ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Steerable wavelet transforms and monogenic image analysis

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Joint work with
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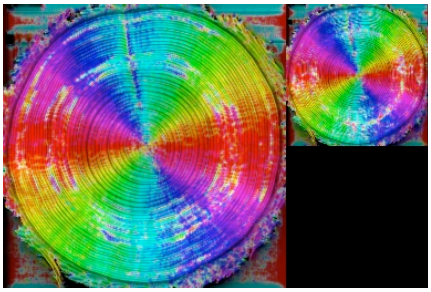


Engineering Science Seminar, Oxford Univ., January 15, 2010

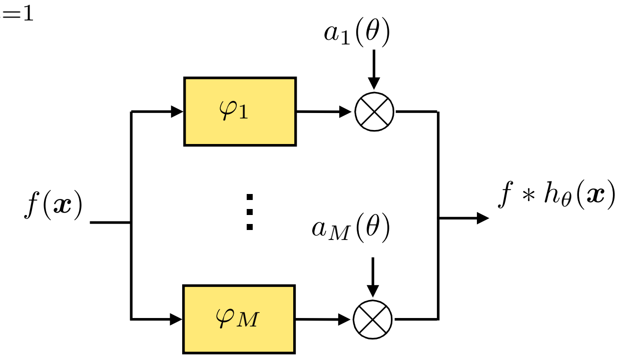
Steerable filters

(Freeman & Adelson, 1991)

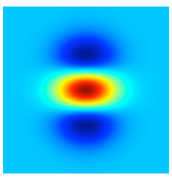
Definition. A 2D filter $h(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$ is steerable of order M iff. there exist some basis filters $\varphi_m(\boldsymbol{x})$ and coefficients $a_m(\theta)$ such that

$$\forall \theta \in [-\pi, \pi], \quad h_\theta(\boldsymbol{x}) := h(\boldsymbol{R}_\theta \boldsymbol{x}) = \sum_{m=1}^{M} a_m(\theta)\, \varphi_m(\boldsymbol{x})$$

- Fast filterbank implementation

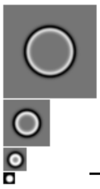


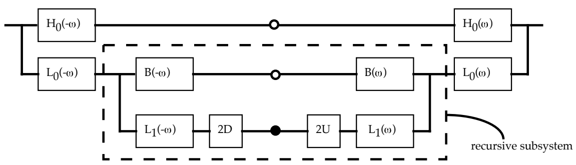
Optimized ridge detector (*M*=3)

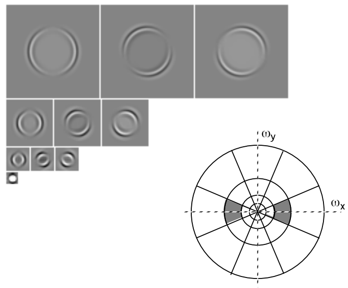


(Jacob-U., IEEE-PAMI, 2004)

Simoncelli's steerable pyramid (1995)







- Many successful applications
  - Contour detection
  - Image filtering and denoising
  - Orientation analysis
  - Texture analysis and synthesis

- Limitations
  - Purely discrete framework (no functional counterpart)
  - Does not extend to dimensions higher than two

CONTENT

- **Riesz transform and its higher-order extensions**
  - Hilbert transform
  - Riesz transform and its properties
  - Higher-order Riesz transform
  - Steerability and directional Hilbert transform
- General construction of steerable wavelet frames
- Steerable Riesz-Laplace wavelet transforms
- Monogenic wavelet analysis

Hilbert transform

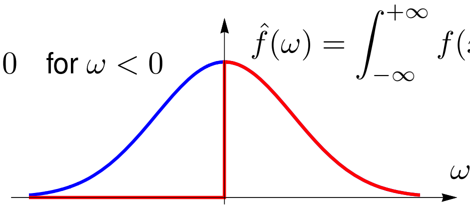
- Definition: $\mathcal{H}f(x) = (h*f)(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\,\mathrm{sgn}(\omega)\,\hat{f}(\omega) = -j\frac{\omega}{|\omega|}\hat{f}(\omega)$

- Key properties
  - Mapping of cosines into sines: $\mathcal{H}\{\cos(\omega_0 \cdot)\}(x) = \sin(\omega_0 x)$
  - Impulse response: $h(x) = \frac{1}{\pi x}$
  - Unitary transform: $\forall \varphi_k, \varphi_l \in L_2(\mathbb{R}), \quad \langle \varphi_k, \varphi_l \rangle_{L_2} = \langle \mathcal{H}\varphi_k, \mathcal{H}\varphi_l \rangle_{L_2}$

- Analytical signal: $f_{\mathrm{ana}}(x) = f(x) + j\mathcal{H}f(x) = A(x)e^{j\xi(x)}$

$\hat{f}_{\mathrm{ana}}(\omega) = 0$ for $\omega < 0$

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-j\omega x}\mathrm{d}x$$



Riesz transform

- Definition: $\boldsymbol{\mathcal{R}}f(\boldsymbol{x}) = \begin{pmatrix} \mathcal{R}_1 f(\boldsymbol{x}) \\ \vdots \\ \mathcal{R}_d f(\boldsymbol{x}) \end{pmatrix} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}\hat{f}(\boldsymbol{\omega})$

Multi-dimensional Fourier transform

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-j\langle \boldsymbol{\omega}, \boldsymbol{x}\rangle}\mathrm{d}x_1 \cdots \mathrm{d}x_d$$

with $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$

- Multi-channel convolution

$$\mathcal{R}_n f(\boldsymbol{x}) = (h_n * f)(\boldsymbol{x}) \quad \text{with} \quad h_n = \mathcal{R}_n\{\delta\} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\frac{\omega_n}{\|\boldsymbol{\omega}\|}$$

- Special case $d = 1$: the Hilbert transform

$$\mathcal{H}f(x) = (h * f)(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -j\,\mathrm{sgn}(\omega)\,\hat{f}(\omega) = -j\frac{\omega}{|\omega|}\hat{f}(\omega)$$

Space-domain characterization

TABLE I

RIESZ TRANSFORM COMPONENTS AND RELATED DIFFERENTIAL OPERATORS FOR $d = 1, 2, 3, 4$

Operator	$(-\Delta)^{-\frac{1}{2}}$	$\mathcal{R}_n = -\frac{\partial}{\partial x_n}(-\Delta)^{-\frac{1}{2}}$	$(-\Delta)^{\frac{1}{2}}$
Frequency response	$\frac{1}{\|\boldsymbol{\omega}\|}$	$\frac{-j\omega_n}{\|\boldsymbol{\omega}\|}$	$\|\boldsymbol{\omega}\|$
Impulse responses			
$d = 1$	$\frac{\log \|x\|}{\pi}$	$\frac{1}{\pi x}$	$\frac{-1}{\pi \|x\|^2}$
$d = 2$	$\frac{1}{2\pi\|\boldsymbol{x}\|}$	$\frac{x_n}{2\pi\|\boldsymbol{x}\|^3}$	$\frac{-1}{2\pi\|\boldsymbol{x}\|^3}$
$d = 3$	$\frac{1}{2\pi^2\|\boldsymbol{x}\|^2}$	$\frac{x_n}{\pi^2\|\boldsymbol{x}\|^4}$	$\frac{-1}{\pi^2\|\boldsymbol{x}\|^4}$
$d = 4$	$\frac{1}{4\pi^2\|\boldsymbol{x}\|^3}$	$\frac{3x_n}{4\pi^2\|\boldsymbol{x}\|^5}$	$\frac{-3}{4\pi^2\|\boldsymbol{x}\|^5}$

Riesz transform in maths, SP and optics

- Riesz transform in mathematics
  - Fonctions conjuguées (Riesz 1920)
  - Singular integral operators (Calderon-Zygmund, 1955; Stein, 1970)
- Hilbert and Riesz transform in signal processing
  - Analytical signal (Gabor, 1946; Ville 1948)
  - 2D extension: Monogenic signal analysis (Felsberg, 2001)
  - Phased-based feature detection (Noble-Brady *et al.*, 2004)
- Riesz transform in optics
  - Radial Hilbert transform (Davis, 2000)
  - Spiral phase quadrature transform (Larkin, 2001)

Steerability and directional Hilbert transform

- Directional Hilbert transform

Unit vector: $\boldsymbol{u} = (u_1, \cdots, u_d)$

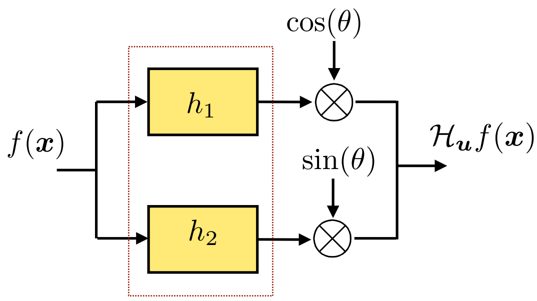
$$\mathcal{H}_{\boldsymbol{u}} f(\boldsymbol{x}) = \sum_{n=1}^{d} u_n \mathcal{R}_n f(\boldsymbol{x}) = \langle \boldsymbol{u}, \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) \rangle$$

Hilbert-like behavior in direction $\boldsymbol{u}$: $\widehat{\mathcal{H}_{\boldsymbol{u}}}(\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\omega\boldsymbol{u}} = -j\mathrm{sgn}(\omega)$

- Implementation in 2-D

$\boldsymbol{u} = (\cos\theta, \sin\theta)$

Gradient-like steerable filterbank



2-D Riesz transform

Reversibility of the Riesz transform

- Adjoint operator

$$\boldsymbol{\mathcal{R}}^* \boldsymbol{r}(\boldsymbol{x}) = \mathcal{R}_1^* r_1(\boldsymbol{x}) + \cdots + \mathcal{R}_d^* r_d(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad j \frac{\boldsymbol{\omega}^T}{\|\boldsymbol{\omega}\|} \hat{\boldsymbol{r}}(\boldsymbol{\omega})$$

- Self-reversibility

$$\boldsymbol{\mathcal{R}}^* \boldsymbol{\mathcal{R}} f(\boldsymbol{x}) = \sum_{i=1}^{d} \mathcal{R}_i^* \mathcal{R}_i f(\boldsymbol{x}) = f(\boldsymbol{x})$$

- What about iterating ?
  - Combining Nth-order components of the form $\mathcal{R}_{i_1} \mathcal{R}_{i_2} \cdots \mathcal{R}_{i_N} f$ with $i_1, \cdots i_N \in \{1, \cdots, d\}$
  - n-fold iteration: $\mathcal{R}_i^n = \mathcal{R}_i \mathcal{R}_i^{n-1}$ with $\mathcal{R}_i^0 = \mathrm{Id}$

Higher-order Riesz transform

Theorem (Decomposition of the identity)

$$\sum_{\substack{n_1,\dots,n_d \geq 0 \\ n_1+\dots+n_d=N}} \frac{N!}{n_1!\,n_2!\cdots n_d!} \left(\mathcal{R}_1^{n_1}\cdots\mathcal{R}_d^{n_d}\right)^* \left(\mathcal{R}_1^{n_1}\cdots\mathcal{R}_d^{n_d}\right) = \mathrm{Id}$$

- Proper definition of Nth-order transform

$p(N,d) = \binom{N+d-1}{d-1}$ distinct Riesz components with $n_1 + \cdots + n_d = N$

$$\boldsymbol{\mathcal{R}}^{(N)} f(\boldsymbol{x}) = \begin{pmatrix} \mathcal{R}^{(N,0,\cdots,0)} f(\boldsymbol{x}) \\ \vdots \\ \mathcal{R}^{(n_1,\cdots,n_d)} f(\boldsymbol{x}) \\ \vdots \\ \mathcal{R}^{(0,\cdots,0,N)} f(\boldsymbol{x}) \end{pmatrix} \quad \text{where} \quad \mathcal{R}^{(n_1,\dots,n_d)} = \sqrt{\frac{N!}{n_1!\cdots n_d!}}\,\mathcal{R}_1^{n_1}\cdots\mathcal{R}_d^{n_d}$$

Properties of higher-order Riesz transform

- Shift invariance: $\forall \boldsymbol{x}_0 \in \mathbb{R}^d, \quad \boldsymbol{\mathcal{R}}^{(N)}\{f(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}}^{(N)}\{f(\cdot)\}(\boldsymbol{x} - \boldsymbol{x}_0)$

- Scale invariance: $\forall a \in \mathbb{R}^+, \quad \boldsymbol{\mathcal{R}}^{(N)}\{f(\cdot/a)\}(\boldsymbol{x}) = \boldsymbol{\mathcal{R}}^{(N)}\{f(\cdot)\}(\boldsymbol{x}/a)$

- Parseval-like identity: $\forall f, \phi \in L_2(\mathbb{R}^d)$

$$\begin{aligned} \langle \boldsymbol{\mathcal{R}}^{(N)} f, \boldsymbol{\mathcal{R}}^{(N)} \phi \rangle_{L_2} &= \sum_{n_1+\cdots+n_d=N} \langle \mathcal{R}^{(n_1,\dots,n_d)} f, \mathcal{R}^{(n_1,\dots,n_d)} \phi \rangle_{L_2} \\ &= \langle f, \phi \rangle_{L_2} \end{aligned}$$

Energy conservation: $\|\boldsymbol{\mathcal{R}}^{(N)} f\|_{L_2}^2 = \sum_{n_1+\cdots+n_d=N} \|\mathcal{R}^{(n_1,\cdots,n_d)} f\|_{L_2}^2 = \|f\|_{L_2}^2$

Riesz transform and derivatives

- Fractional Laplacian

$$(-\Delta)^{\alpha} f(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \|\boldsymbol{\omega}\|^{2\alpha} \hat{f}(\boldsymbol{\omega})$$

- Composition rule

$$(-\Delta)^{\alpha_1}(-\Delta)^{\alpha_2} = (-\Delta)^{\alpha_1+\alpha_2}$$

with $(-\Delta)^0 =$ Identity

- Riesz transform and partial derivatives

$$\boldsymbol{\mathcal{R}} f(\boldsymbol{x}) = (-1)(-\Delta)^{-\frac{1}{2}} \boldsymbol{\nabla} f(\boldsymbol{x})$$

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = -\boldsymbol{\mathcal{R}}(-\Delta)^{\frac{1}{2}} f(\boldsymbol{x})$$

"Smoothed version of gradient"

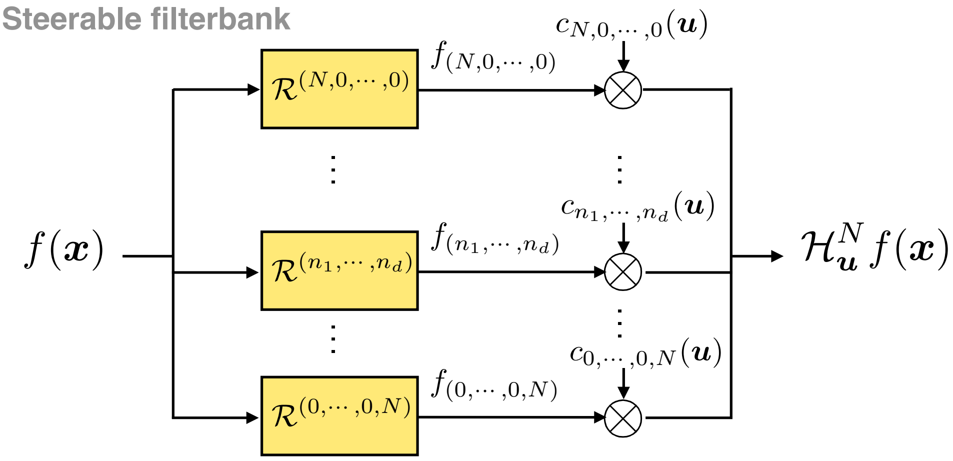
$$\mathcal{R}_1^{n_1} \cdots \mathcal{R}_d^{n_d} f(\boldsymbol{x}) = (-1)^N (-\Delta)^{-\frac{N}{2}} \frac{\partial^N f}{\partial^{n_1} x_1 \cdots \partial^{n_d} x_d}(\boldsymbol{x})$$

Higher-order steerability

- Nth-order directional Hilbert transform along $\boldsymbol{u} = (u_1, \cdots, u_d)$

$$\mathcal{H}_{\boldsymbol{u}}^N f(\boldsymbol{x}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \left(-j \frac{\langle \boldsymbol{u}, \boldsymbol{\omega} \rangle}{\|\boldsymbol{\omega}\|}\right)^N \hat{f}(\boldsymbol{\omega})$$

Steerable filterbank



"steering" coefficients: $c_{n_1,\ldots,n_d}(\boldsymbol{u}) = \sqrt{\dfrac{N!}{n_1! n_2! \cdots n_d!}} (u_1^{n_1} \cdots u_d^{n_d})$

CONTENT

- Riesz transform and its higher-order extensions ✔
- **General construction of steerable wavelet frames**
- Steerable Riesz-Laplace wavelet transform
- Monogenic wavelet analysis

Frame = redundant extension of a basis

- Definition
  - A family of functions $\{\phi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is called a frame of $L_2(\mathbb{R}^d)$ iff.

$$\forall f \in L_2(\mathbb{R}^d), \quad A\,\|f\|_{L_2}^2 \leq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\langle \phi_{\boldsymbol{k}}, f\rangle_{L_2}|^2 \leq B\,\|f\|_{L_2}^2$$

  - Tight frame: $A = B$
  - Parseval frame: $A = B = 1$

- Analysis/synthesis formula
  - $\forall f \in L_2(\mathbb{R}^d), \quad f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle \phi_{\boldsymbol{k}}, f\rangle_{L_2}\,\tilde{\phi}_{\boldsymbol{k}}$
  - $\{\tilde{\phi}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$: dual frame (minimum-norm inverse)
  - Parseval frame: $\tilde{\phi}_{\boldsymbol{k}} = \phi_{\boldsymbol{k}}$

Construction of steerable wavelet frames

- Wavelet frame of $L_2(\mathbb{R}^d)$

$$\forall f \in L_2(\mathbb{R}^d), \quad f(\boldsymbol{x}) = \sum_{i\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle f, \psi_{i,\boldsymbol{k}}\rangle_{L_2} \, \tilde{\psi}_{i,\boldsymbol{k}}(\boldsymbol{x})$$

Wavelet property: $\psi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^{-\frac{id}{2}} \psi_{0,\boldsymbol{k}}(\boldsymbol{x}/2^i)$

Multi-index: $\boldsymbol{n} = (n_1, \dots, n_d)$ with $|\boldsymbol{n}| = \sum_{i=1}^{d} n_i = N$

Theorem

Let $\{\psi_{i,\boldsymbol{k}}\}$ be a primal wavelet frame of $L_2(\mathbb{R}^d)$. Then, $\{\psi^{\boldsymbol{n}}_{i,\boldsymbol{k}} = \mathcal{R}^{\boldsymbol{n}}\psi_{i,\boldsymbol{k}}\}_{|\boldsymbol{n}|=N}$ and $\{\tilde{\psi}^{\boldsymbol{n}}_{i,\boldsymbol{k}} = \mathcal{R}^{\boldsymbol{n}}\tilde{\psi}_{i,\boldsymbol{k}}\}_{|\boldsymbol{n}|=N}$ form a dual set of wavelet frames such that

$$\forall f \in L_2(\mathbb{R}^d), \quad f(\boldsymbol{x}) = \sum_{i\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \sum_{|\boldsymbol{n}|=N} \langle f, \psi^{\boldsymbol{n}}_{i,\boldsymbol{k}}\rangle_{L_2} \, \tilde{\psi}^{\boldsymbol{n}}_{i,\boldsymbol{k}}(\boldsymbol{x})$$

Justification

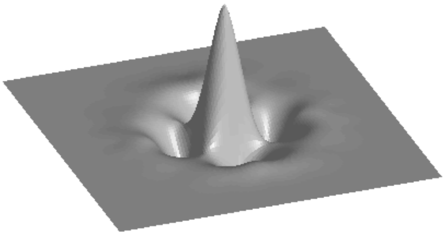
Inner product preservation $\Rightarrow$ $\langle \psi_{i,\boldsymbol{k}}, \psi_{i',\boldsymbol{k}'}\rangle_{L_2} = \langle \boldsymbol{\mathcal{R}}^{(N)}\psi_{i,\boldsymbol{k}}, \boldsymbol{\mathcal{R}}^{(N)}\psi_{i',\boldsymbol{k}'}\rangle_{L_2}$

Shift and scale invariance $\Rightarrow$ $\mathcal{R}^{\boldsymbol{n}}\psi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^{-\frac{id}{2}} \psi^{\boldsymbol{n}}(\boldsymbol{x}/2^i - \boldsymbol{k})$ with $\psi^{\boldsymbol{n}} = \mathcal{R}^{\boldsymbol{n}}\psi$

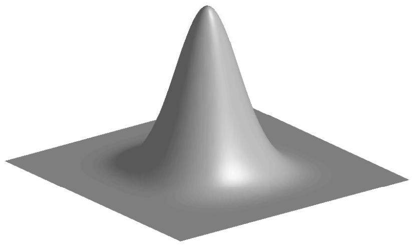
CONTENT

- Riesz transform and its higher-order extensions ✔
- General construction of steerable frames ✔
- **Steerable Riesz-Laplace wavelet transforms**
  - Construction of quasi-isotropic wavelet bases
  - Nth-order Riesz-Laplace wavelets
  - Marr wavelets (primal wavelet sketch)
- Monogenic wavelet analysis

Primary (fractional) Laplacian wavelet



$\psi_{\mathrm{iso}}(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\beta_{2\gamma}(2\boldsymbol{x})$



Gaussian-like smoothing kernel: $\beta_{2\gamma}(2\boldsymbol{x})$

- Quasi-isotropic, Mexican-hat-like wavelet frame
  - Single analysis wavelet: $\psi_{\mathrm{iso}}(\boldsymbol{x})$
  - Polyharmonic B-spline smoothing kernel
    $\beta_{2\gamma}(\boldsymbol{x}) \to C_\gamma \exp(-\|\boldsymbol{x}\|^2/(\gamma/6))$ (Van De Ville, *IEEE-IP* 2005)
  - Multiscale version of (fractional) Laplace operator
  - Fast, reversible filterbank algorithm

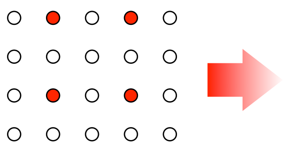
(U.-Van De Ville, *IEEE-IP* 2008)

Laplacian-like multiresolution analysis

- Pyramid decomposition: redundancy $4/3$

$\psi_{\mathrm{iso}}(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\beta_{2\gamma}(\mathbf{D}\boldsymbol{x})$

Dyadic sampling pattern



First decomposition level:

$\varphi(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$ Scaling functions

$\psi(\mathbf{D}^{-1}(\boldsymbol{x} - \boldsymbol{k}))$ Wavelets (redundant by 4/3)

(U.-Van De Ville, *IEEE-IP* 2008)

Steerable Riesz-Laplace wavelets

$$\psi_\gamma^{(n_1,\dots,n_d)}(\boldsymbol{x}) = \sqrt{\frac{N!}{n_1!\cdots n_d!}}\mathcal{R}_1^{n_1}\cdots\mathcal{R}_d^{n_d}(-\Delta)^{\frac{\gamma}{2}}\,\beta_{2\gamma}(\mathbf{D}\boldsymbol{x})$$

$\beta_{2\gamma}(\boldsymbol{x})$: “quasi-isotropic” B-spline of order $2\gamma > d$

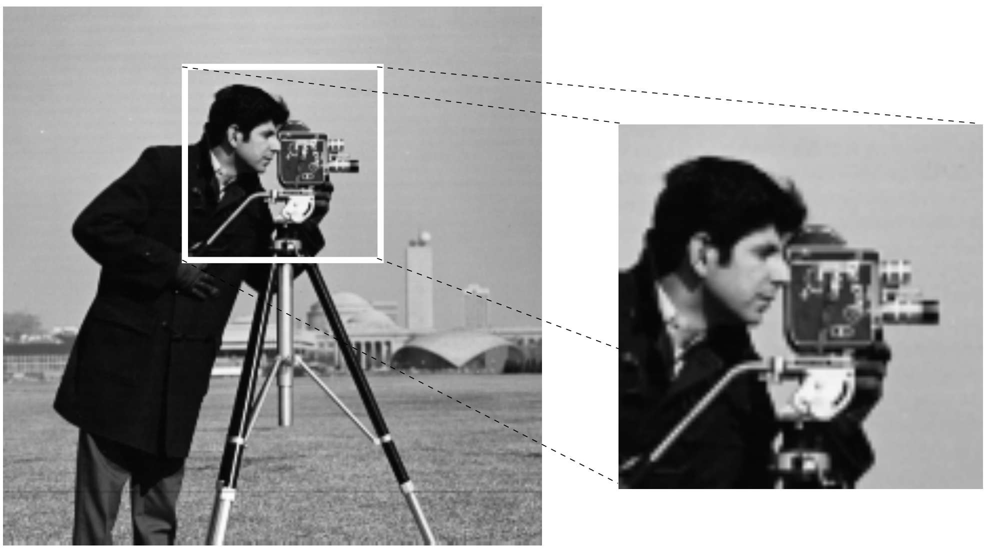
$\mathbf{D}$: admissible dilation matrix

- Analysis wavelets: $\psi^{\boldsymbol{n}}_{(i,\boldsymbol{k})}(\boldsymbol{x}) \triangleq |\det(\mathbf{D})|^{i/2}\psi^{\boldsymbol{n}}_{\gamma}\left(\mathbf{D}^i\boldsymbol{x} - \mathbf{D}^{-1}\boldsymbol{k}\right)$
- The Riesz-Laplace wavelets form a frame of $L_2(\mathbb{R}^d)$
- The Riesz-Laplace wavelets and their duals have $\lceil\gamma\rceil$ vanishing moments
- The Riesz-Laplace wavelet transform provides multiscale estimates of the Nth-order derivatives of the signal

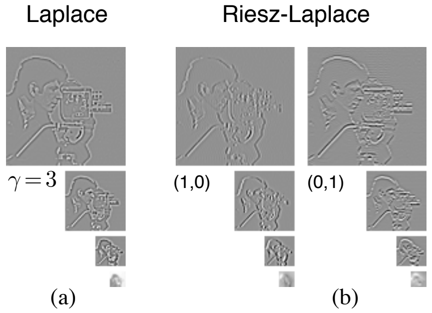
$$\langle f, \psi_{\gamma,i}^{(n_1,\dots,n_d)}(\cdot - \boldsymbol{x}_0)\rangle_{L_2} \propto \left.\frac{\partial^N (f * \xi_i)(\boldsymbol{x})}{\partial^{n_1}x_1\cdots\partial^{n_d}x_d}\right|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

where ξ is a suitable smoothing kernel.

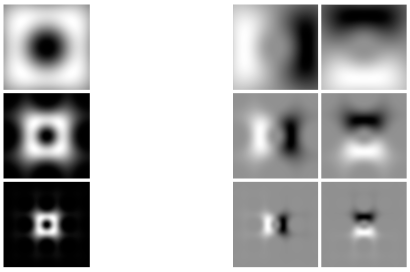
- The wavelet transform has a fast, reversible filterbank algorithm



First-order Riesz-Laplace transform

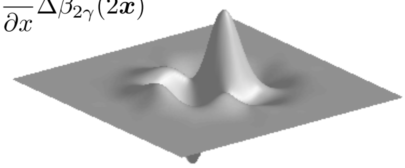


Frequency responses

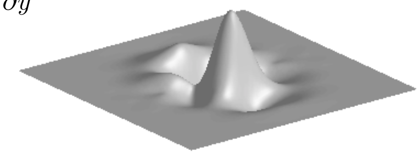


Equivalent to Marr wavelets

$$\psi^{(1,0)}(\boldsymbol{x}) = \frac{\partial}{\partial x}\Delta\beta_{2\gamma}(2\boldsymbol{x})$$

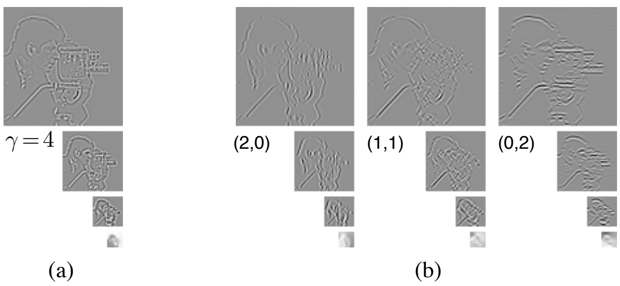


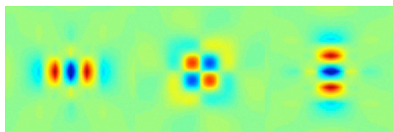
$$\psi^{(0,1)}(\boldsymbol{x}) = \frac{\partial}{\partial y}\Delta\beta_{2\gamma}(2\boldsymbol{x})$$



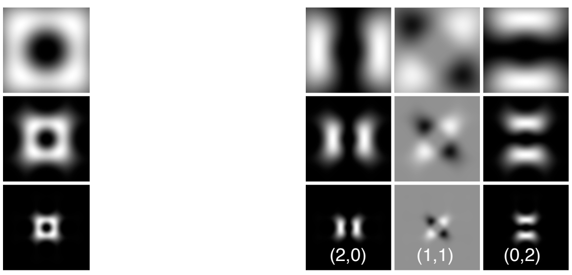
(Van De Ville-U., *IEEE-IP* 2008)

Second-order Riesz-Laplace transforms

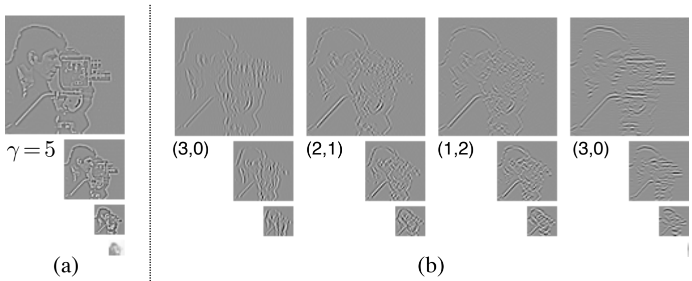




Frequency responses

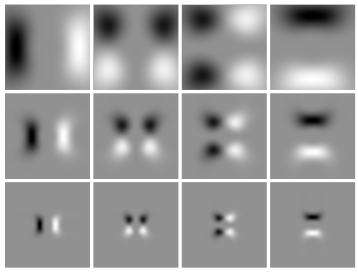


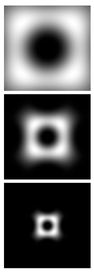
Steerable third-order Riesz-Laplace transform

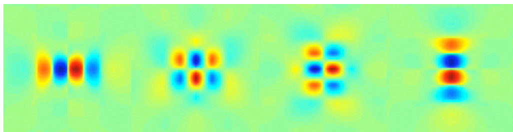


(a) (b)

Frequency responses

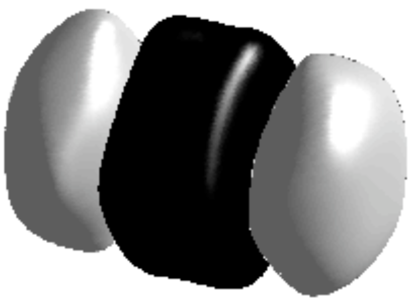






3-D Gradient and Hessian-like wavelets

$$\Delta \frac{\partial}{\partial x_1}$$



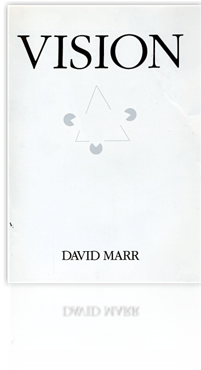
(a) surface detector

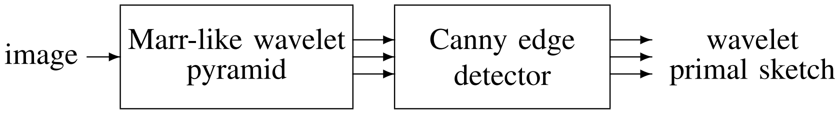
$$\Delta \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$$

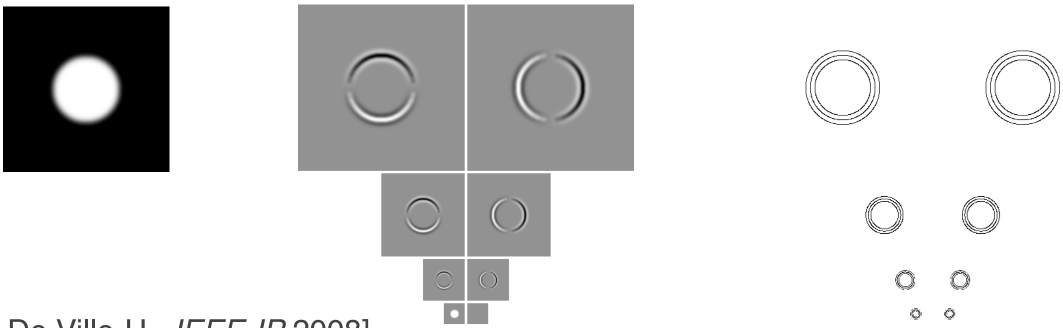
(b) line detector

Processing in early vision - primal sketch

- **Wavelet primal sketch**
  - blurring — smoothing kernel ϕ
  - Laplacian filtering — Δ
  - zero-crossings and orientation — ∇
  - segment detection and grouping — Canny edge detection scheme





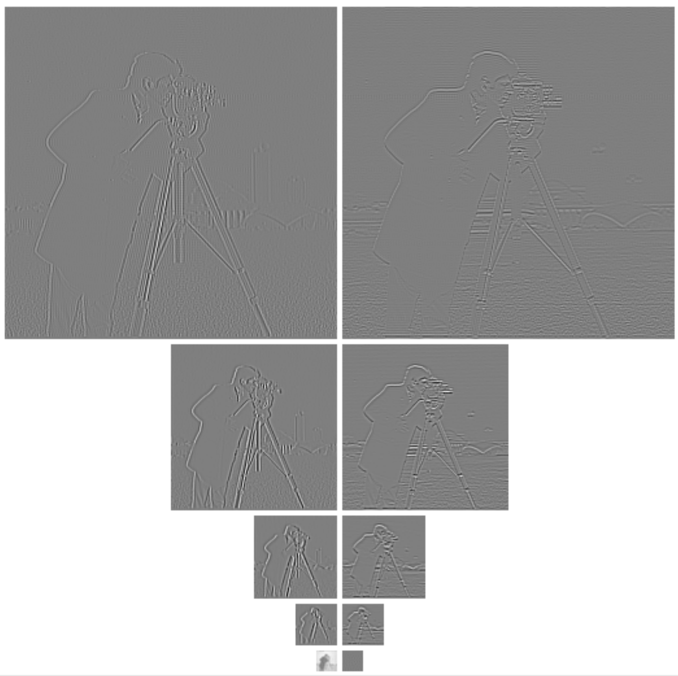


[Van De Ville-U., *IEEE-IP* 2008]

Marr wavelet pyramid

- Pyramid-like decomposition: redundancy $2 \times \frac{4}{3}$

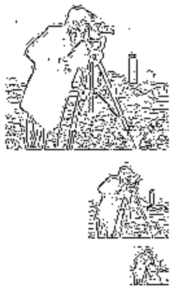
$$\psi_3^{(1,0)}(\boldsymbol{x}) = \frac{\partial}{\partial x}\Delta\beta_6(2\boldsymbol{x}) \qquad \psi_3^{(0,1)}(\boldsymbol{x}) = \frac{\partial}{\partial y}\Delta\beta_6(2\boldsymbol{x})$$



Edge detection in Marr wavelet domain

- Edge map (using Canny's edge detector)
  - Key visual information (Marr's theory of vision)

Similar to Mallat's representation from
wavelet modulus maxima [Mallat-Zhong, 1992]



... but much less redundant !

Iterative reconstruction

- Reconstruction from information on edge map only
  - Better than 30dB PSNR

31.4dB

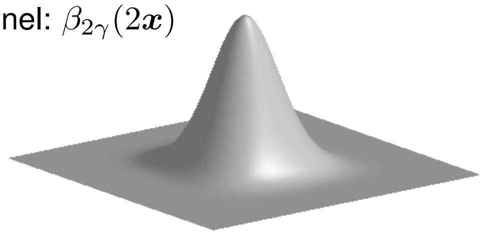
[Van De Ville-U., *IEEE-IP* 2008]

CONTENT

- Riesz transform and its higher-order extensions ✔
- General construction of steerable wavelet frames ✔
- Steerable Riesz-Laplace wavelet transform ✔
- **Monogenic wavelet analysis**
  - Directional AM/FM signal analysis

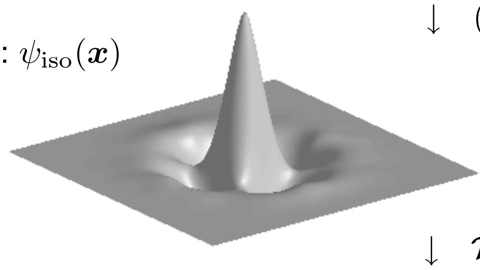
Monogenic splines and wavelets

Gaussian-like kernel: $\beta_{2\gamma}(2\boldsymbol{x})$



$\downarrow$ $(-\Delta)^{\gamma/2}$ (fractional Laplacian)

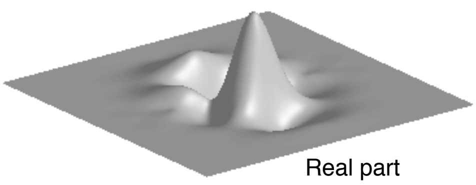
Isotropic wavelet: $\psi_{\mathrm{iso}}(\boldsymbol{x})$



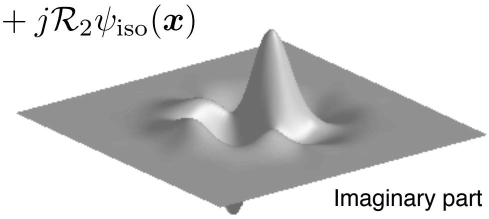
$\gamma = 4$

$\downarrow$ $\boldsymbol{\mathcal{R}}$ (Riesz transform)

Complex Riesz wavelet: $\psi_{\mathrm{Riesz}}(\boldsymbol{x}) = \mathcal{R}_1\psi_{\mathrm{iso}}(\boldsymbol{x}) + j\mathcal{R}_2\psi_{\mathrm{iso}}(\boldsymbol{x})$



Real part



Imaginary part

Felsberg's monogenic signal analysis

- Three-component monogenic signal
  - Input signal: $f(\boldsymbol{x})$
  - Complex Riesz transform: $\mathcal{R}f(\boldsymbol{x}) = (\mathcal{R}_1 + j\mathcal{R}_2)f(\boldsymbol{x}) = r(\boldsymbol{x})\, e^{j\theta(\boldsymbol{x})}$
  - Monogenic signal: $\mathbf{f}_{\mathrm{m}}(\boldsymbol{x}) = (f(\boldsymbol{x}),\, \mathcal{R}_1 f(\boldsymbol{x}),\, \mathcal{R}_2 f(\boldsymbol{x})) = (f,\, r\cos\theta,\, r\sin\theta)$
- Local Orientation: $\theta(\boldsymbol{x}) = \angle(\mathcal{R}f(\boldsymbol{x}))$
- Directional Hilbert analysis: $f_\theta(\boldsymbol{x}) = f(\boldsymbol{x}) + j\mathcal{H}_\theta f(\boldsymbol{x}) = A\, e^{j\xi}$
- Local Amplitude: $A(\boldsymbol{x}) = |f_\theta(\boldsymbol{x})| = \sqrt{|f(\boldsymbol{x})|^2 + |\mathcal{R}_1 f(\boldsymbol{x})|^2 + |\mathcal{R}_2 f(\boldsymbol{x})|^2}$
- Local phase and wavenumber
  - Local phase: $\xi(\boldsymbol{x}) = \angle(f_\theta(\boldsymbol{x}))$
  - Local wavenumber: $\nu(\boldsymbol{x}) = \mathrm{D}_{\boldsymbol{\theta}}\xi(\boldsymbol{x}) = \langle \boldsymbol{\theta}, \nabla\xi(\boldsymbol{x})\rangle$ with $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$

Wavelet-domain monogenic analysis

- Three-component monogenic wavelet transform
  - Real wavelet coefficients: $w_i[\boldsymbol{k}] = \langle f, \psi_{\mathrm{iso}(i,\boldsymbol{k})}\rangle$
  - Complex wavelet coefficients: $w_{\mathrm{Riesz},i}[\boldsymbol{k}] = \langle f, \mathcal{R}\psi_{\mathrm{iso}(i,\boldsymbol{k})}\rangle = r\, e^{j\theta}$
  - Monogenic wavelet vector: $\mathbf{w}_i[\boldsymbol{k}] = (w_i[\boldsymbol{k}], \mathrm{Re}(w_{\mathrm{Riesz},i}[\boldsymbol{k}]), \mathrm{Im}(w_{\mathrm{Riesz},i}[\boldsymbol{k}]))$

$$= (A\cos\varphi, \underbrace{A\sin\xi}_{r}\cos\theta, \underbrace{A\sin\xi}_{r}\sin\theta)$$

- Local Orientation: $\theta_i(\boldsymbol{k}) = \arg(w_{\mathrm{Riesz},i}(\boldsymbol{k}))$
- Local Amplitude: $A_i(\boldsymbol{k}) = \|\mathbf{w}_i(\boldsymbol{k})\| = \sqrt{|w_i(\boldsymbol{k})|^2 + |w_{\mathrm{Riesz},i}(\boldsymbol{k})|^2}$
- Local phase: $\xi_i[\boldsymbol{k}] = \arctan\left(\dfrac{|w_{\mathrm{Riesz},i}(\boldsymbol{k})|}{w_i(\boldsymbol{k})}\right)$

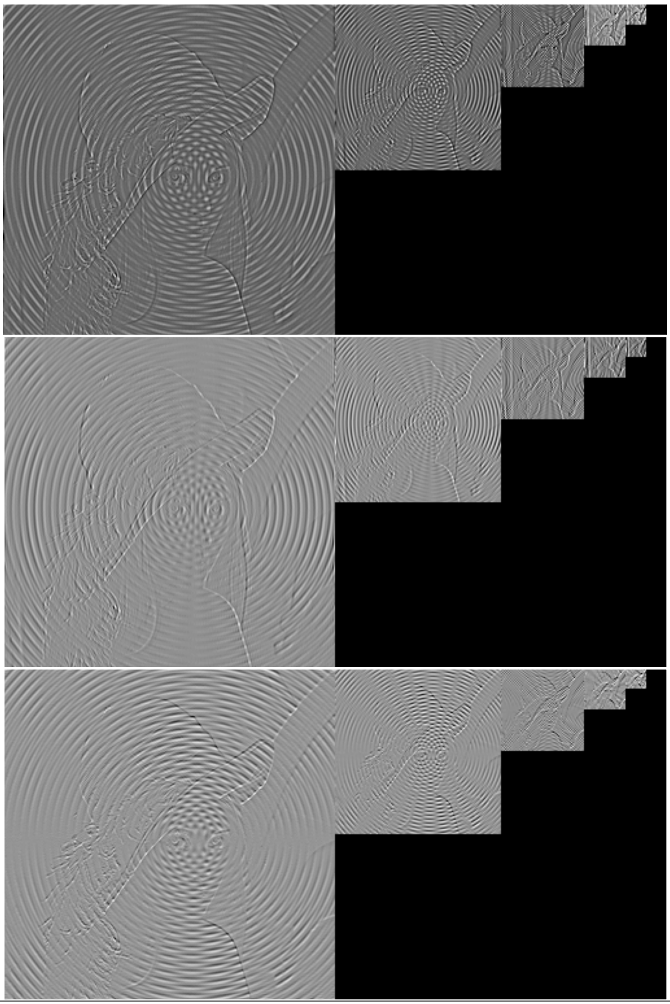
(U.-Sage-Van De Ville, *IEEE-IP* 2009)

Psychedelic Lena

Laplacian

Riesz X

Riesz Y



Wavelet-domain structure analysis

- Gradient-like Marr wavelet transform

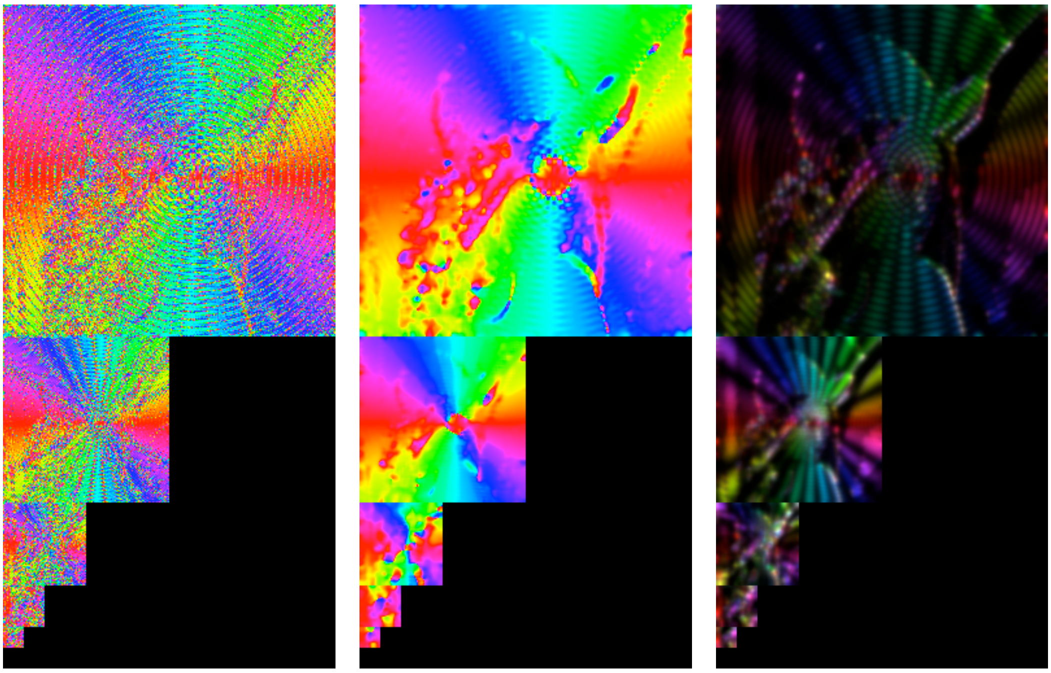
$$u_i[\boldsymbol{k}] = \langle f, \psi^{(1,0)}_{(i,\boldsymbol{k})} \rangle \quad \text{and} \quad v_i[\boldsymbol{k}] = \langle f, \psi^{(0,1)}_{(i,\boldsymbol{k})} \rangle$$

- Structure tensor: $\mathbf{J}_i(\boldsymbol{k}) = \begin{bmatrix} J_{u_i u_i} & J_{u_i v_i} \\ J_{u_i v_i} & J_{v_i v_i} \end{bmatrix}$

with $J_{uv}[\boldsymbol{k}] = \sum_{\boldsymbol{n} \in \mathbb{Z}^2} e^{-\frac{\|\boldsymbol{n}\|^2}{2\sigma^2}} u[\boldsymbol{k}+\boldsymbol{n}] v[\boldsymbol{k}+\boldsymbol{n}]$

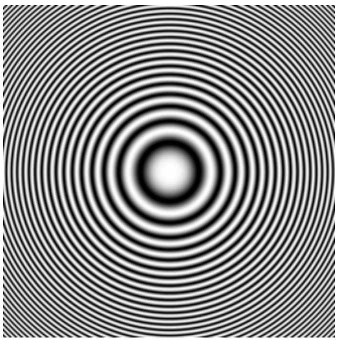
- Local features
  - Hilbert energy: $E = \mathrm{trace}(\mathbf{J}) = J_{uu} + J_{vv}$
  - Orientation: $\mathbf{u}_1 = (\cos\theta, \sin\theta)$ with $\theta = \frac{1}{2} \arctan\left(\frac{2J_{uv}}{J_{vv} - J_{uu}}\right)$
  - Coherency: $0 \le C = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = \frac{\sqrt{(J_{vv} - J_{uu})^2 + 4J_{uv}^2}}{J_{vv} + J_{uu}} \le 1$

Example: Psychedelic Lena

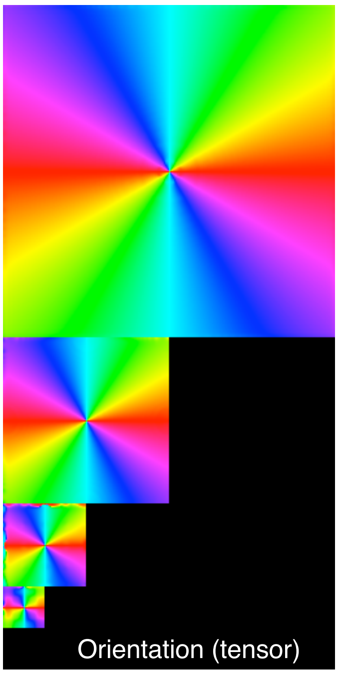


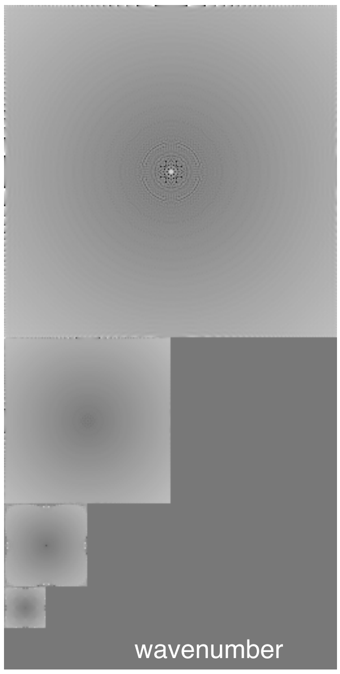
Pointwise orientation | tensor orientation | Coherency in saturation Hilbert energy in brightness

Example: Zoneplate



Synthetic zoneplate

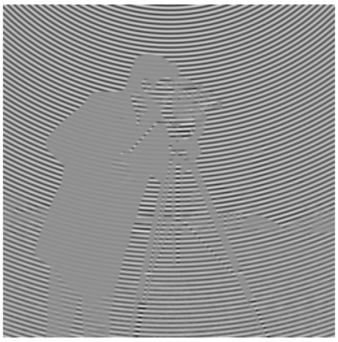




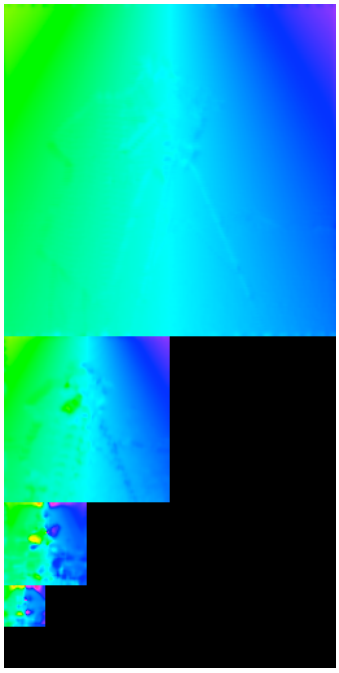
Local phase: $\xi_i[\boldsymbol{k}] = \arctan\left(\frac{|w_{\mathrm{Riesz},i}(\boldsymbol{k})|}{w_i(\boldsymbol{k})}\right)$

Local wavenumber: $\nu_i = \mathrm{D}_{\boldsymbol{\theta}}\xi_i$

Example: Modulated Cameraman



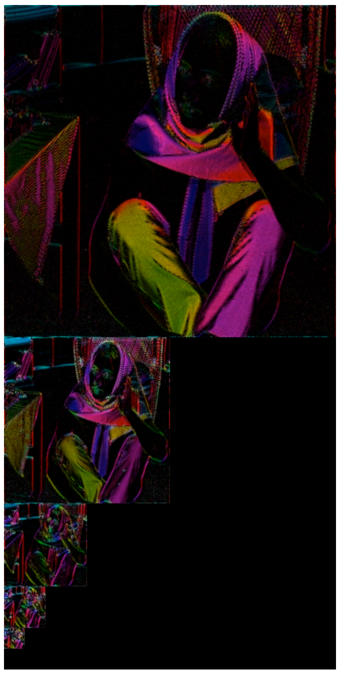
amplitude



tensor orientation

Example: Coherence analysis of Barbara

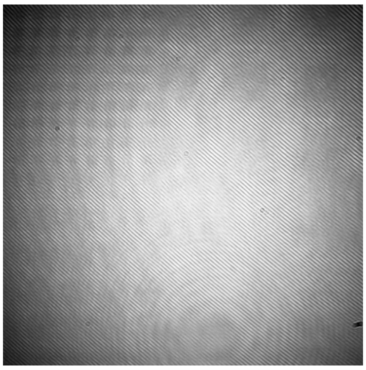
Coherency



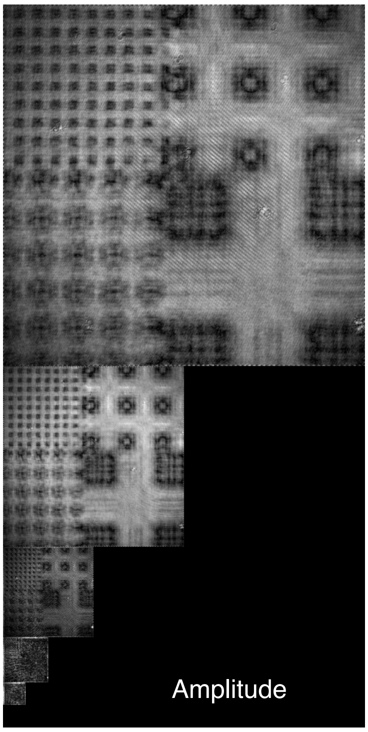
HSB
Hue: Orientation
Saturation: Coherency
Brightness: Modulus

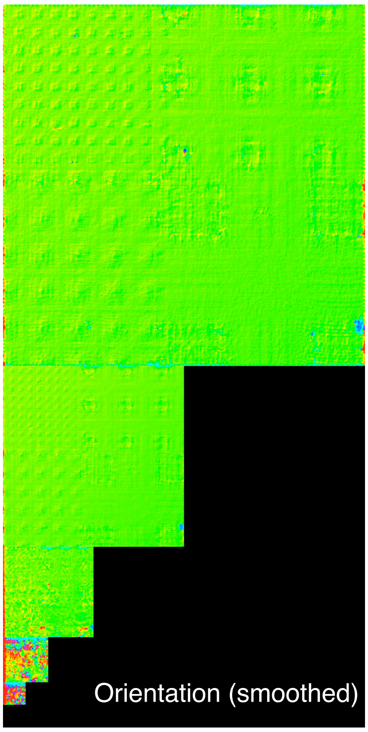
$\gamma = 2, \sigma = 2$

Example: Digital holography microscopy



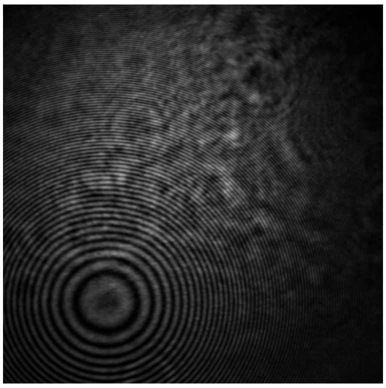
DHM image



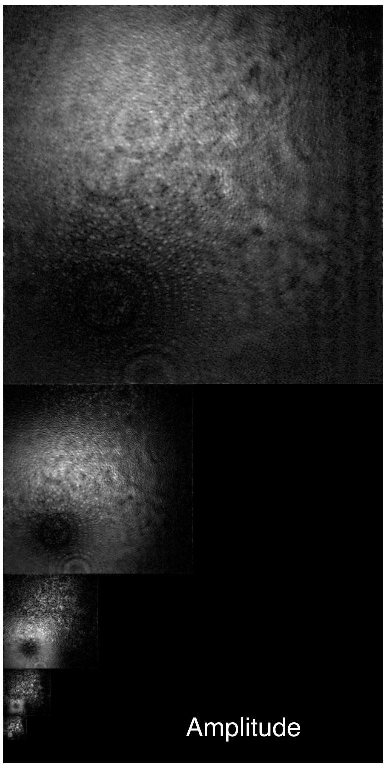


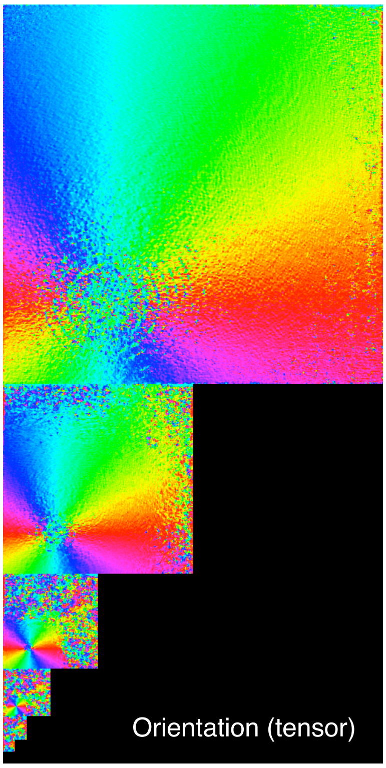
Data courtesy of Prof. Depeursinge, EPFL

Example: Digital holography microscopy



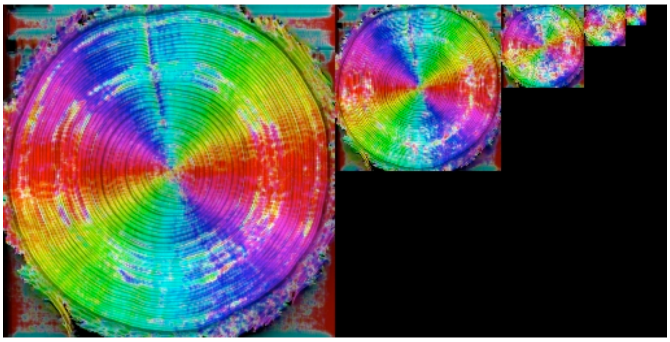
Original DHM image





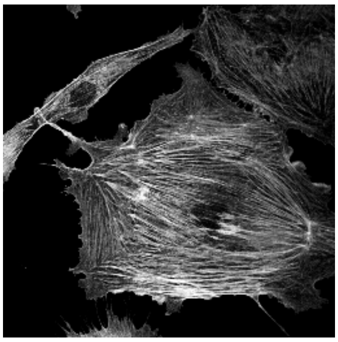
Data courtesy of Prof. Depeursinge, EPFL

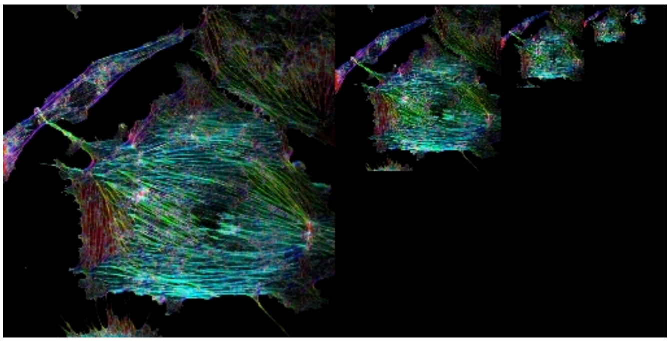
Examples of multiscale orientation analysis



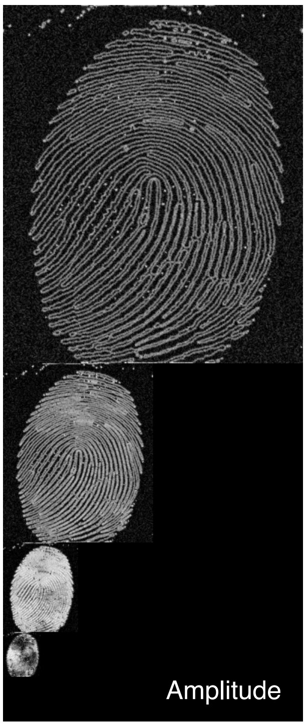
Color HSB
H: Orientation
S: Coherency
B: Input

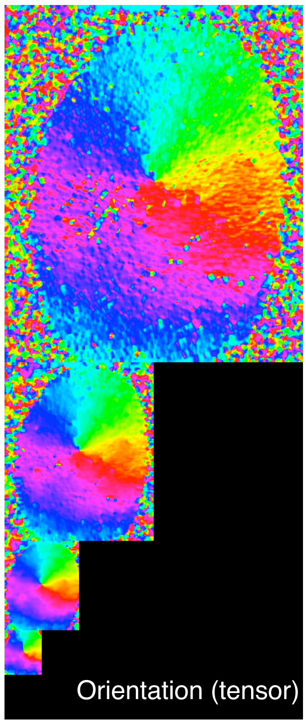
Fluorescence micrograph





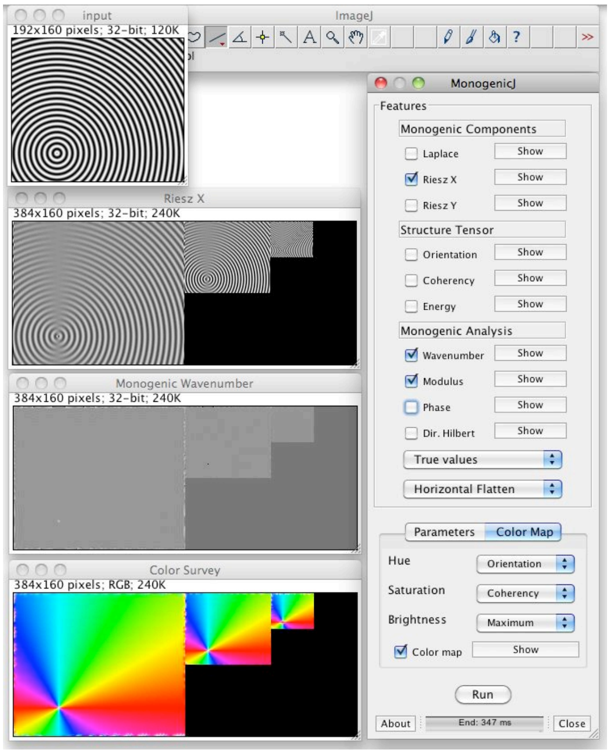
Directional wavelet analysis: Fingerprint





Wavelet-domain monogenic and structure analysis

MonogenicJ: a plugin for ImageJ (JAVA)



ImageJ
Image Processing and Analysis in Java

Author: Daniel Sage

http://bigwww.epfl.ch/demo/monogenic/

CONCLUSION

- General operator-based design of steerable wavelets
- Key features of the approach
  - Decoupling between multiresolution and multiorientation properties
  - Simplicity of implementation (FFT, multirate filterbank)
  - Exact reconstruction property
  - Generalization to higher dimensions
- Riesz-Laplace wavelet transform
  - Quasi-isotropic generator (polyharmonic B-spline)
  - Multiscale partial derivatives
  - Directional Hilbert transform
- Novel perspectives for wavelet-domain image processing
  - Rotation-invariant processing/feature extraction
  - Primal wavelet sketch (Marr wavelets)
  - Wavelet-domain monogenic analysis

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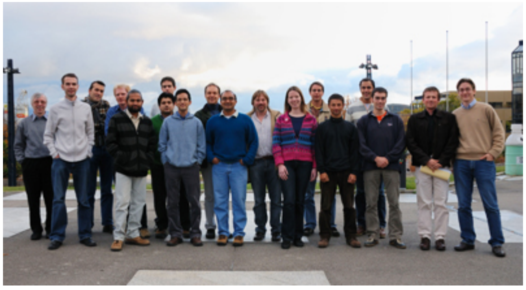
- M. Unser, D. Sage, D. Van De Ville, "Multiresolution monogenic signal analysis using the Riesz-Laplace wavelet transform," *IEEE Trans. Image Processing*, vol. 18, no. 11, pp. 2402-2418, November 2009.
- M. Unser, D. Van De Ville, "Wavelet steerability and the higher-order Riesz transform," *IEEE Trans. Image Processing*, in press.

EPFL's Biomedical Imaging Group



- Preprints and demos: **http://bigwww.epfl.ch/**