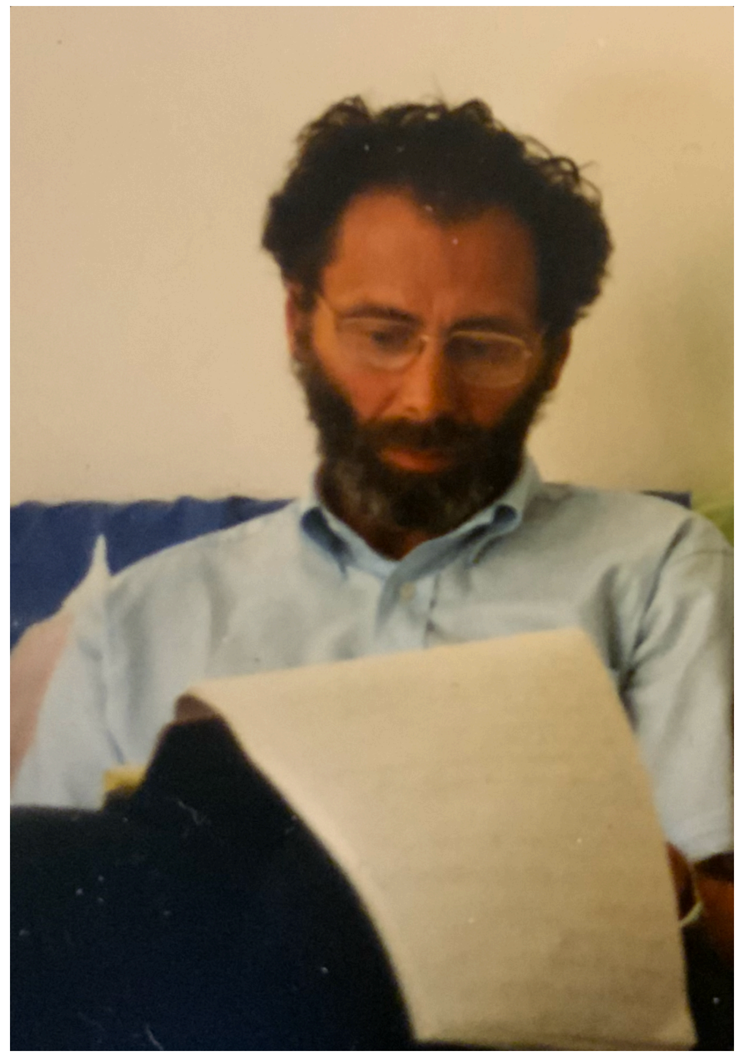
$$f(t) = \frac{1}{C_\psi}\int_0^\infty \int_{\mathbb{R}} W_\psi\{f\}(\tau,s)\psi_{\mathbf{s},\tau}(t)\mathrm{d}\tau\frac{\mathrm{d}s}{s^2}$$

$$\psi_{\mathbf{s},\tau} \triangleq \frac{1}{\sqrt{s}}\psi\left(\frac{\cdot-\tau}{s}\right)$$

Nice, but (strongly) **overcomplete** …

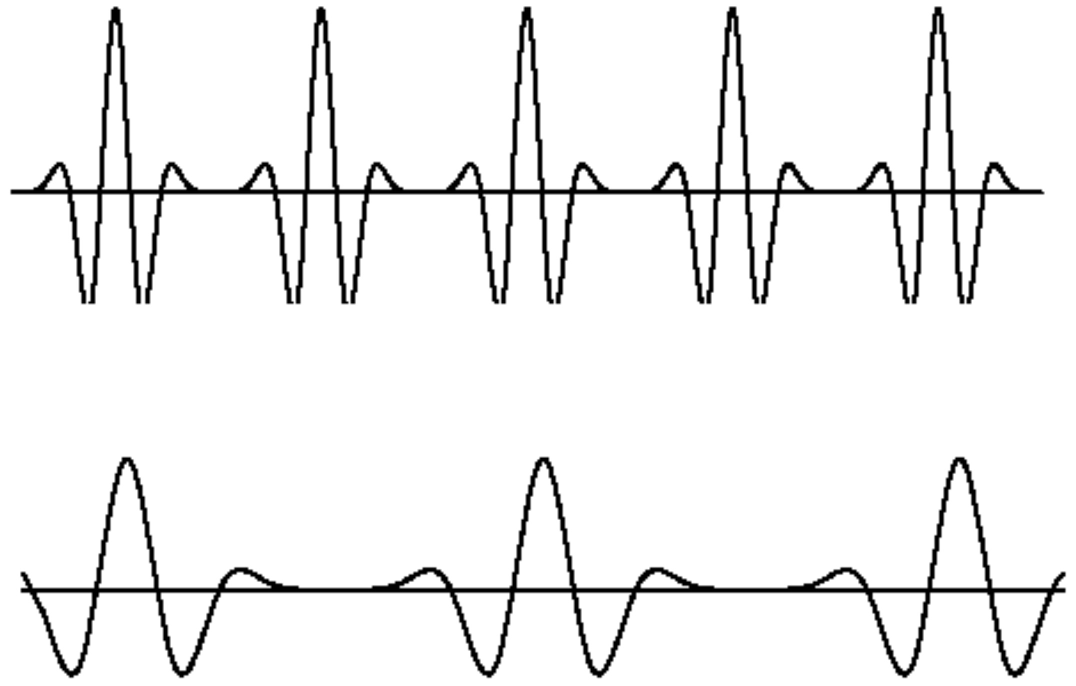
$$C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{\omega}\mathrm{d}\omega < \infty$$

1. CONSTRUCTION OF WAVELET BASES



- Wavelet basis functions
  - Dilation and translation of a single prototype, but with critical sampling

$$\psi_{i,k} = 2^{-i/2}\psi\left(\frac{x - 2^i k}{2^i}\right)$$

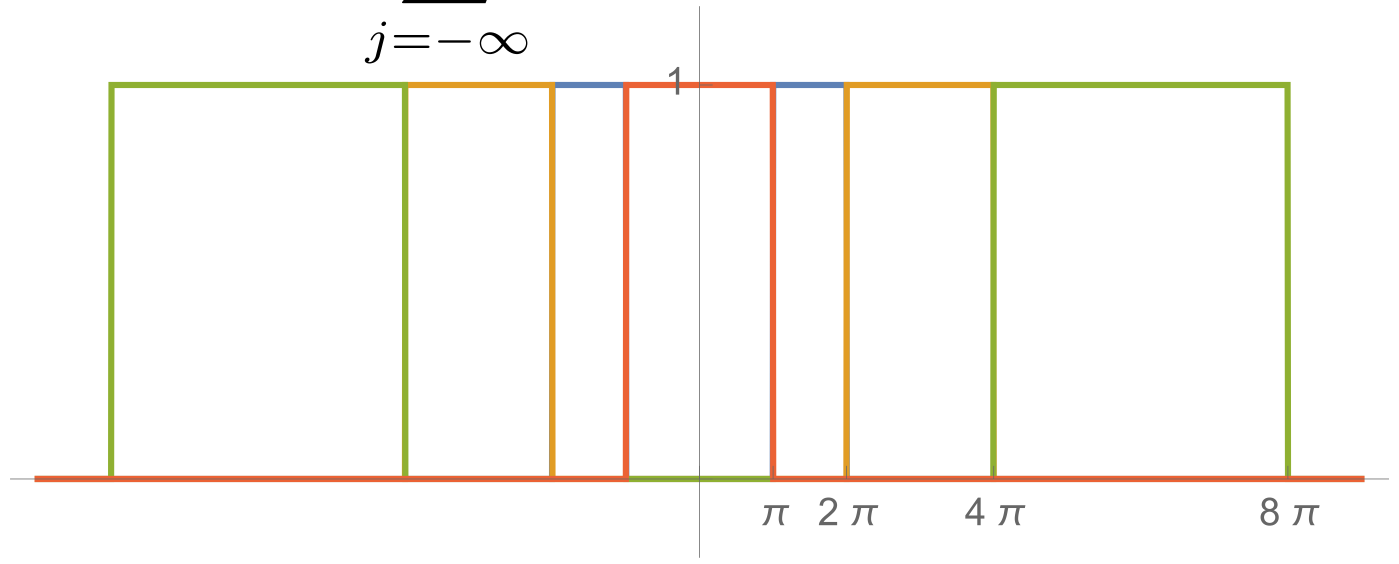


- The Meyer wavelet
- Multi-resolution analysis
- Applications to image coding

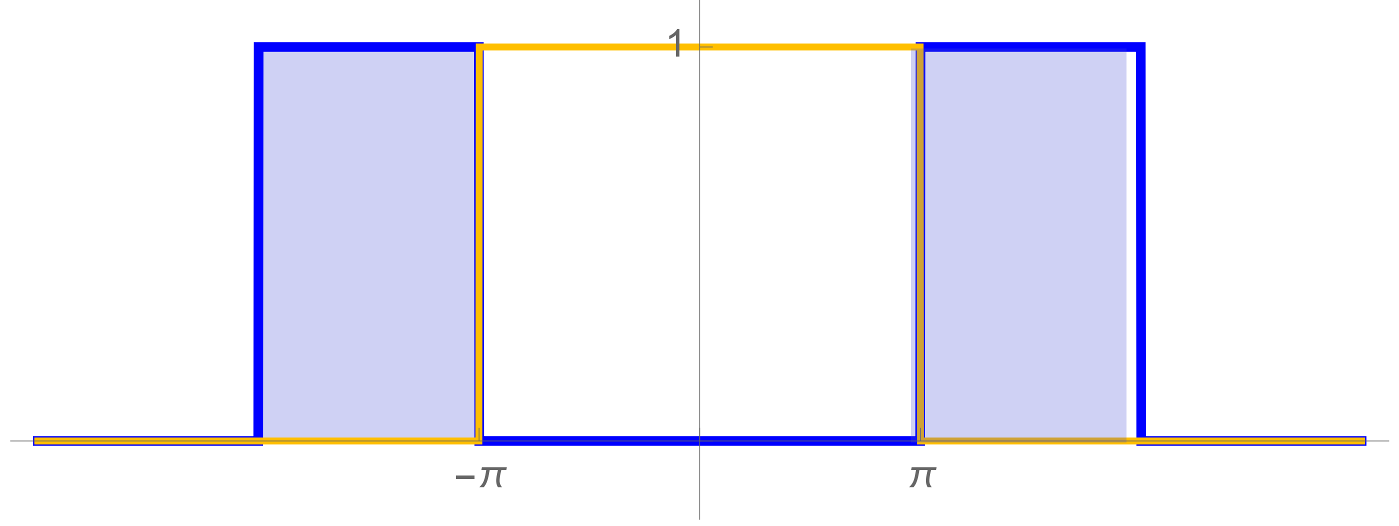
Littlewood-Paley decomposition / Shannon wavelet

$$B_j = [-2^{j+1}\pi, -2^j\pi] \bigcup [2^j\pi, 2^{j+1}\pi]$$

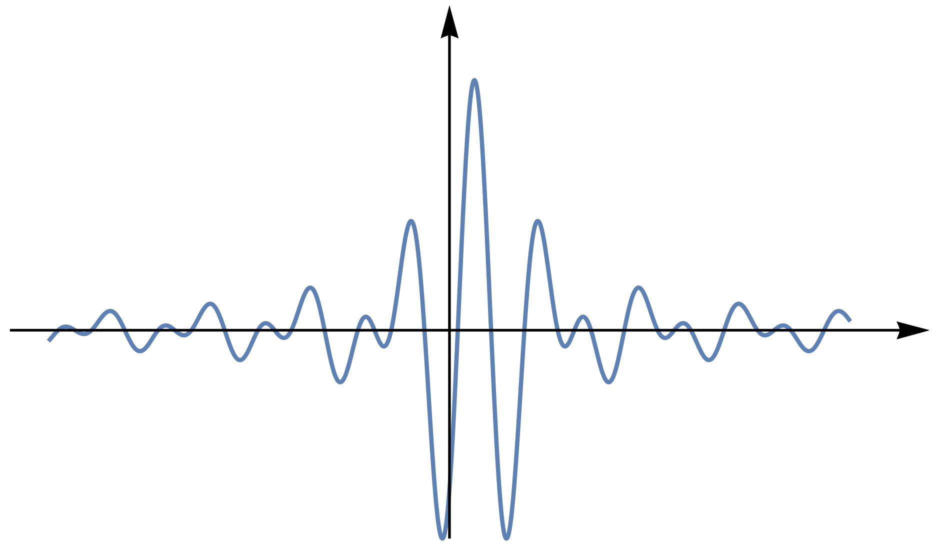
$$\sum_{j=-\infty}^{\infty} |\hat{\psi}_{\mathrm{Sha}}(2^j\omega)|^2 = 1$$



$\hat{\psi}_{\mathrm{Sha}}(\omega)$



$$\psi_{\mathrm{Sha}}(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)$$



$$f(t) = \sum_{j=-\infty}^{\infty} \underbrace{\frac{1}{2\pi}\int_{B_j} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega}_{r_j}$$

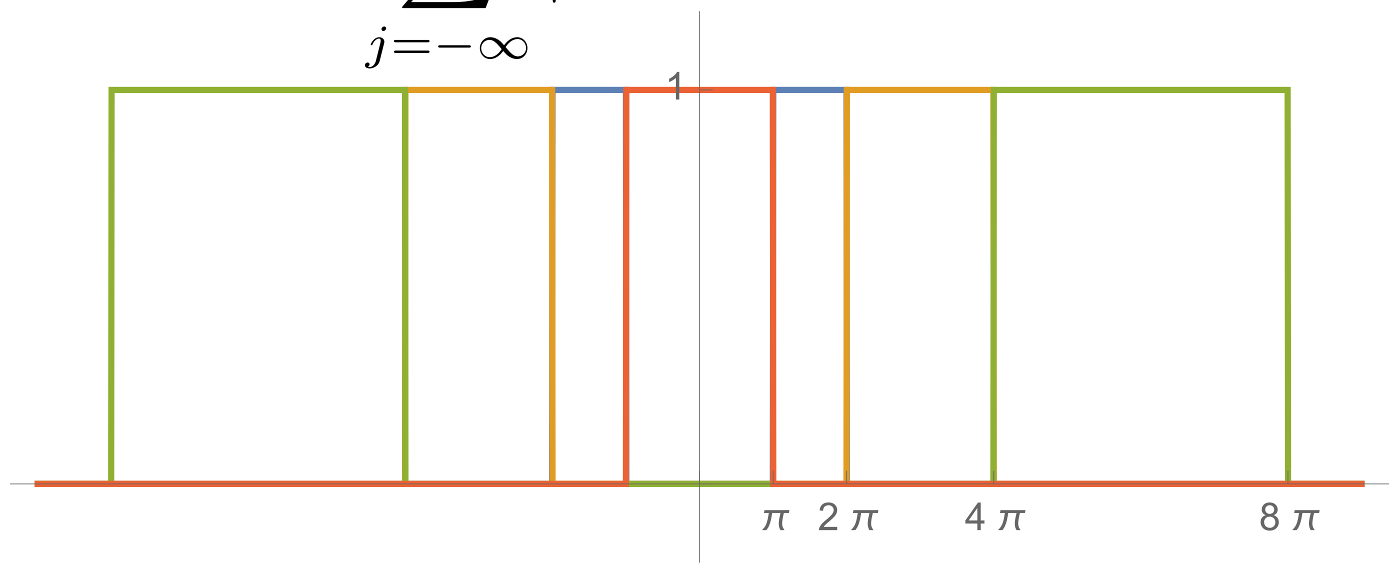
$$r_0(t) = \sum_{k\in\mathbb{Z}} \langle f, \psi_{\mathrm{Sha}}(\cdot - k)\rangle\ \psi_{\mathrm{Sha}}(\cdot - k)$$

$$\forall f \in L_2(\mathbb{R}): \quad f(t) = \sum_{j=\infty}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\ \psi_{j,k}$$

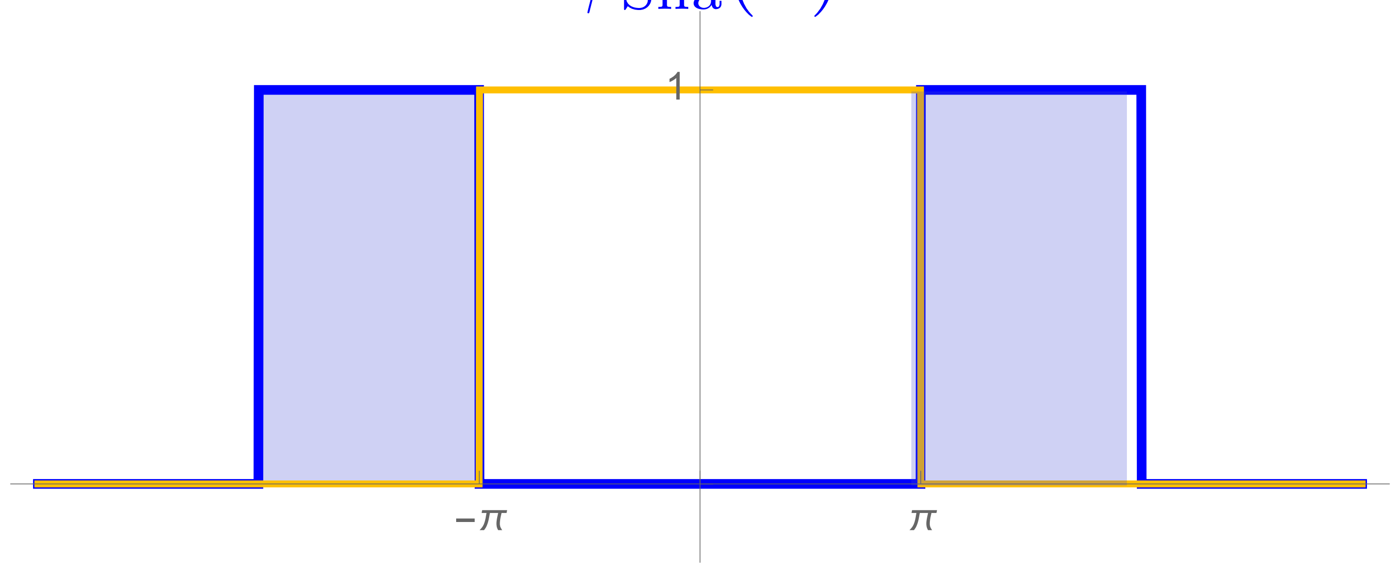
$$\psi_{j,k}(t) = 2^{-j/2}\psi_{\mathrm{Sha}}\left(\frac{t-2^jk}{2^j}\right)$$

The Meyer wavelet

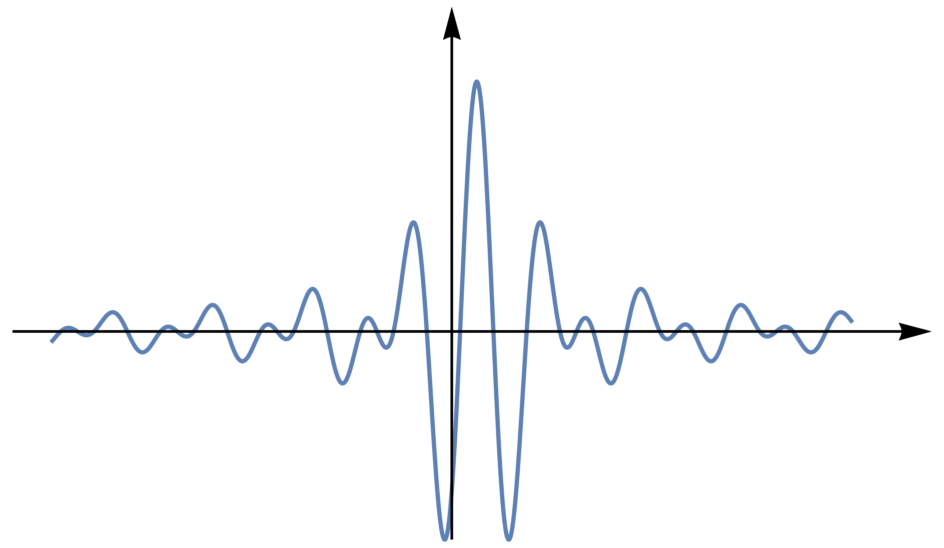
$$\sum_{j=-\infty}^{\infty} |\hat{\psi}_{\text{Sha}}(2^j\omega)|^2 = 1$$



$$\hat{\psi}_{\text{Sha}}(\omega)$$

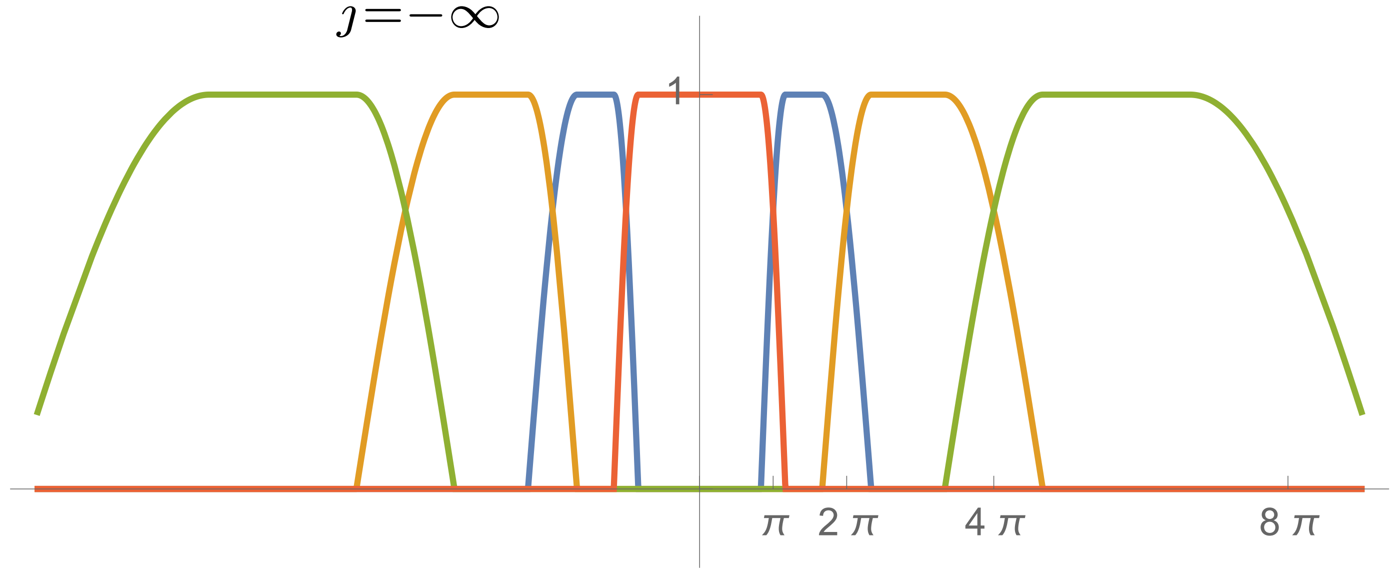


$$\psi_{\text{Sha}}(t) = 2\,\text{sinc}(2t) - \text{sinc}(t)$$

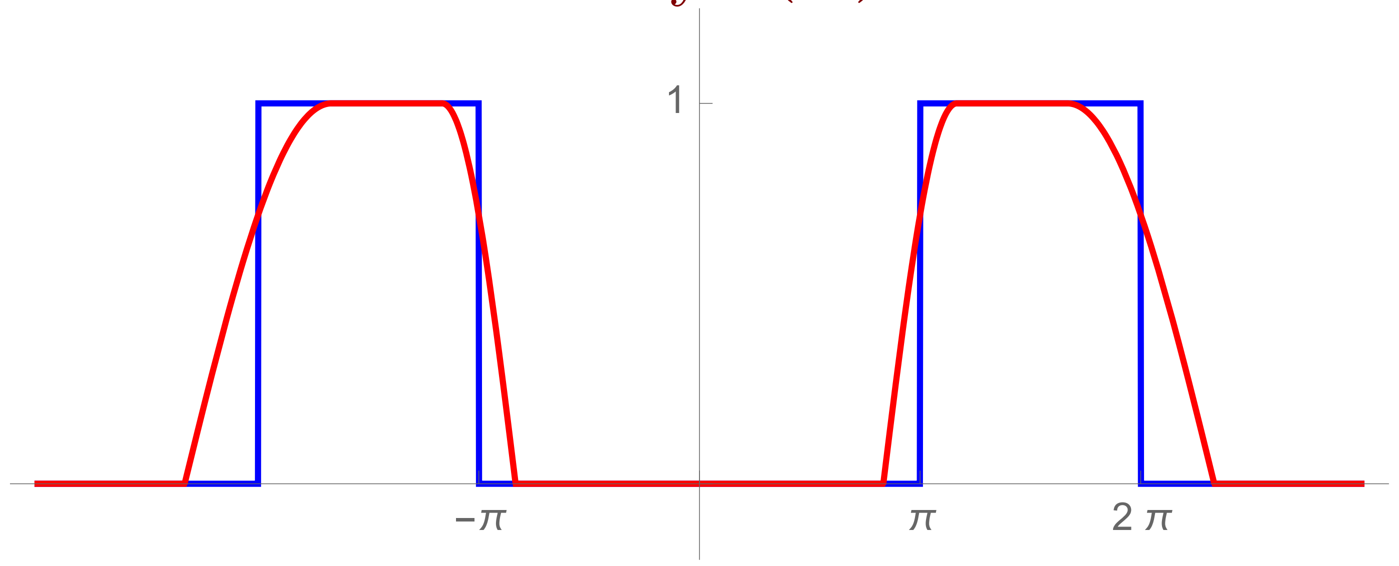


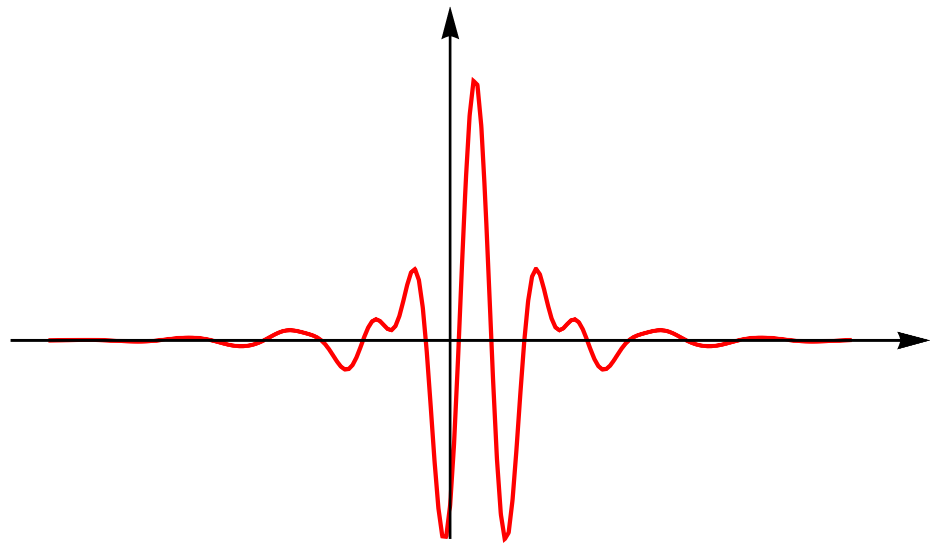
$$\psi_{\text{Sha}} \notin L_p(\mathbb{R}), \quad p \in [1, 2)$$

$$\sum_{j=-\infty}^{\infty} |\hat{\psi}_{\text{Meyer}}(2^j\omega)|^2 = 1$$



$$\hat{\psi}_{\text{Meyer}}(\omega)$$





$$\psi_{\text{Meyer}} \in \mathcal{S}(\mathbb{R}) \subset L_p(\mathbb{R})$$

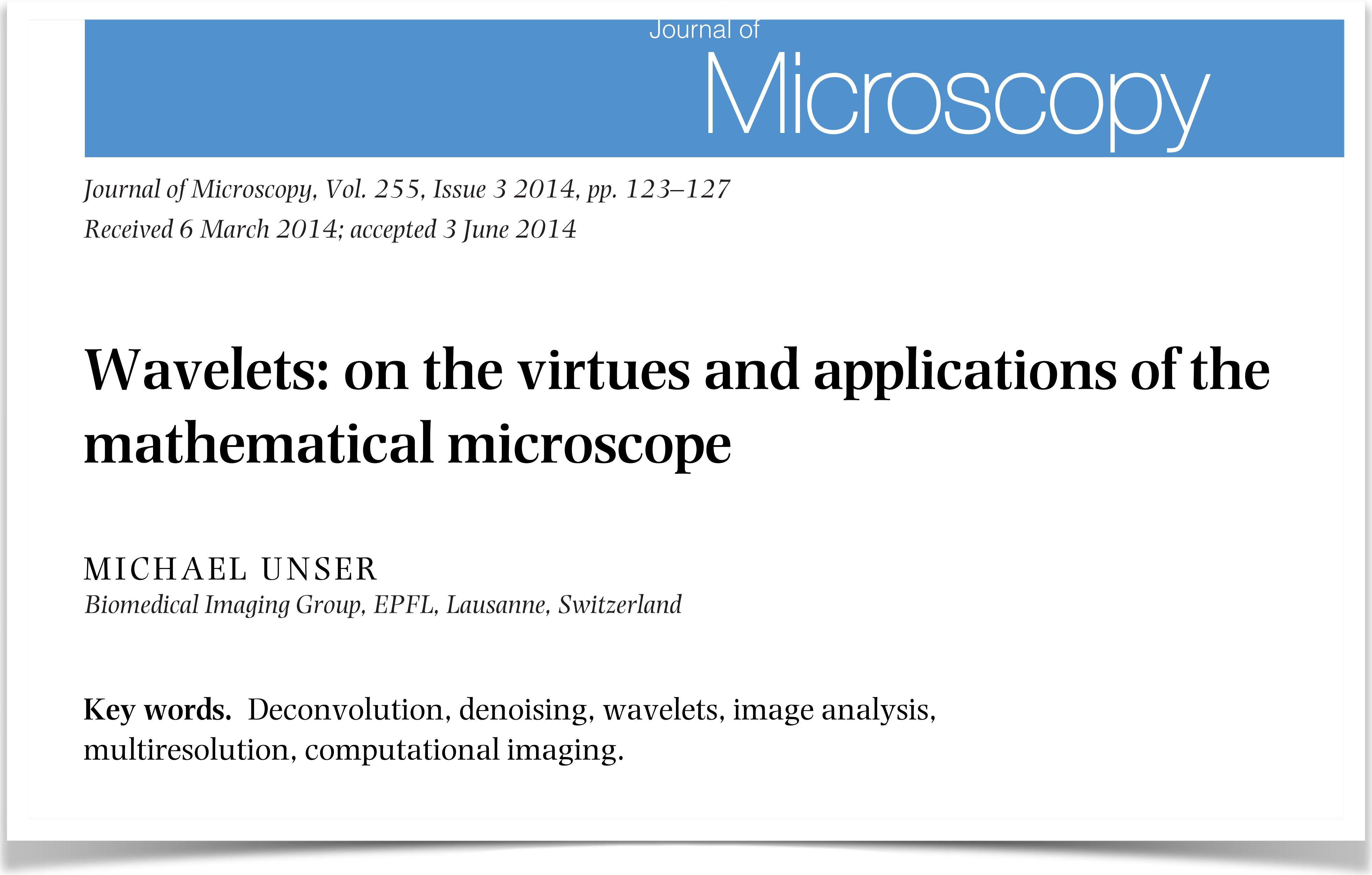
$$\forall f \in L_2(\mathbb{R}): \quad f(t) = \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\, \psi_{j,k}$$

$$\psi_{j,k}(t) = 2^{-j/2}\psi_{\text{Meyer}}\left(\frac{t-2^j k}{2^j}\right)$$

Wavelet transform as a mathematical microscope

Wavelet = Point Spread Function (PSF) of mathematical microscope

- Shape of PSF is the same at all scales
- Magnification by powers of two: 2^i
- Sampling is critical (no redundancy)
- Analysis functions (PSF) are orthogonal
- Resolution can be pushed to ultimate limit
  $\Rightarrow$ existence of wavelet bases of $L_2(\mathbb{R})$



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Wavelets: on the virtues and applications of the mathematical microscope

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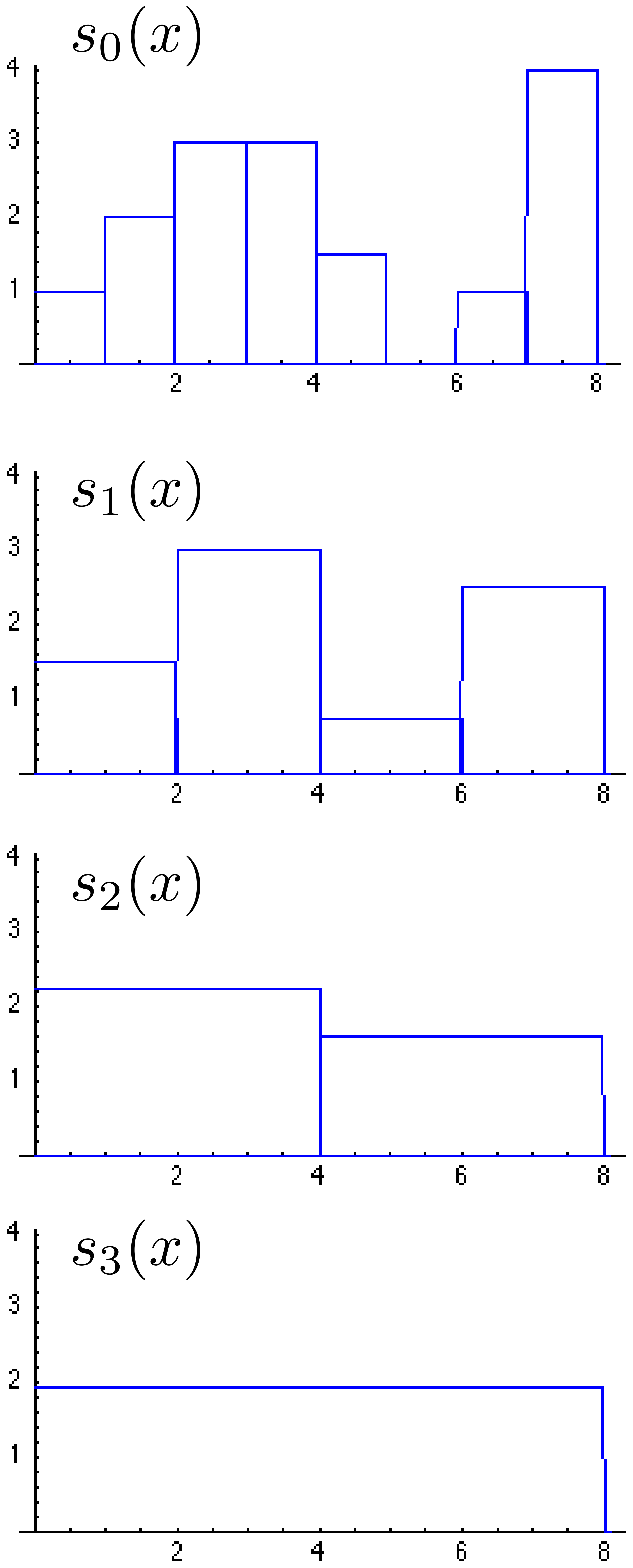
Biomedical Imaging Group, EPFL, Lausanne, Switzerland

Key words. Deconvolution, denoising, wavelets, image analysis, multiresolution, computational imaging.

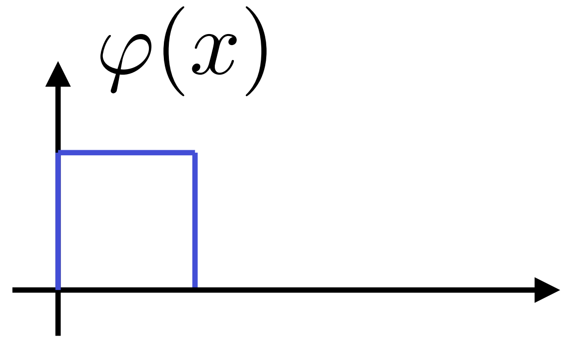
Multiresolution analysis: Haar transform revisited



Signal representation

$$s_0(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k)$$

Scaling function



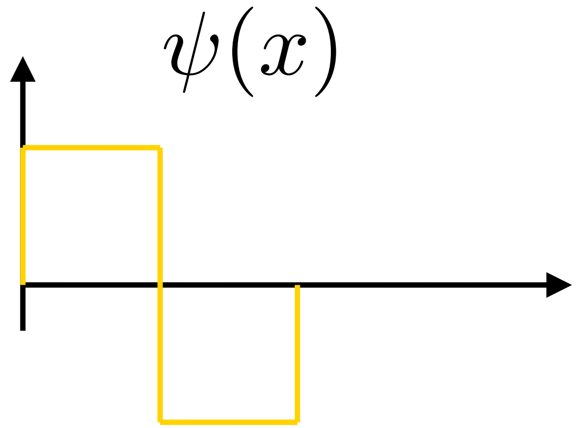
Multi-scale signal representation

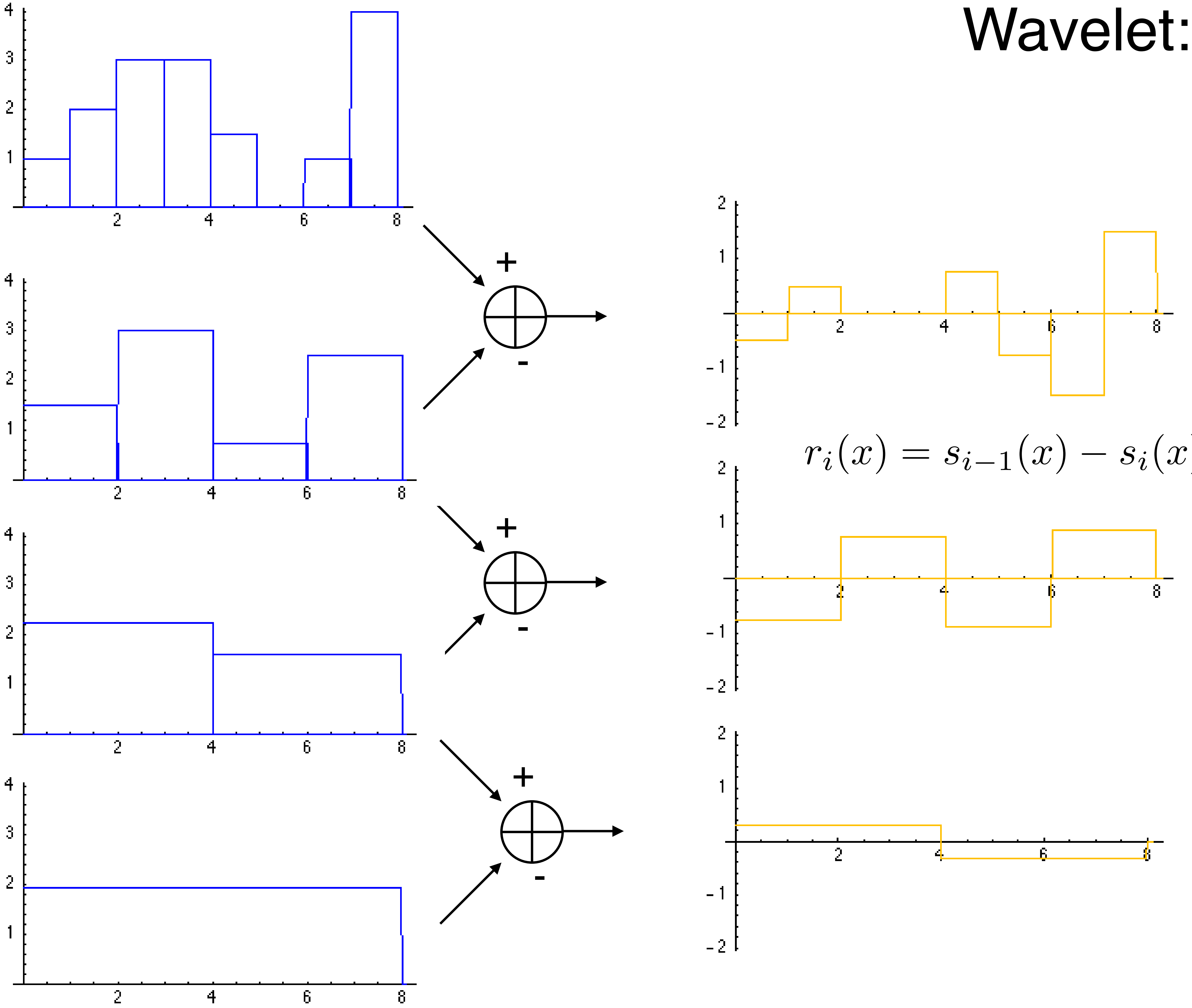
$$s_i(x) = \sum_{k\in\mathbb{Z}} c_i[k]\varphi_{i,k}(x)$$

Multi-scale basis functions

$$\varphi_{i,k}(x) = \varphi\left(\frac{x-2^i k}{2^i}\right)$$

Wavelets: Haar transform revisited

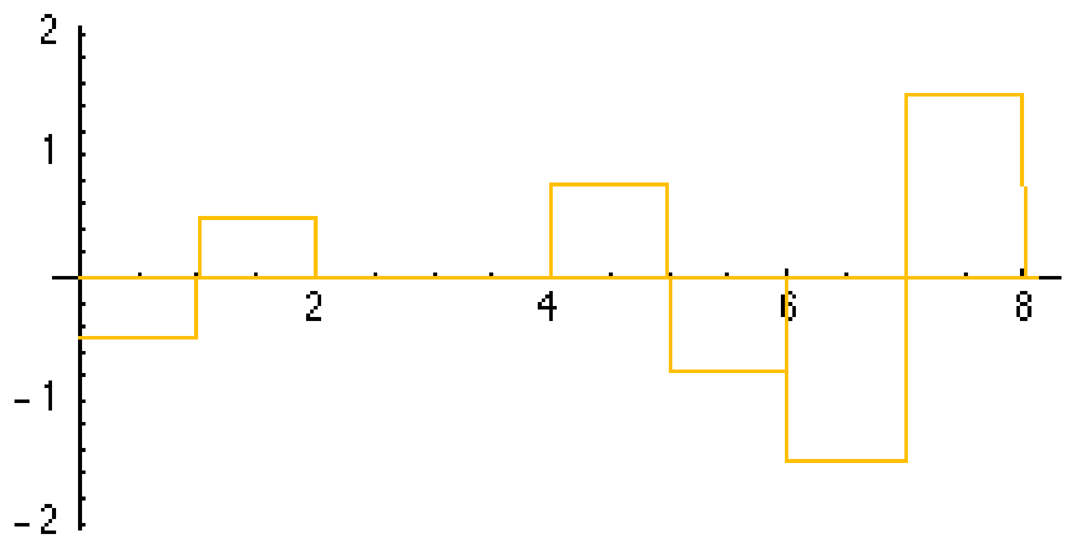
Wavelet: 



$$r_i(x) = s_{i-1}(x) - s_i(x)$$

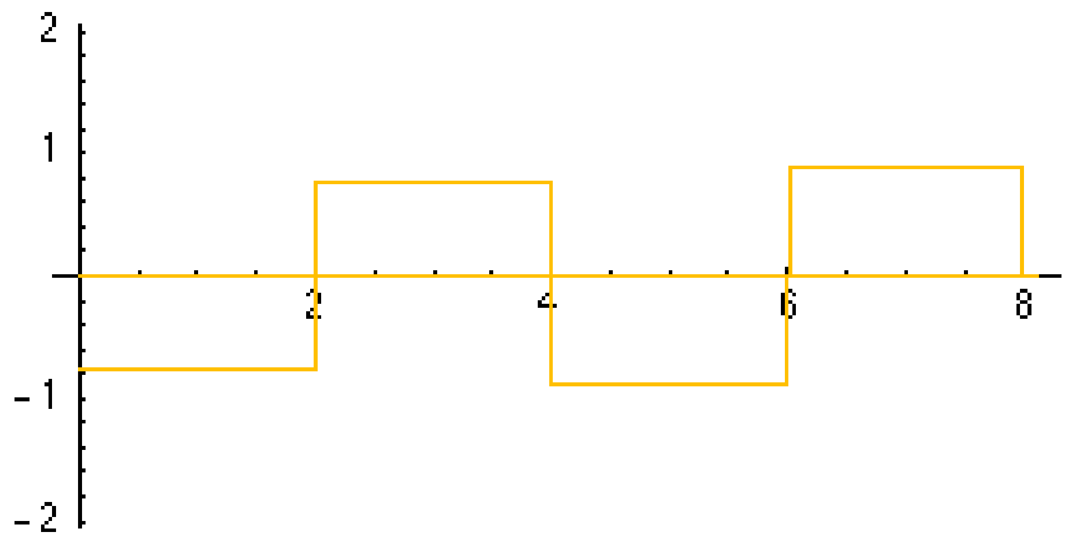
Wavelets: Haar transform revisited

$$r_1(x) = \sum_k w_1[k]\psi_{1,k}$$



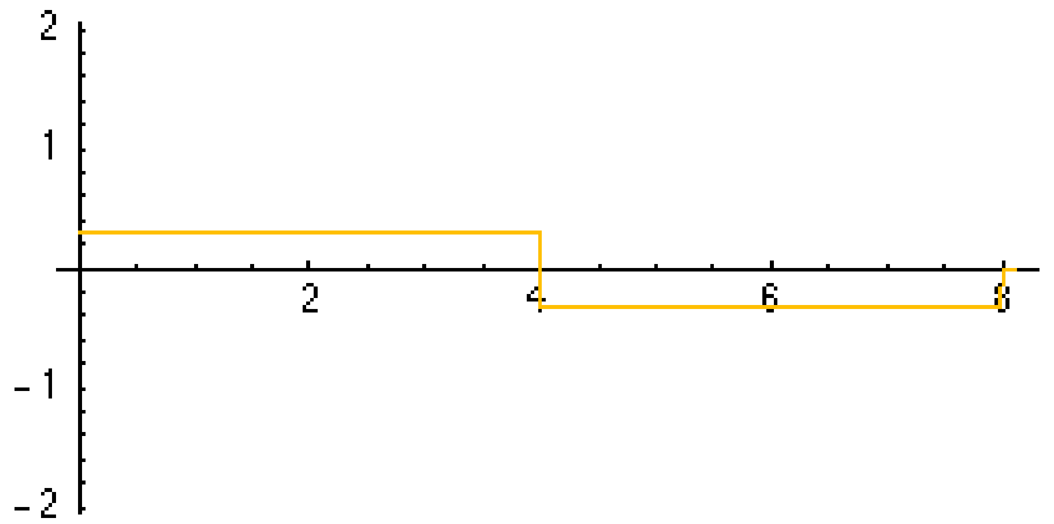
+

$$r_2(x) = \sum_k w_2[k]\psi_{2,k}$$



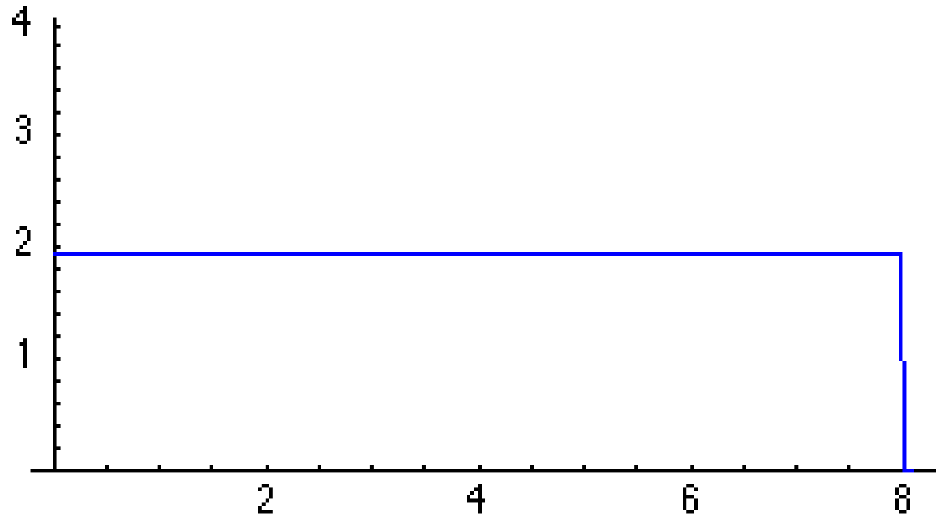
+

$$r_3(x) = \sum_k w_3[k]\psi_{3,k}$$

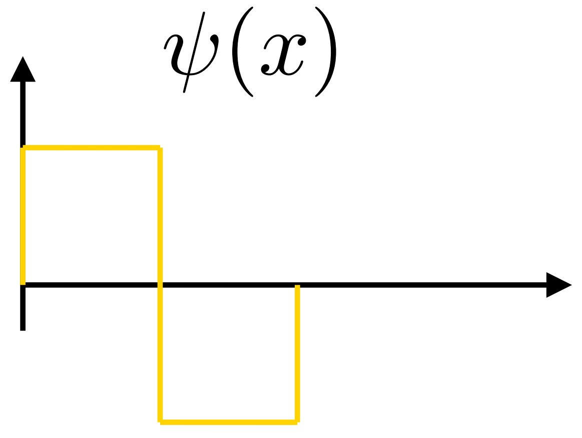


+

$$s_3(x) = \sum_k c_3[k]\varphi_{3,k}$$

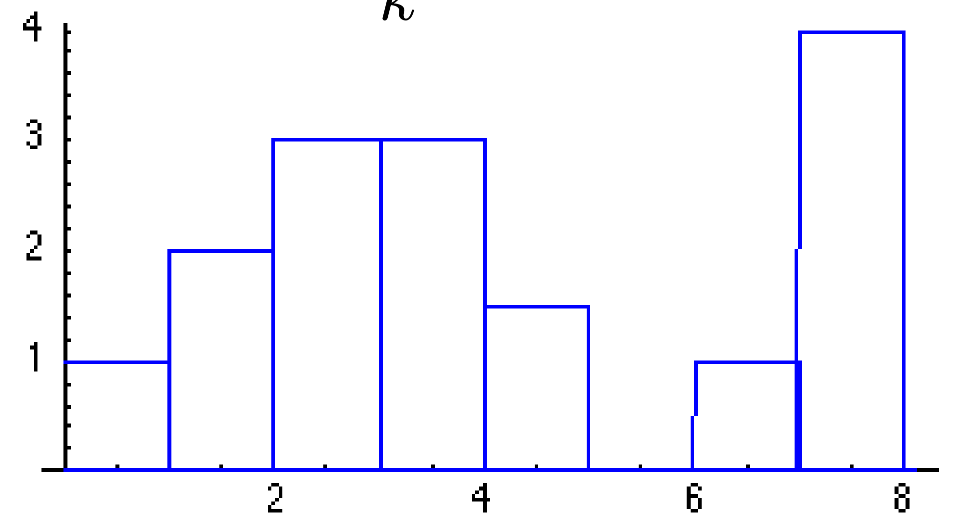


Wavelet:



=

$$s(x) = \sum_k c[k]\varphi(x-k)$$



Scaling function

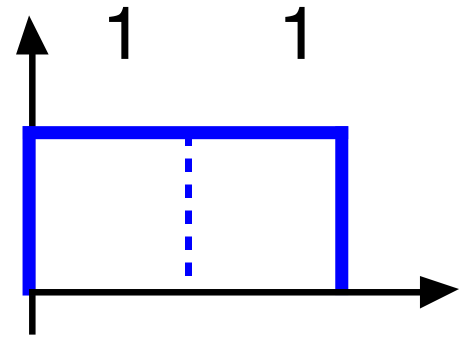
Definition: $\varphi(x)$ is a valid scaling function of $L_2(\mathbb{R})$ iff:

- Riesz basis condition

$$\forall c \in \ell_2, \qquad A \cdot \|c\|_{\ell_2} \leq \left\| \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) \right\|_{L_2} \leq B \cdot \|c\|_{\ell_2}$$

- Two-scale relation

$$\varphi(x) = \frac{2}{H(1)} \sum_{k\in\mathbb{Z}} h[k]\varphi(2x-k)$$

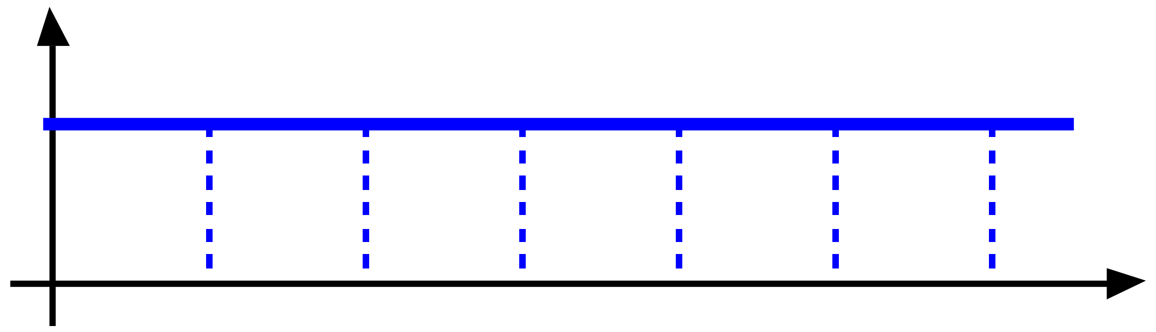


$$H(z) = \sum_{k\in\mathbb{Z}} h[k]z^{-k}$$

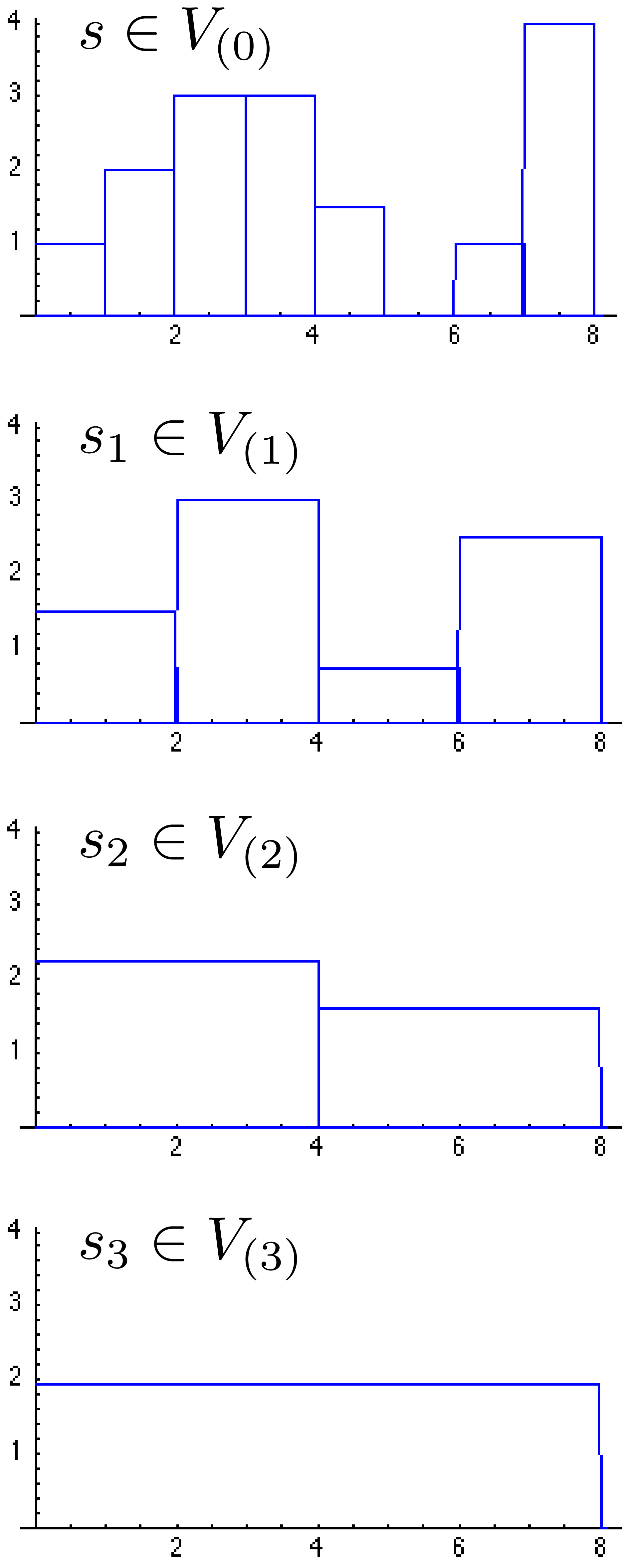
$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} \frac{H(\mathrm{e}^{\mathrm{i}\omega/2^j})}{H(1)}$$

- Partition of unity

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = 1$$



Multiresolution analysis of *L*$_2$



- Multiresolution basis functions: $\varphi_{i,k}(x) = 2^{-i/2}\varphi\left(\frac{x-2^i k}{2^i}\right)$
- Subspace at resolution i: $V_{(i)} = \mathrm{span}\left\{\varphi_{i,k}\right\}_{k\in\mathbb{Z}}$

Two-scale relation $\Rightarrow$ $V_{(i)} \subset V_{(j)}$, for $i \geq j$

Partition of unity $\Leftrightarrow$ $\overline{\bigcup_{i\in\mathbb{Z}} V_{(i)}} = L_2(\mathbb{R})$

From scaling functions to wavelets

- Wavelet bases of L_2 (Mallat-Meyer, 1989)

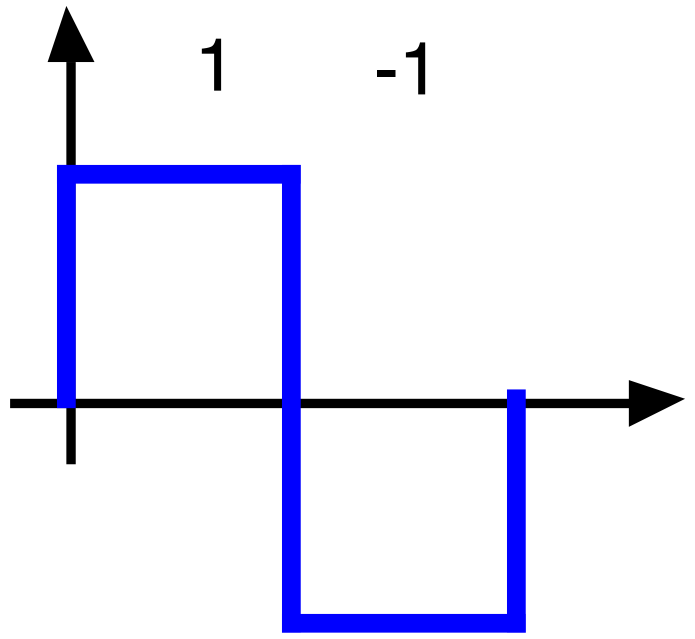
Theorem

For any valid scaling function $\varphi \in L_2(\mathbb{R})$, there exists a wavelet $\psi(x) = \dfrac{2}{H(1)} \sum_{k\in\mathbb{Z}} g[k]\varphi(2x-k)$

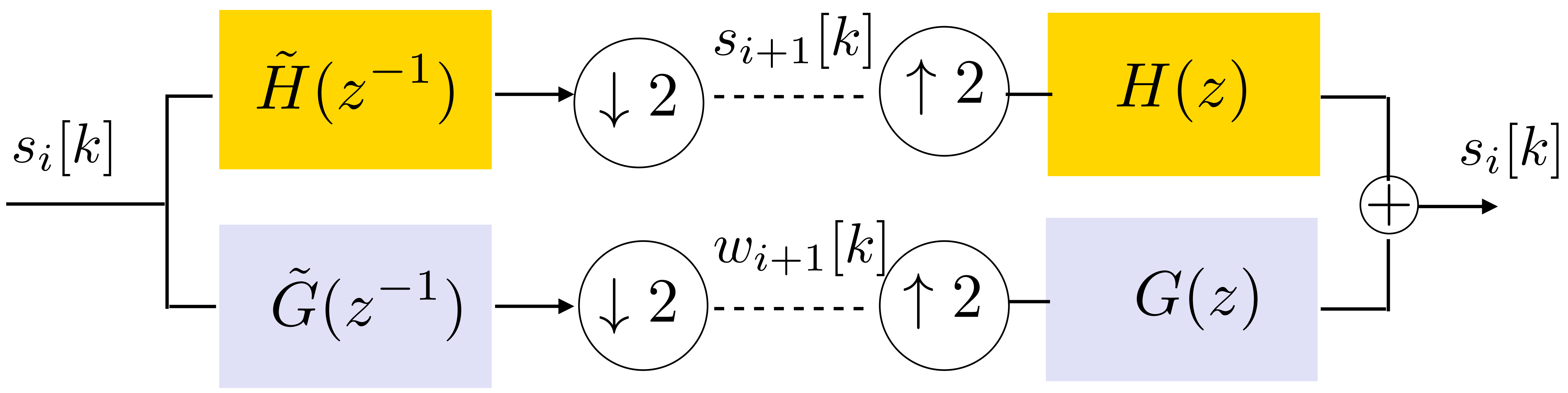
with $g[\cdot] \in \ell_2(\mathbb{Z})$ such that the family of functions

$$\left\{2^{-i/2}\psi\left(\frac{x-2^i k}{2^i}\right)\right\}_{i\in\mathbb{Z},k\in\mathbb{Z}}$$

forms a Riesz (or an orthogonal) basis of $L_2(\mathbb{R})$.



- Constructive approach: perfect-reconstruction filterbank

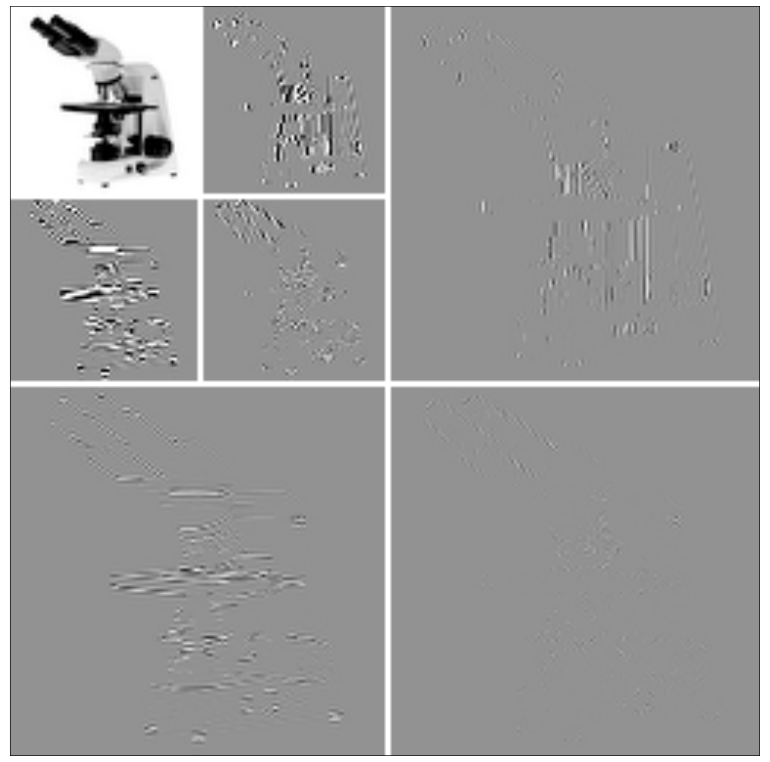


$$G(z) = \sum_{k\in\mathbb{Z}} g[k] z^{-k}$$

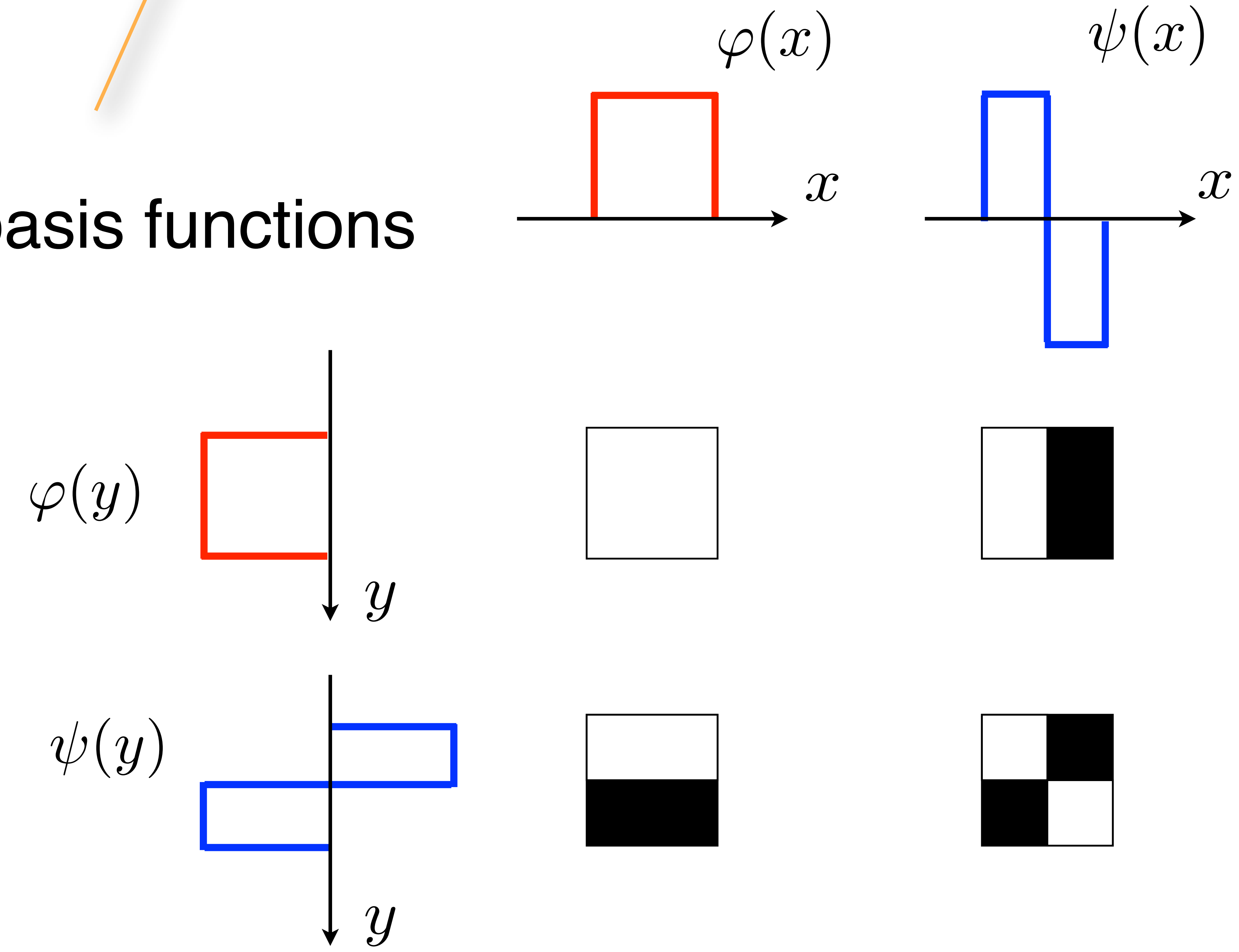
Haar wavelet and 2D basis functions

Expansion coefficients

$$f(x,y) = \sum_{i,\boldsymbol{k}} w_{i,\boldsymbol{k}}\, \psi_{i,\boldsymbol{k}}(x,y)$$



Tensor-product basis functions



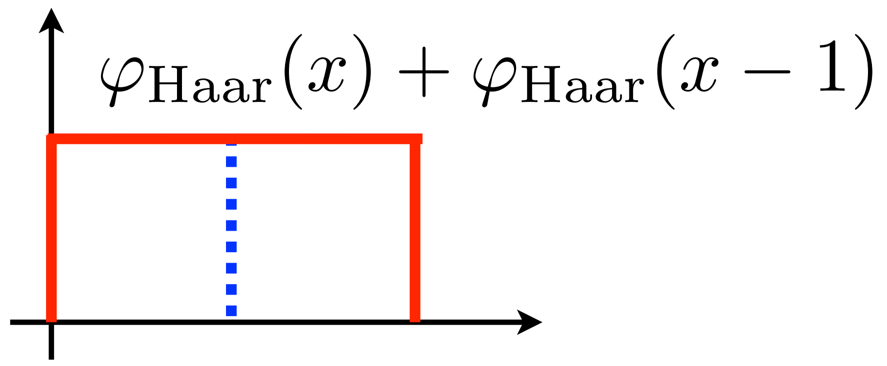
$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} s \\ d \end{bmatrix}$$

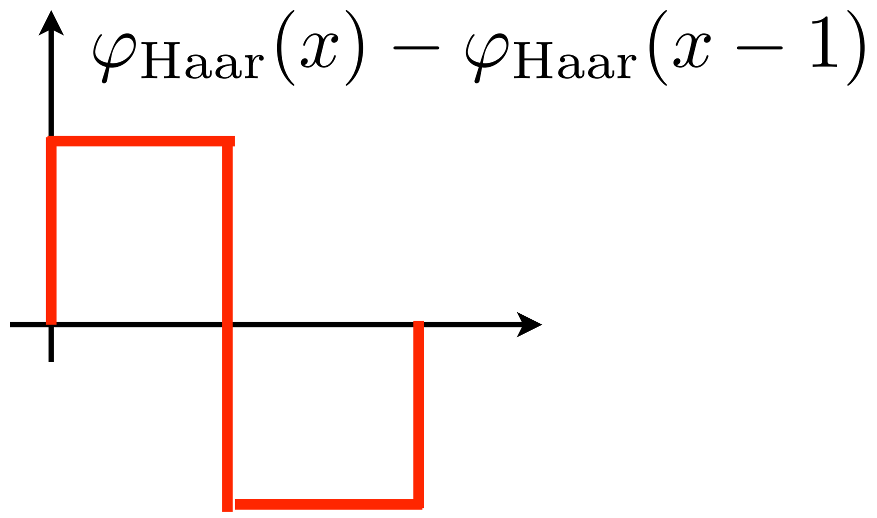
Shortest, orthogonal solutions of the two-scale relation

Two-scale relation: $\varphi(x/2) = \sum_{k \in \mathbb{Z}} h[k]\varphi(x-k)$ (without normalization)

Haar transform (order 1):

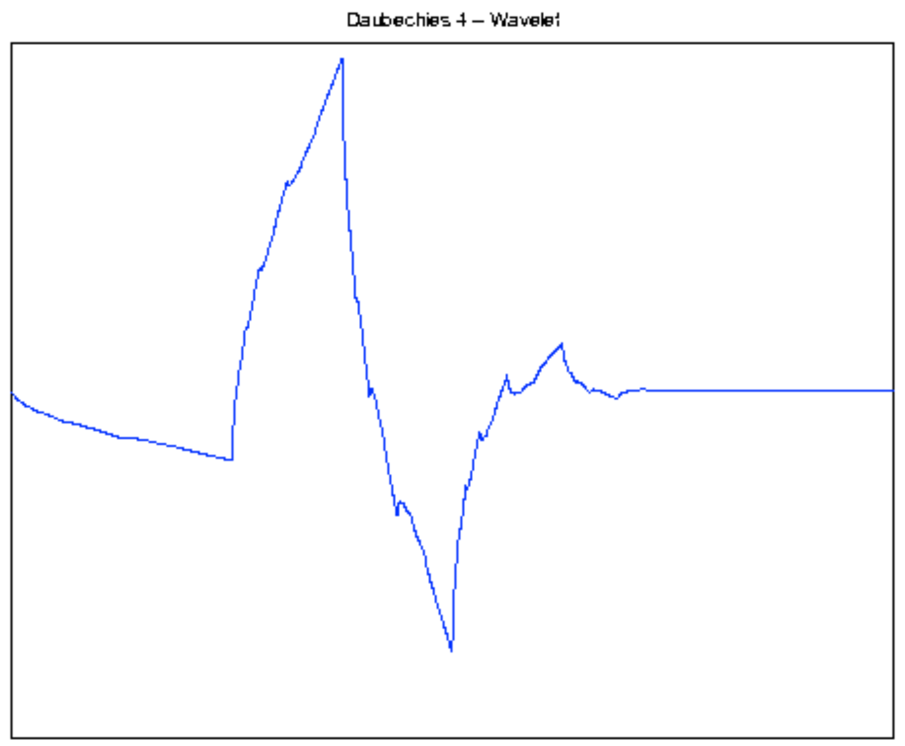
$$H_1(z) = 1 + z^{-1}$$





Daubechies of order 2:

$$H_2(z) = \frac{1}{4}\left[\left(1+\sqrt{3}\right) + \left(3+\sqrt{3}\right)z^{-1} + \left(3-\sqrt{3}\right)z^{-2} + \left(1-\sqrt{3}\right)z^{-3}\right]$$
$$= (1+z^{-1})^2\, P_2(z)$$



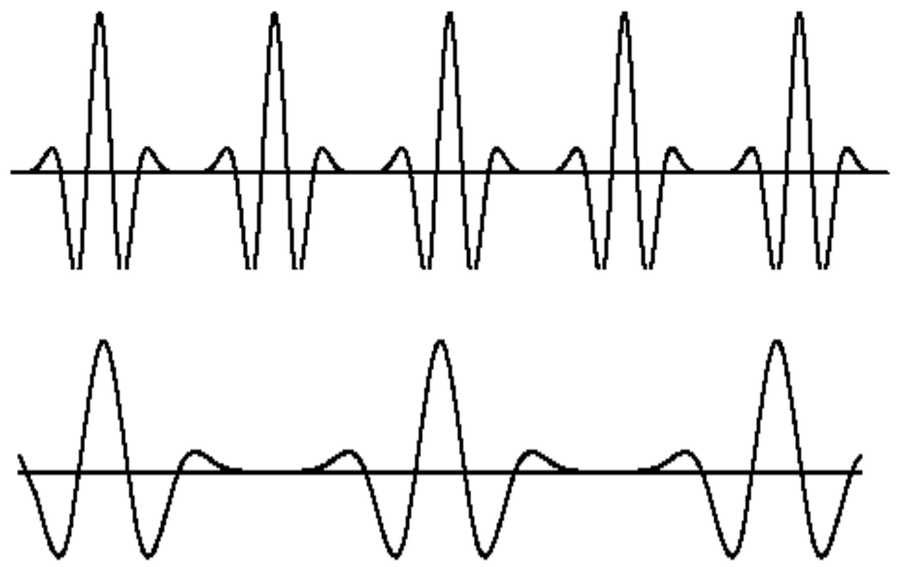
Daubechies of order N:

$$H_N(z) = (1+z^{-1})^N P_N(z)$$

$\Leftrightarrow$ ψ has N vanishing moments; i.e., $\int_{\mathbb{R}} x^n \psi(x)\mathrm{d}x = 0, n = 0, \ldots, N-1$

Application: Wavelet coding (outperforms DCT-based jpeg)

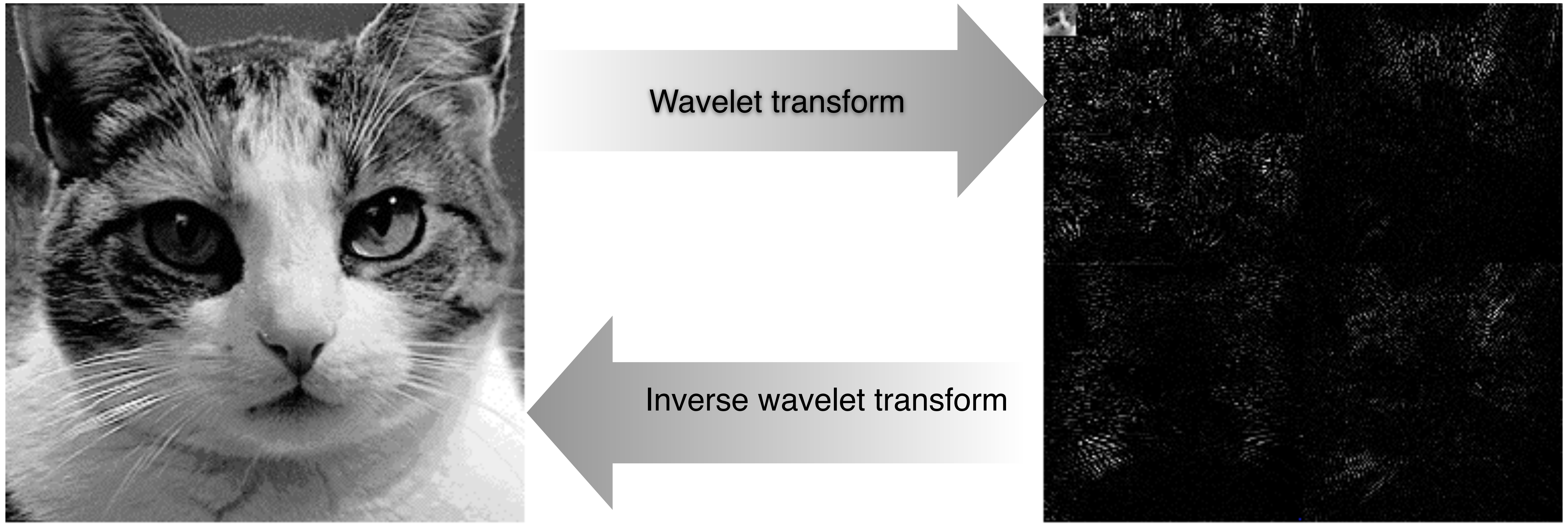
Wavelet expansion of an image



$$f(\boldsymbol{x}) = \sum_{i,\boldsymbol{k}} \psi_{i,\boldsymbol{k}}(\boldsymbol{x})\, w_{i,\boldsymbol{k}}$$

Space-domain representation: $\mathbf{f} = \mathbf{W}\mathbf{w}$

Wavelet-domain representation: $\mathbf{w} = \mathbf{W}^{-1}\mathbf{f}$



66.4 dB

0.00%

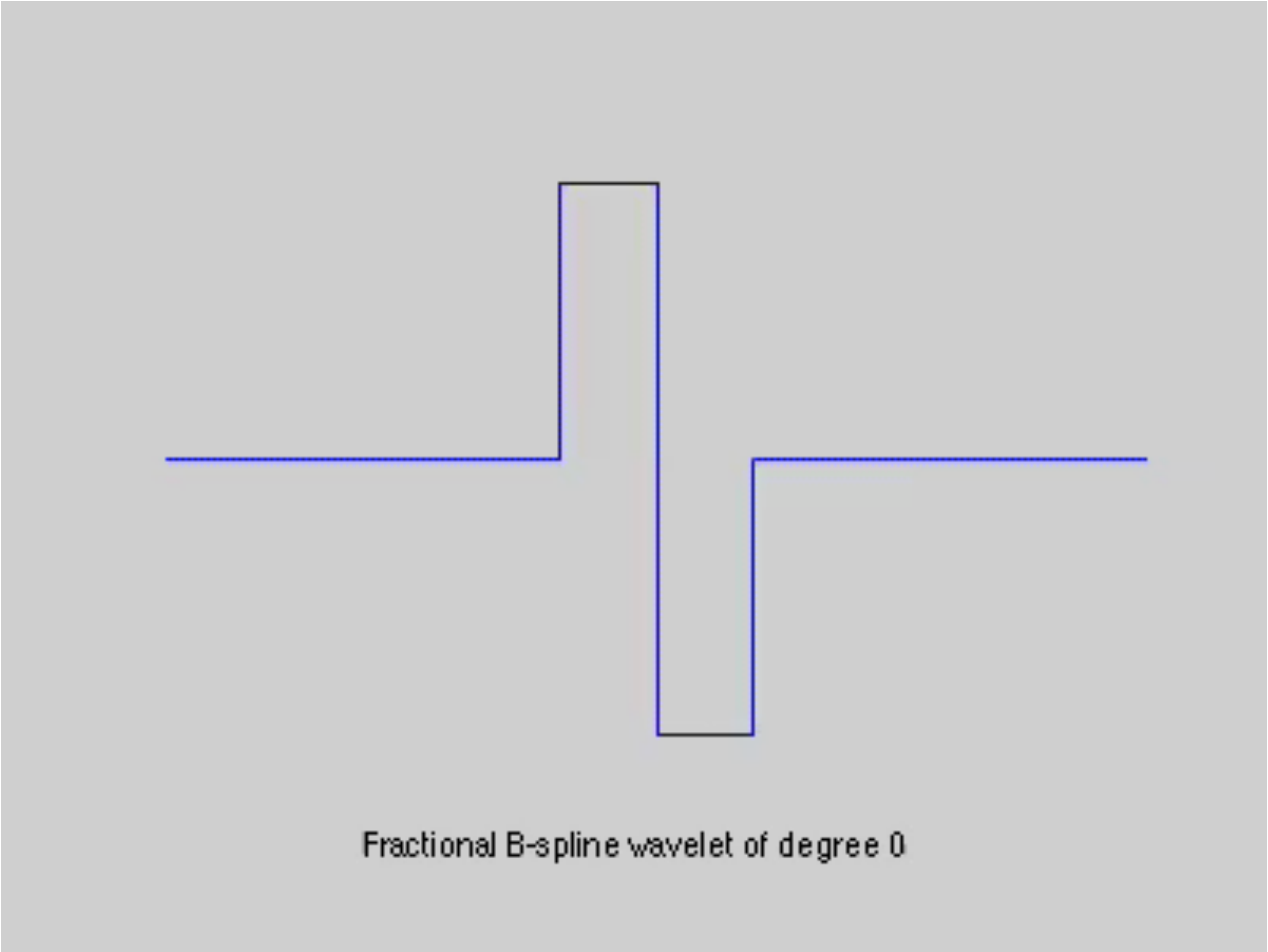
Discarding “small coefficients”

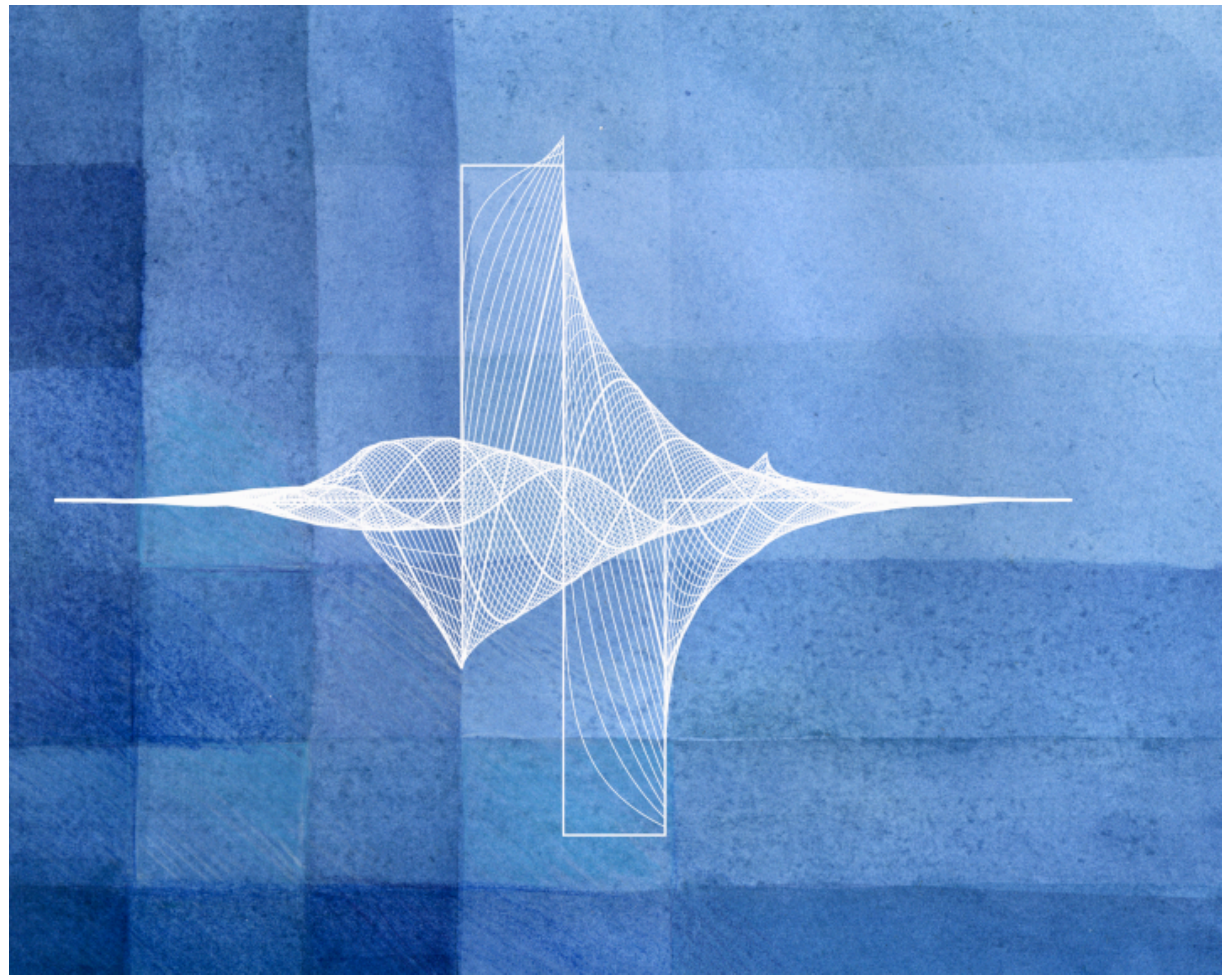
CDF 9/7 Filters:
Cohen-Daubechies-Feauveau

Reconstruction: $\mathbf{f}_N = \mathbf{W}\mathbf{w}_N$

Thresholding: $\mathbf{w} \rightarrow \mathbf{w}_N$

Fractional B-spline wavelets





(Unser & Blu, *SIAM Rev,* 2000)

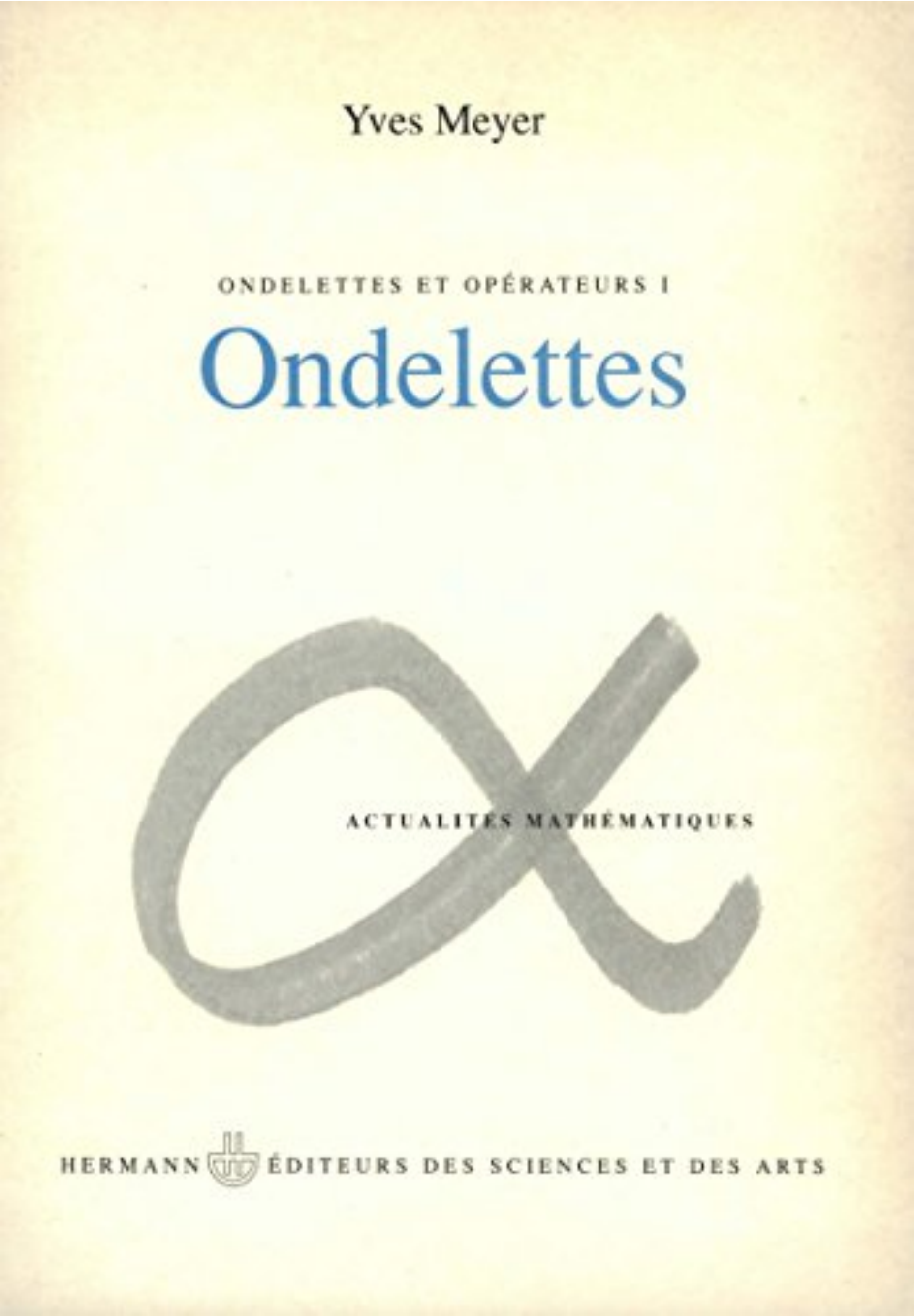
- Remarkable property

Each of these wavelets generates a Riesz basis of $L_2(\mathbb{R})$

$$\psi_+^\alpha(x/2) = \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{2^\alpha} \sum_{n\in\mathbb{N}} \binom{\alpha+1}{n} \beta_*^{2\alpha+1}(n+k-1) \frac{\Delta_+^{\alpha+1}(x-k)_+^\alpha}{\Gamma(\alpha+1)}$$

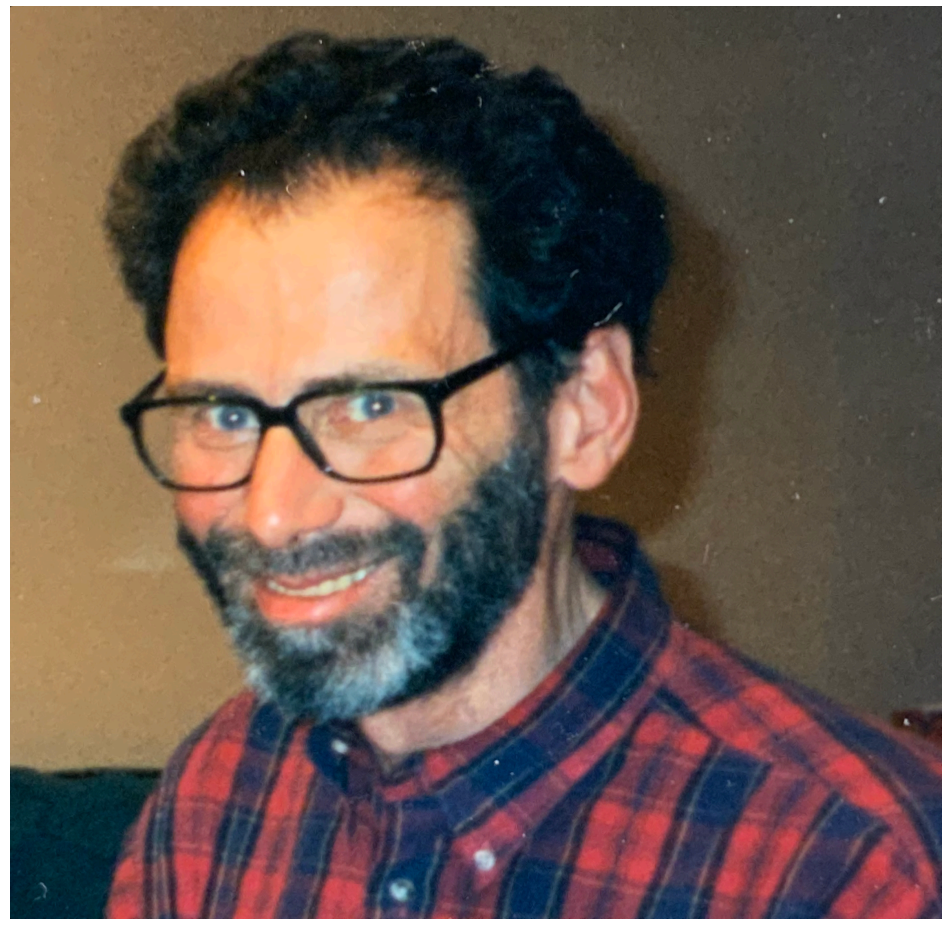
Only known wavelet bases that have an explicit time-domain formula !

2. WAVELETS AND FUNCTIONAL ANALYSIS



- Wavelets and differentiation
- Wavelets and Besov spaces
- Best N-term approximations

What makes wavelets attractive for mathematicians



- Existence of wavelet bases of $L_2(\mathbb{R}^d)$ (one-to-one representation)

  Basis functions are dilations and translates of a single template

- Vanishing moments, derivative-like behavior

  $\Rightarrow$ Sparse representation of piecewise-smooth signals

- **Unconditional basis** of many function spaces: L_p-Sobolev, Hölder, Besov, ...

- Assessment of **local/global regularity** from wavelet decay/mixed ℓ_p-norms

- **Sparsity** and (non-linear) N-term approximation of functions

  Natural images tend to have few large wavelet coefficients

  $\Rightarrow$ Wavelet-domain regularization: ℓ_1-sparsity, compressed sensing, ...

Wavelets and functional spaces

"everything takes place as if the wavelets $\psi(x/a)$ *were eigenvectors of the differential operator* ∂^s*, with corresponding eigenvalue* a^{-s} *"*

Yves Meyer (Wavelets and Operators)

Wavelets and fractional differentiation

- Fractional differentiation

Fourier transform: $\hat{f}(\omega) = \mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}} f(x) e^{-\mathrm{i}\omega x} \mathrm{d}x$

Fractional derivative of order r: $\partial^r f(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (\mathrm{i}\omega)^r \hat{f}(\omega)$

- Differentiation and scaling

$$\partial^r \psi(x) = \psi^{(r)}(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (\mathrm{i}\omega)^r \hat{\psi}(\omega)$$

$$\partial^r \psi(x/a) = a^{-r} \psi^{(r)}(x/a) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (\mathrm{i}\omega)^r |a| \hat{\psi}(a\omega) = a^{-r} |a| (\mathrm{i}a\omega)^r \hat{\psi}(a\omega)$$

Property: the “derivative” wavelet $\psi^{(r)}$ also generates a biorthogonal basis of $L_2(\mathbb{R})$ provided that $\psi \in W_2^r$ and $r < N$ (number of vanishing moments of ψ).

- Differentiation of wavelet expansion

a^{-r}

$$\partial^r f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} w_j[k]\, \partial^r \psi_{j,k}(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \underbrace{2^{-jr} w_j[k]}_{w'_j[k]}\, \psi_{j,k}^{(r)}(x)$$

Wavelets and Sobolev spaces

- Sobolev space of order s — f has s derivatives in L_2-sense

$$f \in W_2^s(\mathbb{R}) \quad \Leftrightarrow \quad f,\ \partial^s f \in L_2(\mathbb{R})$$

$$W_2^s(\mathbb{R}) = \left\{ f = \mathcal{F}^{-1}\{\hat{f}\} : \int_{\omega \in \mathbb{R}} (1 + |\omega|^{2s}) |\hat{f}(\omega)|^2 \frac{\mathrm{d}\omega}{2\pi} \triangleq \|f\|_{W_2^s}^2 < +\infty \right\}$$

- Equivalent norm in wavelet domain

$$\|w\|_{\ell_2,s} \triangleq \left(\|c_{j_0}\|_{\ell_2}^2 + \underbrace{\sum_{j=-\infty}^{j_0} \|2^{-js} w_j\|_{\ell_2}^2}_{\sim \|\partial^s f\|_{L_2}^2} \right)^{\frac{1}{2}}$$

Orthogonal wavelet expansion

$$c_{j_0}[k] = \langle f, \varphi_{j_0,k} \rangle$$

$$w_j[k] = \langle f, \psi_{j,k} \rangle$$

$$\|f\|_{W_2^s} \sim \|w\|_{\ell_2,s} \quad \Leftrightarrow \quad C_1 \|w\|_{\ell_2,s} \le \|f\|_{W_2^s} \le C_2 \|w\|_{\ell_2,s}$$

Wavelets and Besov spaces

- Besov space of order $s > 0$

$$f \in B^s_q(L_p(\mathbb{R})) \Leftrightarrow \begin{cases} (i) \quad f \in L_p(\mathbb{R}) \\ (ii) \text{ there exists a sequence of “smooth” functions } g_j \in L_p(\mathbb{R}), j \in \mathbb{N} \\ \qquad \text{such that } \|f - g_j\|_{L_p} \le 2^{-js}\epsilon_j \text{ and } \|\partial^{m_0} g_j\|_{L_p} \le 2^{(m_0-s)j}\epsilon_j \\ \qquad\qquad \text{with } \epsilon \in \ell_q \text{ and } m_0 \ge s \end{cases}$$

- Besov space and multiresolution analysis (Meyer 1990)

Explicit approximation sequence: $f_j = \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k}\rangle \, \varphi_{j,k}, \qquad r_j = \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \, \psi_{j,k}$

$$f \in B^s_q(L_p(\mathbb{R})) \Leftrightarrow \begin{cases} f \in L_p(\mathbb{R}) \\ \|f - f_j\|_{L_p} \le 2^{js}\epsilon_j, \ \epsilon \in \ell_q \end{cases} \Leftrightarrow \begin{cases} f_{j_0} \in L_p(\mathbb{R}) \\ \|\partial^s r_j\|_{L_p} \le \epsilon_j, \ \epsilon \in \ell_q \end{cases}$$

s derivatives in L_p-sense

q: fine-tuning parameter that controls decay across scales

- Equivalent norm in wavelet domain

$$\|f\|_{B^s_q(L_p(\mathbb{R}))} \sim \left[\left(2^{-j_0(\frac{1}{2}-\frac{1}{p})} \|c_{j_0}\|_{\ell_p} \right)^q + \sum_{j=-\infty}^{j_0} \underbrace{\left(2^{-j(\frac{1}{2}-\frac{1}{p})} \|2^{-js} w_j\|_{\ell_p} \right)^q}_{\sim \|\partial^s r_i\|_{L_p}} \right]^{\frac{1}{q}}$$

Wavelets and non-linear approximation

- Equivalent Besov norm in wavelet domain $\quad$ (d = dimension of domain)

$$\|f\|_{B^s_q(L_p(\mathbb{R}^d))} \sim \left[\sum_{j=-\infty}^{+\infty} \left(2^{-j(s+\frac{d}{2}-\frac{d}{p})}\|w_j\|_{\ell_p}\right)^q\right]^{\frac{1}{q}} = \|w\|_{\ell_1} \quad \text{If } p=q=1 \text{ and } s=\frac{d}{2}$$

- Critical regularity exponent: $s = \frac{d}{p} - \frac{d}{2}$

For $p=q\in[1,2]$: $\quad \ell_p(\mathbb{Z}^d) \subseteq \ell_2(\mathbb{Z}^d) \quad \Rightarrow \quad B^s_p\big(L_p(\mathbb{R}^d)\big) \subseteq L_2(\mathbb{R}^d)$

N-term wavelet expansions: $\quad \Sigma_N = \left\{g = \sum_{j,k\in\Lambda_N} c_{j,k}\psi_{j,k},\, c_{j,k}\in\mathbb{R}, \mathrm{card}(\Lambda_N)\le N\right\}$

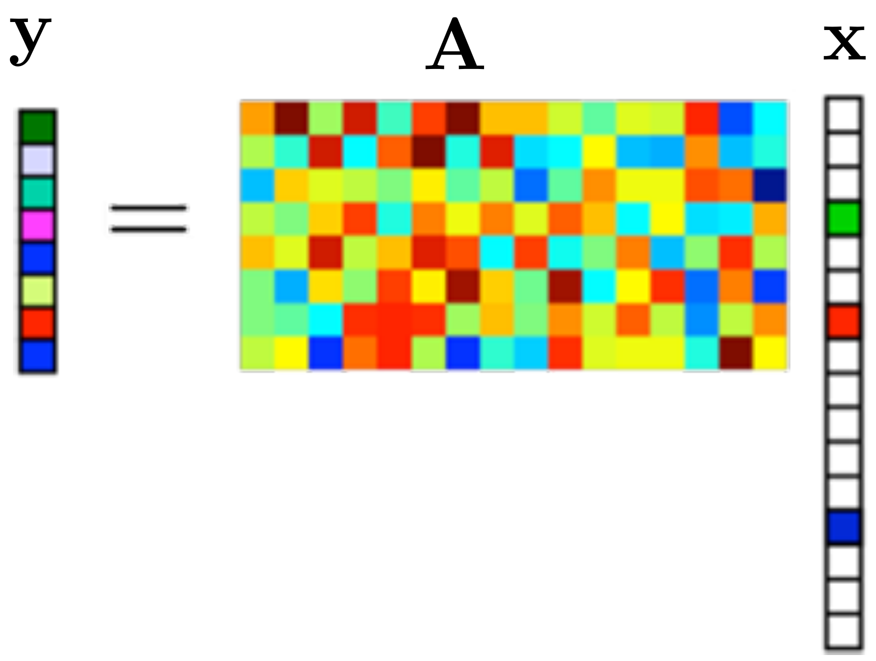
Best N-term wavelet approximation: $\quad f_N = \arg\min_{g\in\Sigma_N} \|f-g\|_{L_2} = \sum_{j,k\in\Lambda_N(f)} w_j[k]\psi_{j,k}$

$$\|f-f_N\|_{L_2(\mathbb{R}^d)} \le \frac{C}{N^r}\|f\|_{B^s_p\left(L_p(\mathbb{R}^d)\right)} \quad \text{with} \quad r=\frac{s}{d}$$

(DeVore, Cohen 1998)

3. WAVELETS AND SPARSITY

- Wavelet-based denoising [Donoho 1995]
- Image reconstruction by iterative shrinkage thresholding [Figuereido et al. 2003, Daubechies et al. 2004]
- Compressed sensing



+ “noise”

[Donoho et al., 2005
Candès-Tao, 2006, ...]

$$\min_{\mathbf{x}} \|\mathbf{x}\|_{\ell_1} \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \leq \sigma^2$$

Denoising and wavelet-domain/Besov regularization

- Measurement model

Orthogonal wavelet transform: $\mathbf{W}^T\mathbf{W} = \mathbf{I}$

Space domain		Wavelet domain	
$\mathbf{y} = \mathbf{f} + \mathbf{n}$	$\Leftrightarrow$	$w_i[\boldsymbol{k}] = \tilde{w}_i[\boldsymbol{k}] + n_i[\boldsymbol{k}]$	(additive white noise)

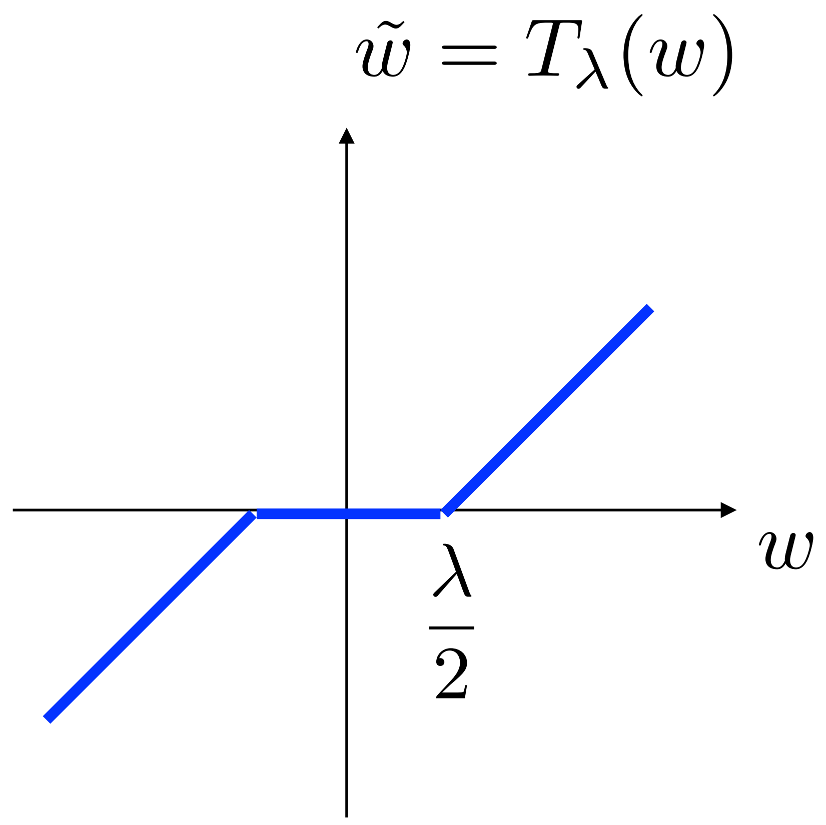
- Regularized least-squares signal estimation
  - Regularization functional: $R(\tilde{f}) = \|\tilde{\mathbf{w}}\|_{\ell_1} = \sum_j \sum_{\boldsymbol{k}} |\tilde{w}_j[\boldsymbol{k}]| \sim \|\tilde{f}\|_{B_1^1(L_1(\mathbb{R}^2))}$
  - Variational formulation of denoising problem:

$$\tilde{\mathbf{f}} = \arg\min_{\mathbf{f}} \left(\|\mathbf{y} - \mathbf{f}\|_2^2 + \lambda \|\mathbf{W}^T\mathbf{f}\|_1 \right)$$

  - Equivalent wavelet-domain solution (by Parseval)

$$\tilde{\mathbf{f}} = \mathbf{W}\tilde{\mathbf{w}} \quad \text{with} \quad \tilde{\mathbf{w}} = \arg\min_{\mathbf{v}} \left(\|\underbrace{\mathbf{W}^T\mathbf{y}}_{\mathbf{w}} - \mathbf{v}\|_2^2 + \lambda \|\mathbf{v}\|_{\ell_1} \right)$$

$\Rightarrow$ Soft-thresholding



Standard Color Image

Input PSNR=18.59 dB

Denoised with OWT SURE-LET

SURE-LET Optimized thresholds

Output PSNR = 31.91 dB

(Luisier et al., *IEEE Trans. Image Proc.* 2007)

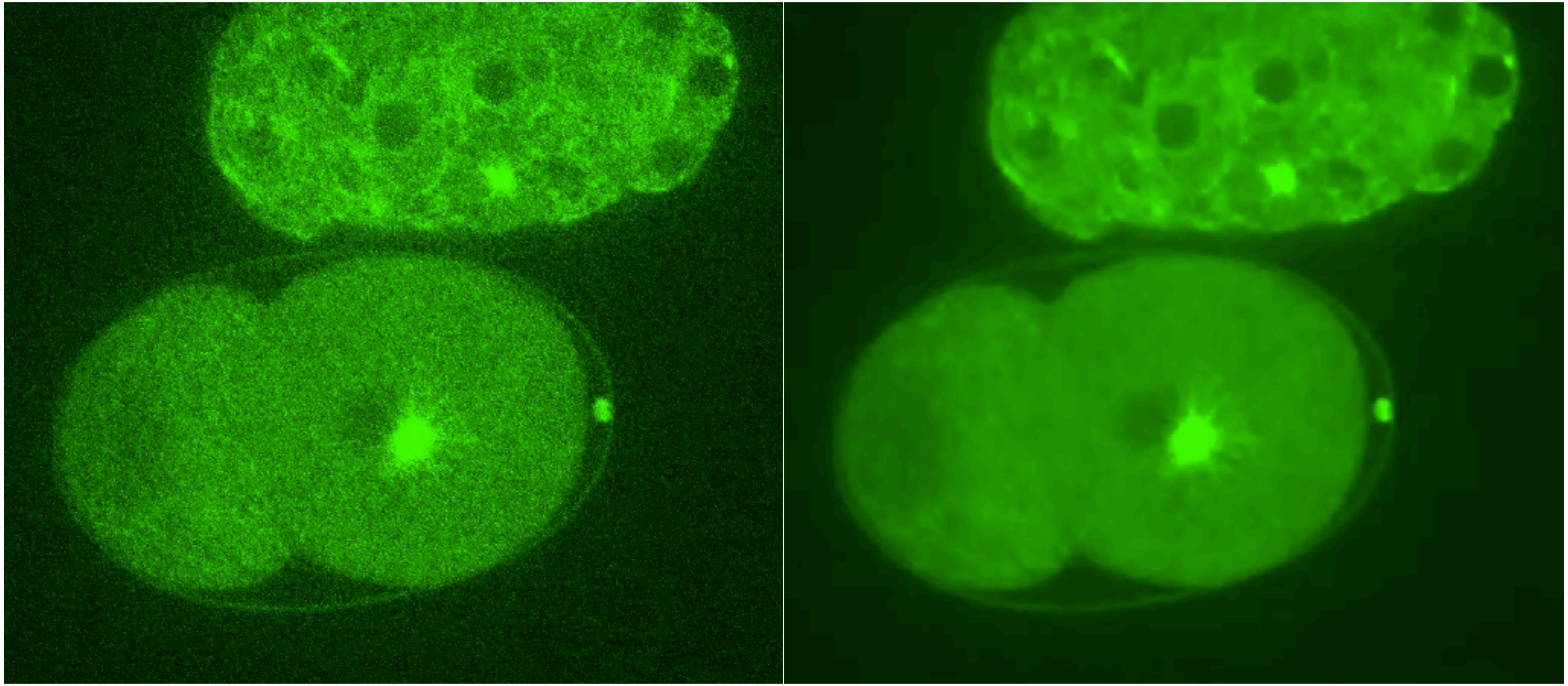
Denoised with UWT SURE-LET

SURE-LET Optimized thresholds
+ redundant wavelet transform

Output PSNR = 33.27 dB

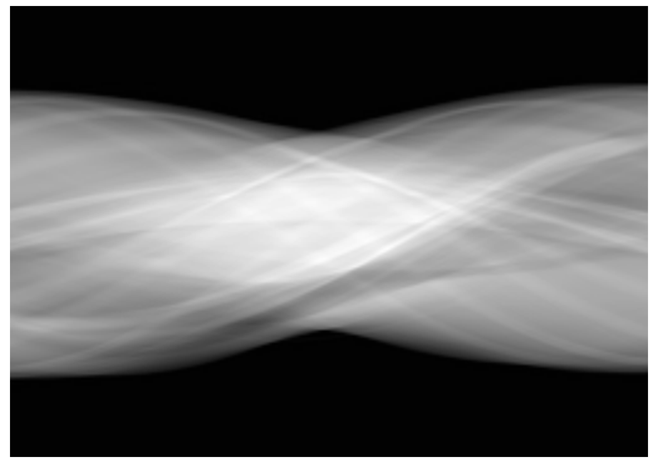
(Luisier et al., *IEEE Trans. Image Proc.* 2007)

2D + time SURE-LET denoising (DWT) : C-elegance embryo



Wavelet-regularized image reconstruction

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n}$$



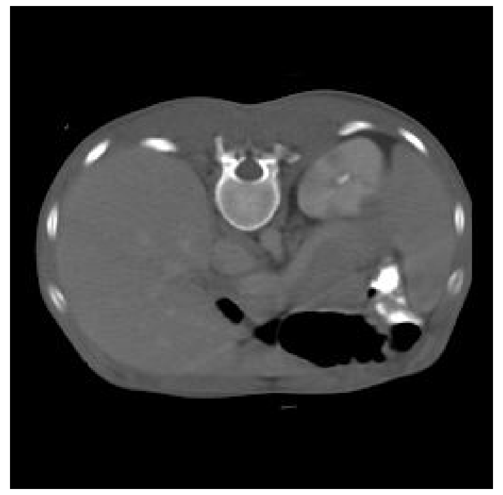
Hypotheses:

- System matrix $\mathbf{H}$ is known (physics)
- $\mathbf{f} = \mathbf{W}\mathbf{w}$ has a “sparse” wavelet expansion

- Reconstruction as a (convex) optimization problem

$$\tilde{\mathbf{f}} = \arg\min_{\mathbf{f}} \Big(\underbrace{\|\mathbf{g} - \mathbf{H}\mathbf{f}\|_2^2}_{\text{data consistency}} + \underbrace{\lambda \| \overbrace{\mathbf{W}^{-1}\mathbf{f}}^{\mathbf{w}} \|_{\ell_1}}_{\text{regularization}} \Big)$$

$$\sim \|\tilde{f}\|_{B_1^1(L_1(\mathbb{R}^2))}$$



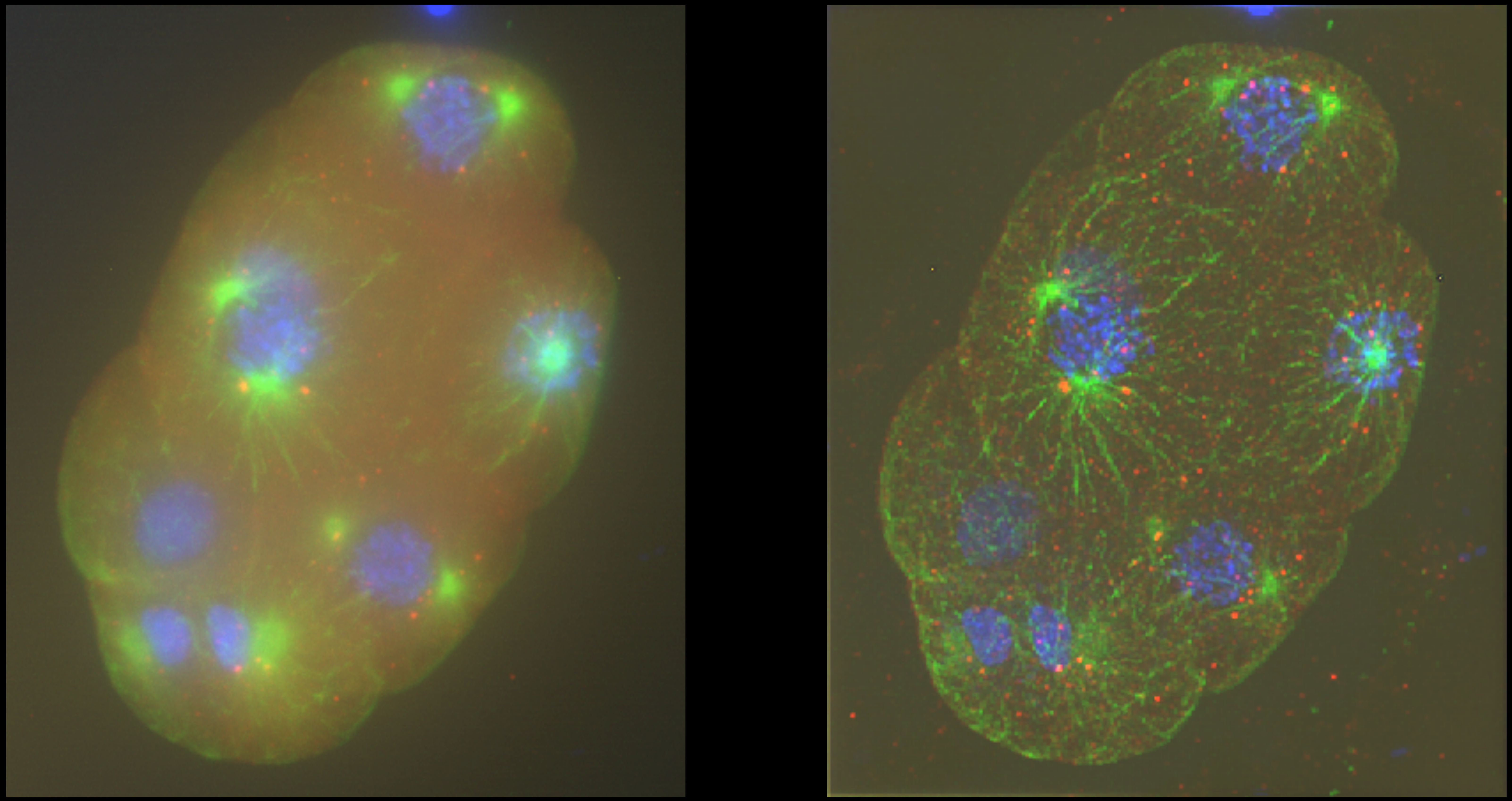
$$\tilde{\mathbf{f}} = \mathbf{W}\tilde{\mathbf{w}} \quad \text{with} \quad \tilde{\mathbf{w}} = \arg\min_{\mathbf{w}} \left(\|\mathbf{g} - \mathbf{A}\mathbf{w}\|_2^2 + \lambda \|\mathbf{w}\|_{\ell_1} \right)$$

Equivalent system matrix: $\mathbf{A} = \mathbf{H}\mathbf{W}$

- Theory of compressed sensing (Donoho et al., 2005, Candès-Tao, 2006]

Conditions on $\mathbf{A}$ for perfect signal recovery from few measurements when $\|\mathbf{w}\|_0 < K_0$

3D deconvolution of widefield stack

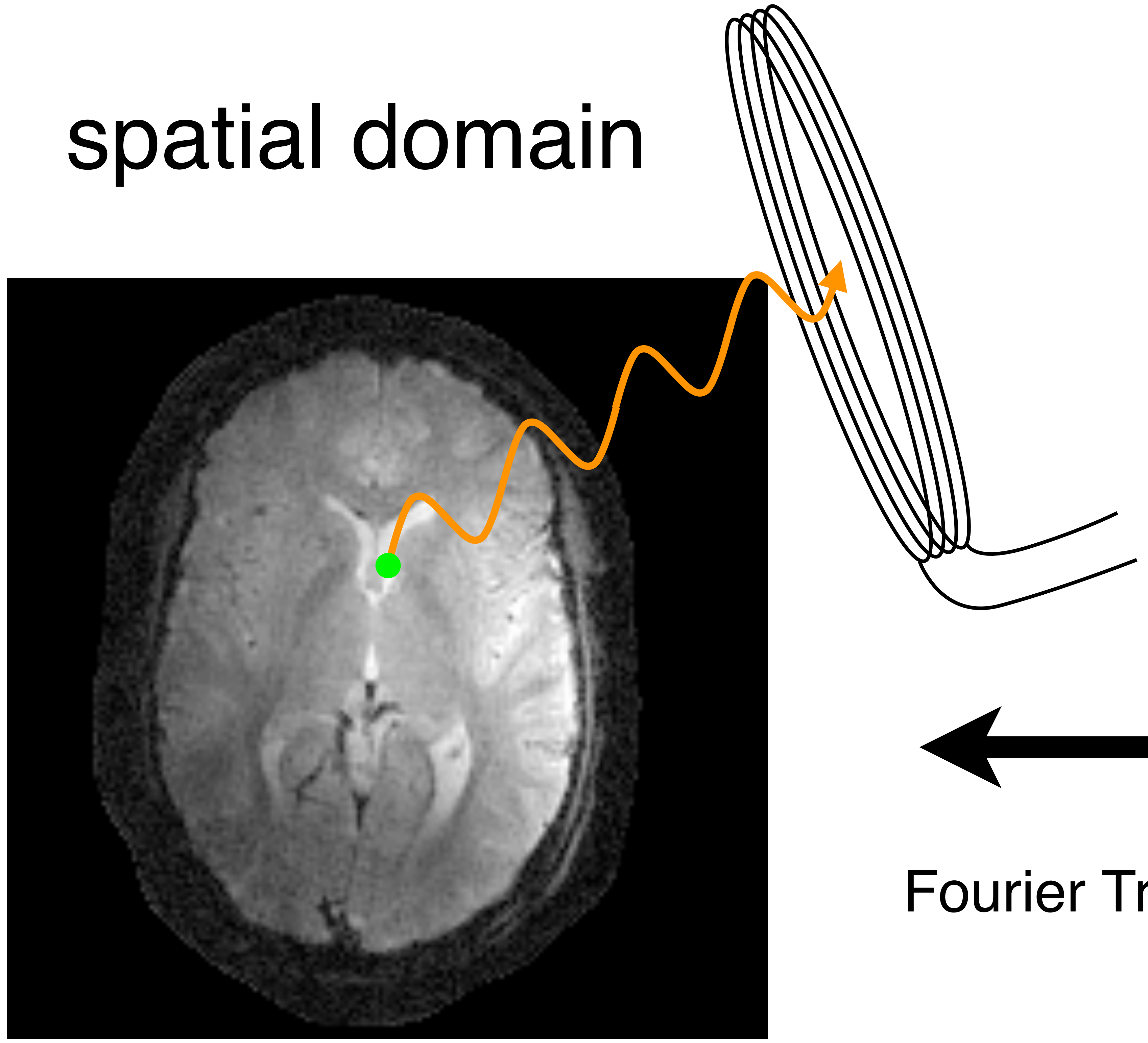


Maximum intensity projections of 384×448×260 image stacks;
Leica DM 5500 widefield epifluorescence microscope with a 63× oil-immersion objective;
C. Elegans embryo labeled with Hoechst, Alexa488, Alexa568;
wavelet regularization (Haar), 3 decomposition levels for X-Y, 2 decomposition levels for Z.

(Vonesch-U., *IEEE TIP* 2009)

Application: Parallel MRI reconstruction

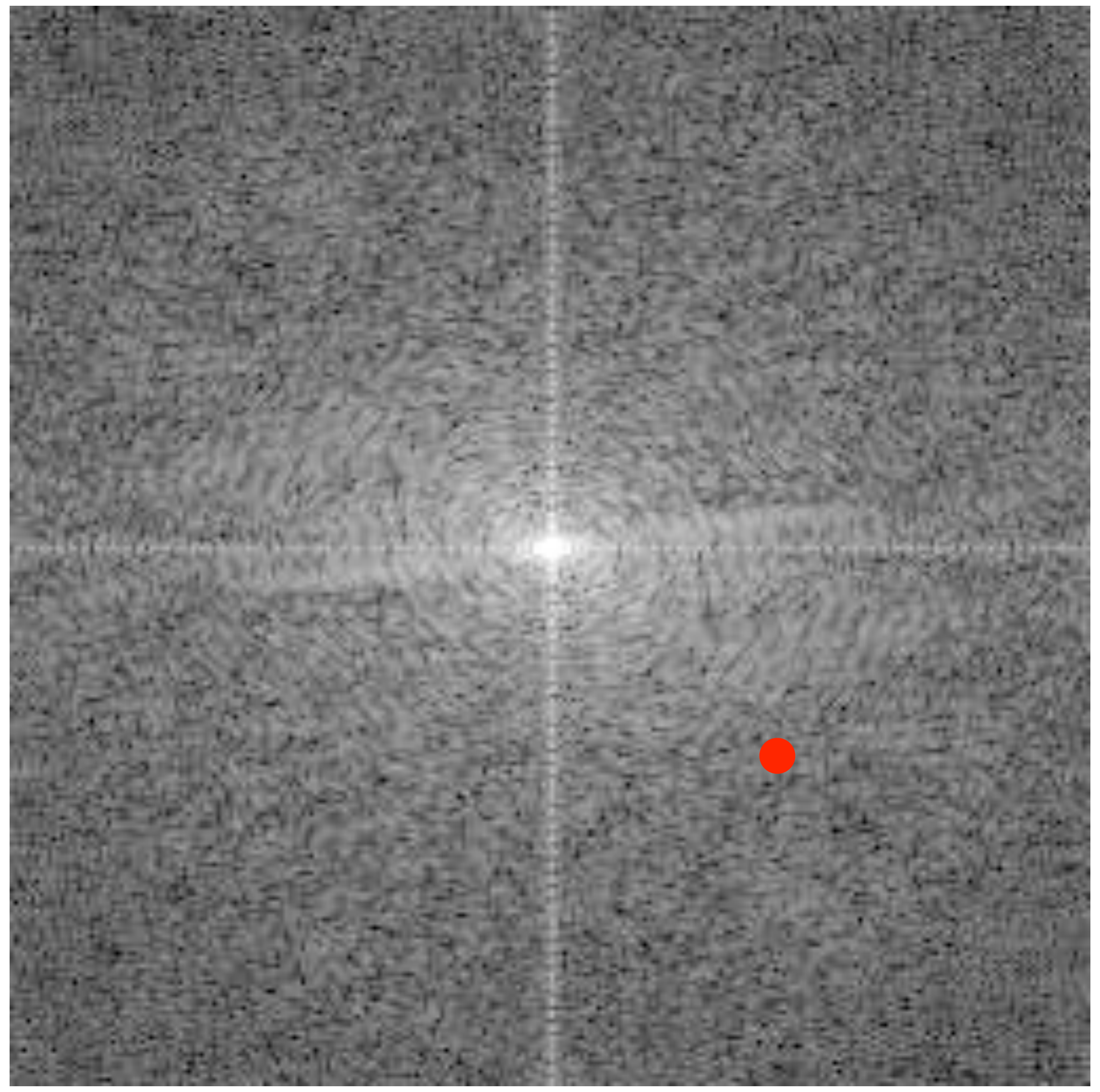
spatial domain



$$S(\boldsymbol{r})f(\boldsymbol{r})$$

Fourier Transform

k-space domain



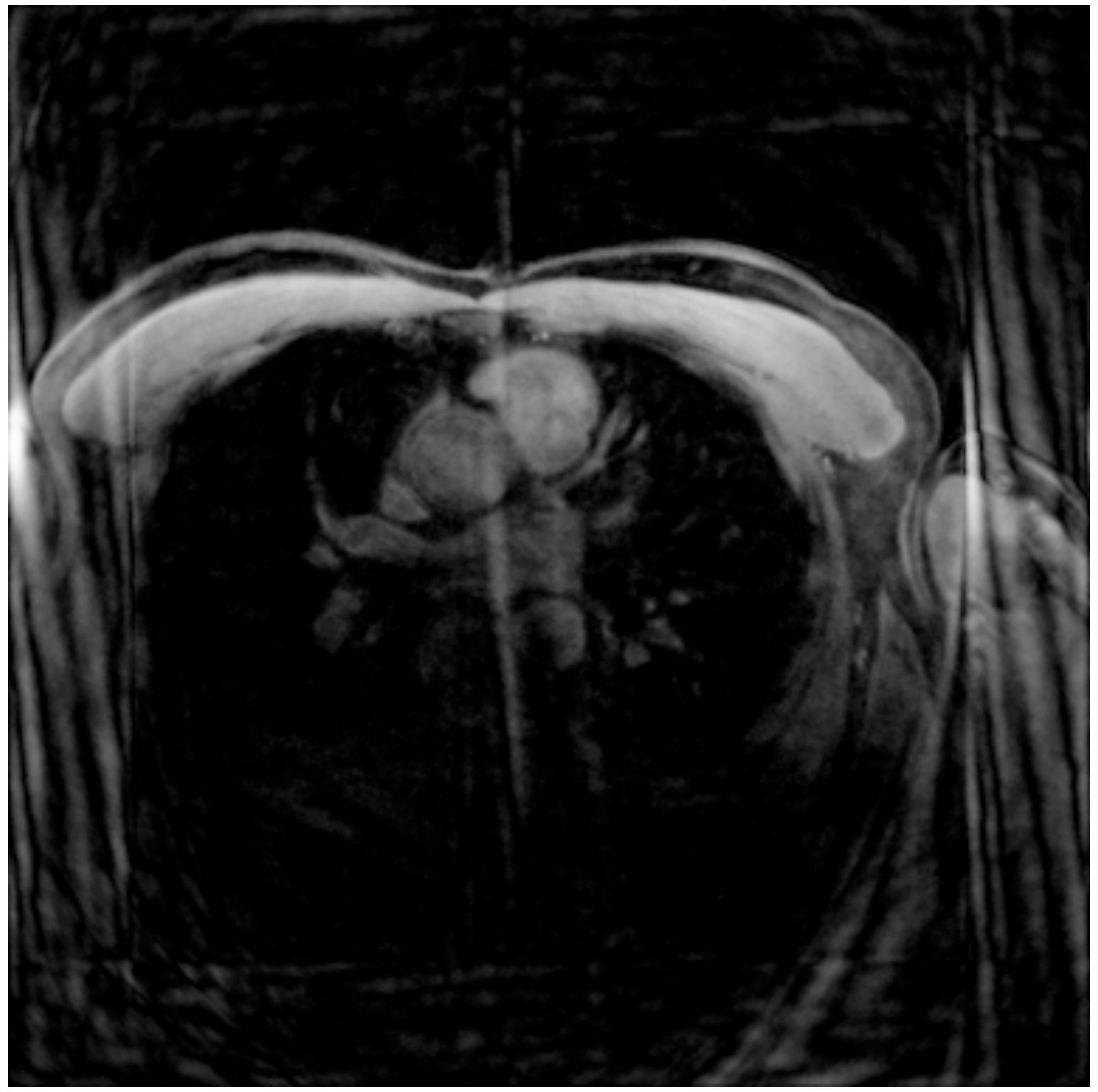
$$g(\boldsymbol{k}) = \int S(\boldsymbol{r})f(\boldsymbol{r})\mathrm{e}^{\mathrm{j}\langle \boldsymbol{r}\,,\,\boldsymbol{k}\rangle}\,\mathrm{d}\boldsymbol{r}$$

Parallel MRI: several receiving coils, known sensitivities

Challenging reconstruction: few k-space samples

Wavelet-regularized reconstruction of MRI

L_2 regularization (Laplacian)



Standard approach (CG)

ℓ_1 wavelet regularization

WFISTA algorithm

(Guerquin-Kern et al. *IEEE Trans. Med. Im.* 2012)

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Ondelettes et bases hilbertiennes

P. G. Lemarié et Y. Meyer

En hommage à A. P. Calderón

Théorème 5. *Une distribution tempérée f (modulo les polynômes) appartient à $B_q^{s,p}$ si et seulement si $\sup_{\epsilon\in E}|\langle f, \psi_Q^{(\epsilon)}\rangle| = \alpha(k,j)$ vérifie*

$$\left(\sum_{j=-\infty}^{\infty}\left\{\left(\sum_{k\in\mathbb{Z}^n}(\alpha(k,j))^p\right)^{1/p}2^{j(s+n(1/2-1/p))}\right\}^q\right)^{1/q} < +\infty \tag{6.1}$$

(avec les changements usuels si $p = +\infty$ ou $q = +\infty$).

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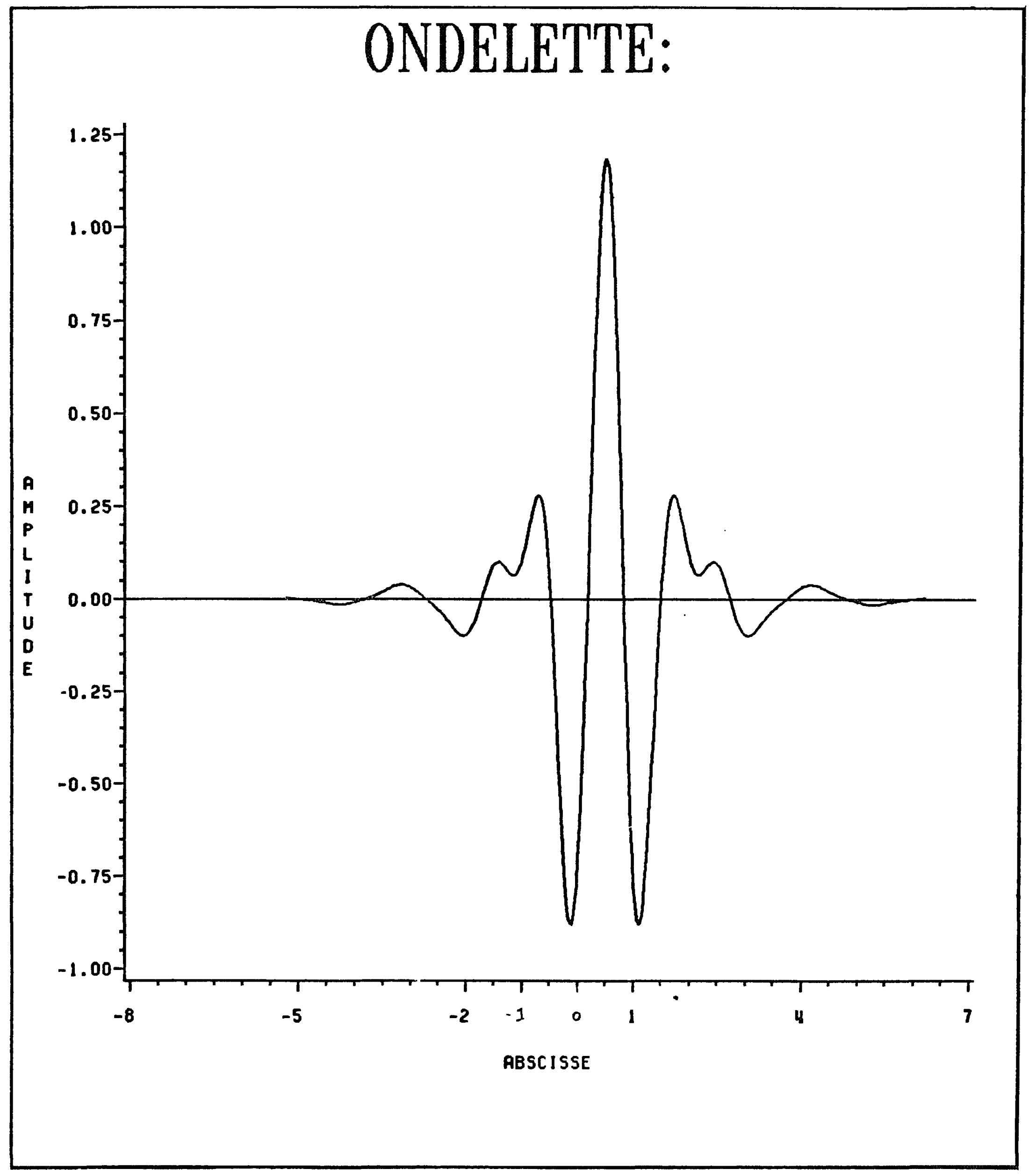
«Unité Associée au C.N.R.S. n.° 169»

Ce travail a été effectué dans le cadre de la «R.C.P. ondelettes» du C.N.R.S. M727.1285, Décembre 1985.



Cette base convient à tous les espaces fonctionnels classiques: espaces de Sobolev, de Besov, de Hardy... qui se traduisent isomorphiquement en des espaces de suites.

L'espace BMO($\mathbb{R}^n$) n'est pas séparable et ne possède donc pas de base inconditionnelle. Nous le remplaçons par la version séparable VMO($\mathbb{R}^n$) qui est la fermeture, pour la norme de BMO($\mathbb{R}^n$), de l'espace $S(\mathbb{R}^n)$ des fonctions de test.

Théorème 3. *La suite $\psi_Q^{(\epsilon)}$, $\epsilon\in E$, $Q\in\mathbb{Q}$, est une base inconditionnelle pour $L^p(\mathbb{R}^n; dx)$, $1 < p < +\infty$, VMO($\mathbb{R}^n$) et $H^1(\mathbb{R}^n)$.*

Conclusion: Impact of wavelets

Yves Meyer

Functional analysis

Numerical analysis

Stochastic processes

Fractals and multi-fractals

⋮

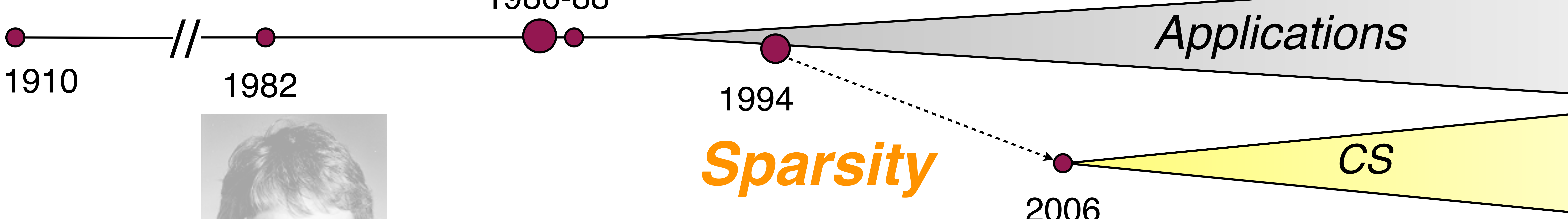
Albert Cohen

Stéphane Jaffard

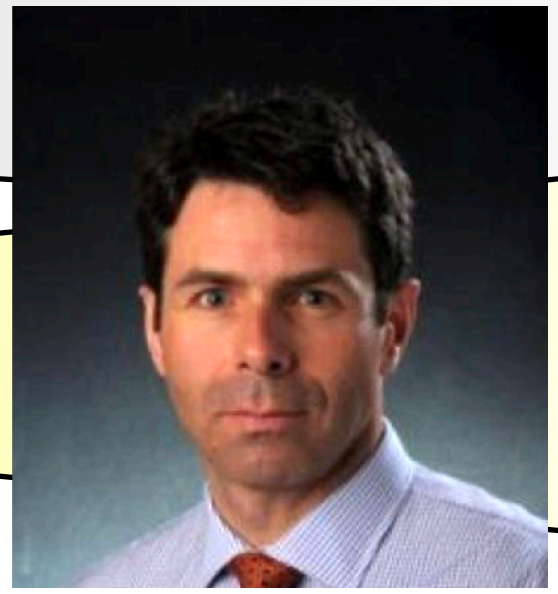
⋮

Alfred Haar

1986-88



Sparsity



François Meyer

Wavelet Art