ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Wavelets and differential operators: from fractals to Marr's primal sketch

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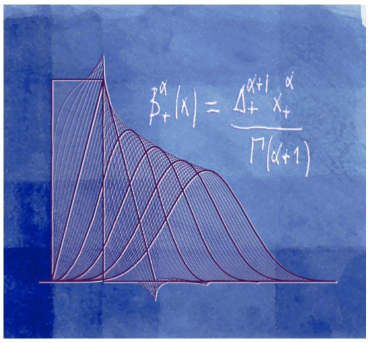


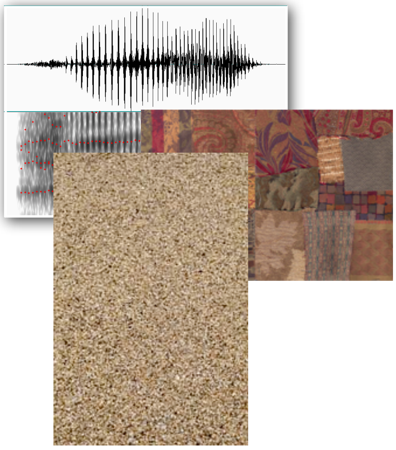
Plenary talk, SMAI 2009, La Colle sur Loup, 25-29 Mai, 2009

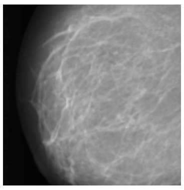
The quest for invariance

- Invariance to coordinate transformations
  - Primary transformations (X): translation (T), scaling (S), rotation (R), affine (similarity) (A=S+R)
  - A continuous-domain operator L is X-invariant iff. it commutes with X; i.e,
    $\forall f \in L_2(\mathbb{R}^d), \mathrm{XL}f = C_{\mathrm{X}} \cdot \mathrm{LX}f$ $\qquad$ C_{X}: normalization constant
  - All classical physical laws are TSR-invariant
- Classical signal/image processing operators are invariant (to various extents)
  - Filters (linear or non-linear): T-invariant
  - Differentiators, wavelet transform: TS-invariant
  - Contour/ridge detectors (Gradient, Laplacian, Hessian): TSR-invariant
  - Steerable filters: TR-invariant

Invariant signals

- Natural signals/images often exhibit some degree of invariance (at least locally, if not globally)
  - Stationarity, texture: T-invariance
  - Isotropy (no preferred orientation): R-invariance
  - Self-similarity, fractality: S-invariance (Pentland 1984; Mumford, 2001)





OUTLINE

- Splines and T-invariant operators
  - Green functions as elementary building blocks
  - Existence of a local B-spline basis
  - Link with stochastic processes
- Imposing affine (TSR) invariance
  - Fractional Laplace operator & polyharmonic splines
  - Fractal processes
- Laplacian-like, quasi-isotropic wavelets
  - Polyharmonic spline wavelet bases
  - Analysis of fractal processes
- The Marr wavelet
  - Complex Laplace/gradient operator
  - Steerable complex wavelets
  - Wavelet primal sketch
  - Directional wavelet analysis

General concept of an *L*-spline

$\mathrm{L}\{\cdot\}$: differential operator (translation-invariant)

$\delta(\boldsymbol{x}) = \prod_{i=1}^{d} \delta(x_i)$: multidimensional Dirac distribution

Definition

The continuous-domain function $s(\boldsymbol{x})$ is a ***cardinal L-spline*** iff.

$$\mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

- Cardinality: the knots (or spline singularities) are on the (multi-)integers
- Generalization: includes polynomial splines as particular case ($\mathrm{L} = \frac{\mathrm{d}^N}{\mathrm{d}x^N}$)

Example: piecewise-constant splines

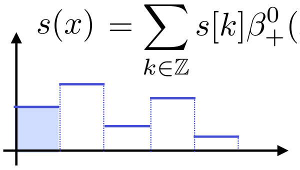
- Spline-defining operators

Continuous-domain derivative: $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \longleftrightarrow \quad j\omega$

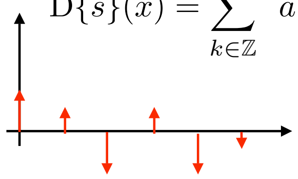
Discrete derivative: $\Delta_+\{\cdot\} \quad \longleftrightarrow \quad 1 - e^{-j\omega}$

- Piecewise-constant or D-spline

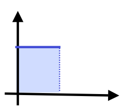
$$s(x) = \sum_{k\in\mathbb{Z}} s[k]\beta_+^0(x-k)$$



$$\mathrm{D}\{s\}(x) = \sum_{k\in\mathbb{Z}} \overbrace{a[k]}^{\Delta_+ s(k)} \delta(x-k)$$



- B-spline function

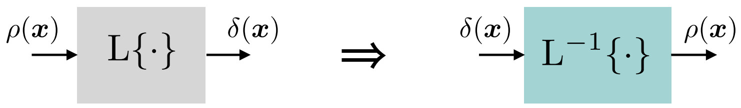


$$\beta_+^0(x) = \Delta_+\mathrm{D}^{-1}\{\delta\}(x) \quad \longleftrightarrow \quad \frac{1-e^{-j\omega}}{j\omega}$$

Splines and Green's functions

Definition

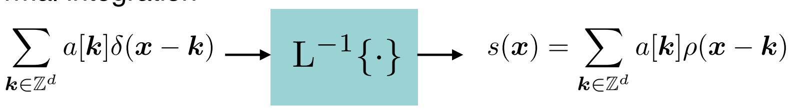
$\rho(\boldsymbol{x})$ is a Green function of the shift-invariant operator L iff $\mathrm{L}\{\rho\} = \delta$



(+ null-space component?)

- Cardinal L-spline: $\mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$

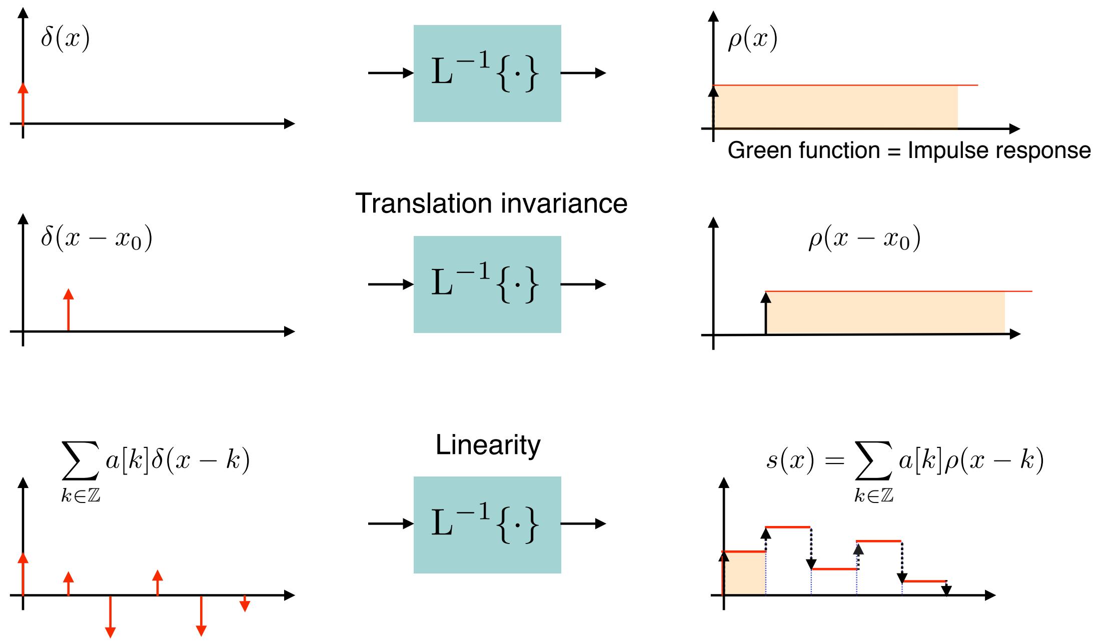
Formal integration



$$\Rightarrow \quad V_{\mathrm{L}} = \mathrm{span}\,\{\rho(\boldsymbol{x}-\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$$

Example of spline synthesis

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}x} = \mathrm{D} \quad \Rightarrow \quad \mathrm{L}^{-1}\text{: integrator}$$



Existence of a local, shift-invariant basis?

- Space of cardinal L-splines

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) : \mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \right\} \cap L_2(\mathbb{R}^d)$$

- Generalized B-spline representation

A "localized" function $\varphi(\boldsymbol{x}) \in V_{\mathrm{L}}$ is called *generalized B-spline* if it generates a Riesz basis of V_{L}; i.e., iff. there exists $(A > 0, B < \infty)$ s.t.

$$A \cdot \|c\|_{\ell_2(\mathbb{Z}^d)} \le \left\| \textstyle\sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \right\|_{L_2(\mathbb{R}^d)} \le B \cdot \|c\|_{\ell_2(\mathbb{Z}^d)}$$

$$\Downarrow$$

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) : \boldsymbol{x}\in\mathbb{R}^d, c \in \ell_2(\mathbb{Z}^d) \right\}$$

continuous-domain signal

discrete signal
(B-spline coefficients)

Link with stochastic processes

Splines are in direct correspondence with stochastic processes (stationary or fractals) that are solution of the same partial differential equation, but with a random driving term.

Defining operator equation: $\mathrm{L}\{s(\cdot)\}(\boldsymbol{x}) = r(\boldsymbol{x})$

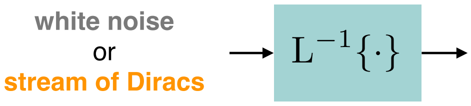
- Specific driving terms
  - $r(\boldsymbol{x}) = \delta(x)$ $\Rightarrow$ $s(\boldsymbol{x}) = \mathrm{L}^{-1}\{\delta\}(\boldsymbol{x})$: Green function
  - $r(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$ $\Rightarrow$ $s(\boldsymbol{x})$: Cardinal L-spline
  - $r(\boldsymbol{x})$: white Gaussian noise $\Rightarrow$ $s(\boldsymbol{x})$: generalized stochastic process

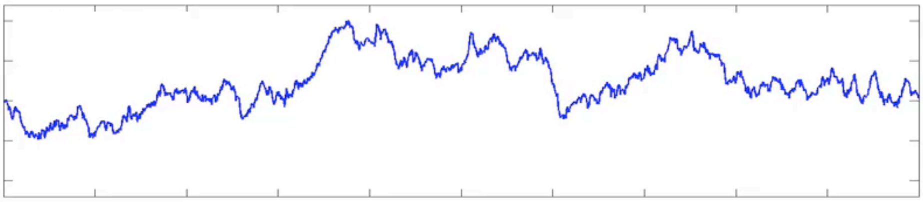
⚠ non-empty null space of L, boundary conditions

References: stationary proc. (U.-Blu, *IEEE-SP* 2006), fractals (Blu-U., *IEEE-SP* 2007)

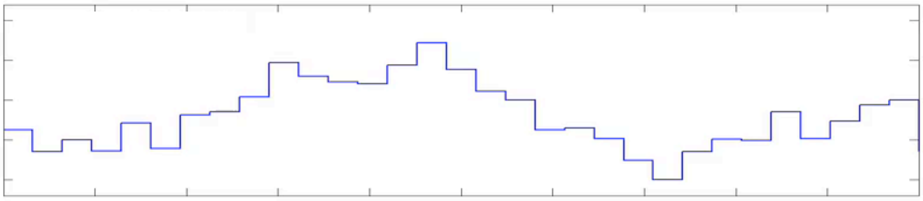
Example: Brownian motion vs. spline synthesis

$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \Rightarrow \quad \mathrm{L}^{-1}$: integrator





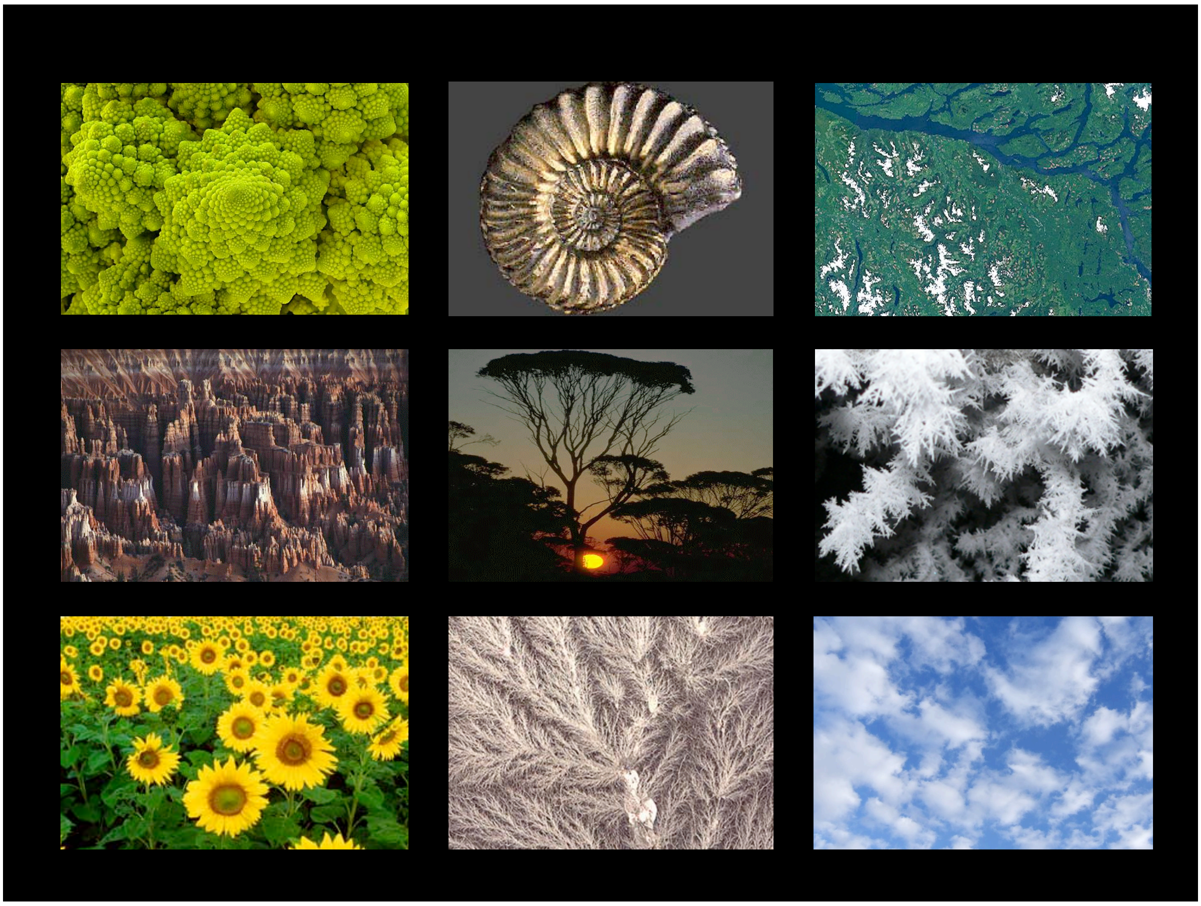
Brownian motion



Cardinal spline (Schoenberg, 1946)

IMPOSING SCALE INVARIANCE

- Affine-invariant operators
- Polyharmonic splines
- Associated fractal random fields: fBms



Scale- and rotation-invariant operators

Definition: An operator L is affine-invariant (or SR-invariant) iff.

$$\forall s(\boldsymbol{x}),\ \mathrm{L}\{s(\cdot)\}(\mathbf{R}_\theta \boldsymbol{x}/a) = C_a \cdot \mathrm{L}\{s(\mathbf{R}_\theta \cdot /a)\}(\boldsymbol{x})$$

where $\mathbf{R}_\theta$ is an arbitrary $d \times d$ unitary matrix and C_a a constant

- **Invariance theorem**

The complete family of real, scale- and rotation-invariant convolution operators is given by the fractional Laplacians

$$(-\Delta)^{\frac{\gamma}{2}} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \|\boldsymbol{\omega}\|^{\gamma}$$

- Invariant Green functions (a.k.a. RBF) (Duchon, 1979)

$$\rho(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|^{\gamma-d} \log \|\boldsymbol{x}\|, & \text{if } \gamma - d \text{ is even} \\ \|\boldsymbol{x}\|^{\gamma-d}, & \text{otherwise} \end{cases}$$

Polyharmonic splines

Spline functions associated with fractional Laplace operator $(-\Delta)^{\gamma/2}$

- **Distributional definition** [Madych-Nelson, 1990]

$s(\boldsymbol{x})$ is a cardinal polyharmonic spline of order γ iff.

$$(-\Delta)^{\gamma/2} s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} d[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

- **Explicit Shannon-like characterization**

$$\mathcal{V}_0 = \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} s[\boldsymbol{k}]\phi_\gamma(\boldsymbol{x}-\boldsymbol{k}) \right\}$$

$\phi_\gamma(\boldsymbol{x})$: Unique polyharmonic spline interpolator s.t. $\phi_\gamma(\boldsymbol{k}) = \delta_{\boldsymbol{k}}$

$$\stackrel{\mathcal{F}}{\longleftrightarrow} \qquad \hat{\phi}_\gamma(\boldsymbol{\omega}) = \frac{1}{1+\sum_{\boldsymbol{k}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}}\left(\frac{\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}+2\pi\boldsymbol{k}\|}\right)^\gamma}$$

Construction of polyharmonic B-splines

Laplacian operator: $\Delta \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -\|\boldsymbol{\omega}\|^2$

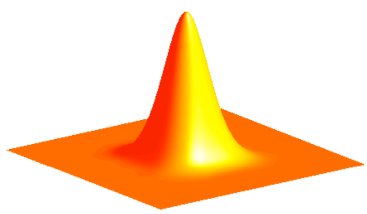
Discrete Laplacian: $\Delta_{\mathrm{d}} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad -\sum_{i=1}^{d} 4\sin^2(\omega_i/2) \triangleq -\|2\sin(\boldsymbol{\omega}/2)\|^2$

0	-1	0
-1	4	-1
0	-1	0

- **Polyharmonic B-splines** (Rabut, 1992)

Discrete operator: localization filter $Q(e^{j\boldsymbol{\omega}})$

$$\frac{\|2\sin(\boldsymbol{\omega}/2)\|^\gamma}{\|\boldsymbol{\omega}\|^\gamma} \quad \stackrel{\mathcal{F}^{-1}}{\longrightarrow} \quad \beta_\gamma(\boldsymbol{x})$$



Continuous-domain operator: $\hat{L}(\boldsymbol{\omega})$

Polyharmonic B-splines properties

- Stable representation of polyharmonic splines (Riesz basis)

$$V_{(-\Delta^{\frac{\gamma}{2}})} = \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\beta_\gamma(\boldsymbol{x}-\boldsymbol{k}) : c[\boldsymbol{k}] \in \ell_2(\mathbb{Z}^d) \right\}$$ Condition: $\gamma > \frac{d}{2}$

- Two-scale relation: $\beta_\gamma(\boldsymbol{x}/2) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} h_\gamma[\boldsymbol{k}]\beta\gamma(\boldsymbol{x}-\boldsymbol{k})$

- Order of approximation γ (possibly fractional)

- Reproduction of polynomials

  The polyharmonic B-splines $\{\varphi_\gamma(\boldsymbol{x}-\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ reproduce the polynomials of degree $n = \lceil \gamma - 1 \rceil$. In particular,

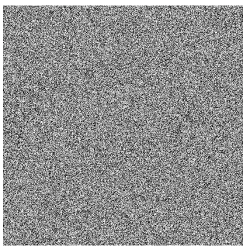
$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_\gamma(\boldsymbol{x}-\boldsymbol{k}) = 1 \qquad \text{(partition of unity)}$$

(Rabut, 1992; Van De Ville, 2005)

Associated random field: multi-D fBm

Formalism: Gelfand's theory of generalized stochastic processes

White noise



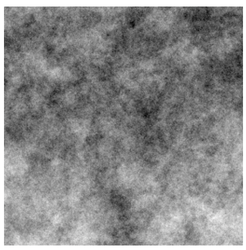
$(-\Delta)^{-\frac{\gamma}{2}}$

fractional integrator
(appropriate boundary conditions)

$(-\Delta)^{\frac{\gamma}{2}}$

Whitening
(fractional Laplacian)

fractional Brownian field



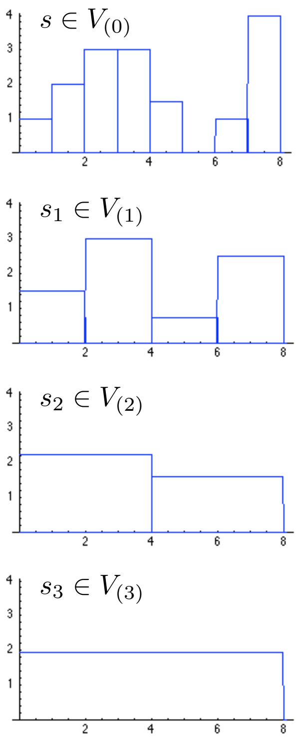
Hurst exponent: $H = \gamma - \frac{d}{2}$

(Tafti *et al.*, IEEE-IP 2009)

LAPLACIAN-LIKE WAVELET BASES

- Operator-like wavelet design
- Fractional Laplacian-like wavelet basis
- Improving shift-invariance and isotropy
- Wavelet analysis of fractal processes
  (multidimensional generalization of pioneering work of Flandrin and Abry)

Multiresolution analysis of $L_2(\mathrm{R}^d)$



- Multiresolution basis functions: $\varphi_{i,\boldsymbol{k}}(\boldsymbol{x}) = 2^{-id/2}\varphi\left(\frac{\boldsymbol{x}-2^i\boldsymbol{k}}{2^i}\right)$
- Subspace at resolution i: $V_{(i)} = \mathrm{span}\,\{\varphi_{i,\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$

Two-scale relation $\Rightarrow$ $V_{(i)} \subset V_{(j)}$, for $i \geq j$

Partition of unity $\Leftrightarrow$ $\overline{\bigcup_{i\in\mathbb{Z}} V_{(i)}} = L_2(\mathbb{R}^d)$

General operator-like wavelet design

Search for a **single wavelet** that generates a basis of $L_2(\mathbb{R}^d)$ and that is a multi-scale version of the operator L; i.e., $\psi = \mathrm{L}^*\phi$ where ϕ is a suitable smoothing kernel

- General operator-based construction
  - Basic space V_0 generated by the integer shifts of the Green function ρ of L:

    $$V_0 = \mathrm{span}\{\rho(\boldsymbol{x}-\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d} \text{ with } \mathrm{L}\rho = \delta$$

  - Orthogonality between V_0 and $W_0 = \mathrm{span}\{\psi(\boldsymbol{x}-\frac{1}{2}\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d\setminus 2\mathbb{Z}^d}$

    $$\begin{aligned}\langle\psi(\cdot-\boldsymbol{x}_0),\rho(\cdot-\boldsymbol{k})\rangle &= \langle\phi,\mathrm{L}\rho(\cdot-\boldsymbol{k}+\boldsymbol{x}_0)\rangle\\ &= \langle\phi,\delta(\cdot-\boldsymbol{k}+\boldsymbol{x}_0)\rangle = \phi(\boldsymbol{k}-\boldsymbol{x}_0) = 0\end{aligned}$$

    (can be enforced via a judicious choice of ϕ (interpolator) and $\boldsymbol{x}_0$)

  - Works in arbitrary dimensions and for any dilation matrix $\mathbf{D}$

Fractional Laplacian-like wavelet basis

$\psi_\gamma(\boldsymbol{x}) = (-\Delta)^{\frac{\gamma}{2}}\phi_{2\gamma}(\mathbf{D}\boldsymbol{x})$

$\phi_{2\gamma}(\boldsymbol{x})$: polyharmonic spline interpolator of order $2\gamma > 1$

$\mathbf{D}$: admissible dilation matrix

- Wavelet basis functions: $\psi_{(i,\boldsymbol{k})}(\boldsymbol{x}) \triangleq |\det(\mathbf{D})|^{i/2}\psi_\gamma\left(\mathbf{D}^i\boldsymbol{x}-\mathbf{D}^{-1}\boldsymbol{k}\right)$
- $\{\psi_{(i,\boldsymbol{k})}\}_{(i\in\mathbb{Z},\,\boldsymbol{k}\in\mathbb{Z}^2\setminus\mathbf{D}\mathbb{Z}^2)}$ forms a semi-orthogonal basis of $L_2(\mathbb{R}^2)$

  $$\forall f\in L_2(\mathbb{R}^2),\quad f = \sum_{i\in\mathbb{Z}}\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\mathbf{D}\mathbb{Z}^2}\langle f,\psi_{(i,\boldsymbol{k})}\rangle\,\tilde{\psi}_{(i,\boldsymbol{k})} = \sum_{i\in\mathbb{Z}}\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\mathbf{D}\mathbb{Z}^2}\langle f,\tilde{\psi}_{(i,\boldsymbol{k})}\rangle\,\psi_{(i,\boldsymbol{k})}$$

  where $\{\tilde{\psi}_{(i,\boldsymbol{k})}\}$ is the dual wavelet basis of $\{\psi_{(i,\boldsymbol{k})}\}$
- The wavelets $\psi_{(i,\boldsymbol{k})}$ and $\tilde{\psi}_{(i,\boldsymbol{k})}$ have $\lceil\gamma\rceil$ vanishing moments
- The wavelet analysis implements a multiscale version of the Laplace operator and is perfectly reversible (one-to-one transform)
- The wavelet transform has a fast filterbank algorithm (based on FFT)

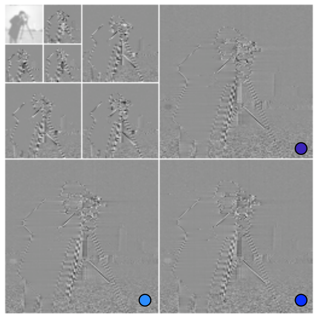
[Van De Ville, IEEE-IP, 2005]

Laplacian-like wavelet decomposition

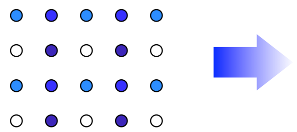
- Nonredundant transform

$f(\boldsymbol{x})$

$d_i[\boldsymbol{k}] = \langle f, \psi_{(i,\boldsymbol{k})} \rangle$



dyadic sampling pattern



$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

first decomposition level (one-to-one)

$\beta_\gamma(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$

scaling functions
(dilated by 2)

$\psi_\gamma(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$

wavelets
(dilated by 2)

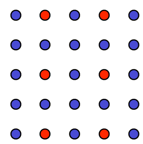
Improving shift-invariance and isotropy

Wavelet sampling patterns

- Wavelet subspace at resolution i

$$\mathcal{W}_i = \text{span}\,\{\psi_{i,\boldsymbol{k}}\}_{(\boldsymbol{k} \in \mathbb{Z}^2 \setminus \mathbf{D}\mathbb{Z}^2)}$$

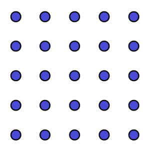
Non-redundant
(basis)



- Augmented wavelet subspace at resolution i

$$\mathcal{W}_i^+ = \text{span}\,\{\psi_{(i,\boldsymbol{k})}\}_{(\boldsymbol{k} \in \mathbb{Z}^2)} = \text{span}\{\psi_{\text{iso},(i,\boldsymbol{k})}\}_{(\boldsymbol{k} \in \mathbb{Z}^2)}$$

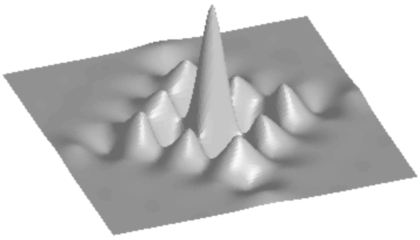
Mildly redundant
(frame)



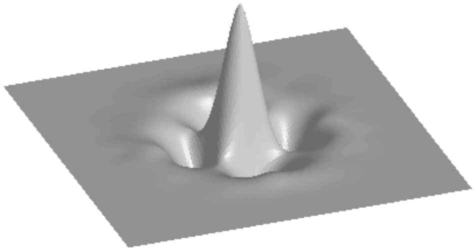
- Admissible polyharmonic spline wavelets
  - Operator-like generator: $\psi_\gamma(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\phi_{2\gamma}(\mathbf{D}\boldsymbol{x})$
  - More isotropic wavelet: $\psi_{\text{iso}}(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\beta_{2\gamma}(\mathbf{D}\boldsymbol{x})$
  - "Quasi-isotropic" polyharmonic B-spline [Van De Ville, 2005]

$$\beta_{2\gamma}(\boldsymbol{x}) \rightarrow C_\gamma \exp(-\|\boldsymbol{x}\|^2/(\gamma/6))$$

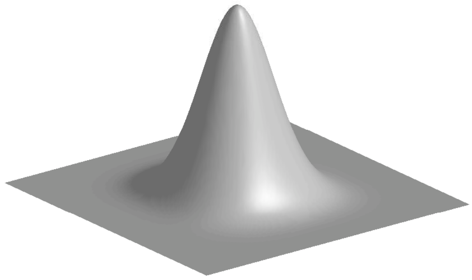
Building Mexican-Hat-like wavelets



$\psi_\gamma(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\phi_{2\gamma}(2\boldsymbol{x})$



$\psi_{\rm iso}(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\beta_{2\gamma}(2\boldsymbol{x})$



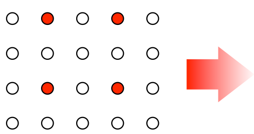
Gaussian-like smoothing kernel: $\beta_{2\gamma}(2\boldsymbol{x})$

Mexican-Hat multiresolution analysis

- Pyramid decomposition: redundancy $4/3$

$\psi_{\rm iso}(\boldsymbol{x}) = (-\Delta)^{\gamma/2}\beta_{2\gamma}(\mathbf{D}\boldsymbol{x})$

Dyadic sampling pattern



First decomposition level:

$\varphi(\mathbf{D}^{-1}\boldsymbol{x} - \boldsymbol{k})$ — Scaling functions

$\psi(\mathbf{D}^{-1}(\boldsymbol{x} - \boldsymbol{k}))$ — Wavelets (redundant by 4/3)

(U.-Van De Ville, *IEEE-IP* 2008)

Wavelet analysis of fBm: whitening revisited

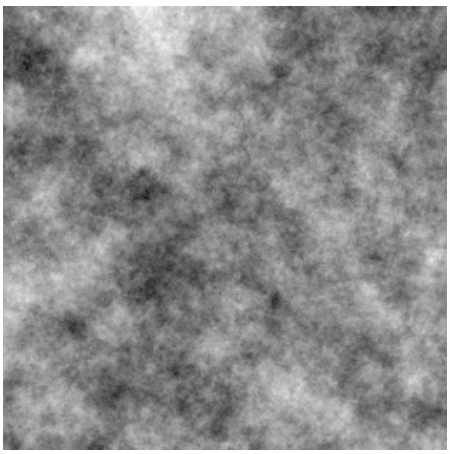
- Operator-like behavior of wavelet
  - Analysis wavelet: $\psi_\gamma = (-\Delta)^{\frac{\gamma}{2}}\phi(\boldsymbol{x}) = (-\Delta)^{\frac{H}{2}+\frac{d}{4}}\psi'_{\gamma'}(\boldsymbol{x})$
  - Reduced-order wavelet: $\psi'_{\gamma'}(\boldsymbol{x}) = (-\Delta)^{\frac{\gamma'}{2}}\phi(\boldsymbol{x})$ with $\gamma' = \gamma - (H + \frac{d}{2}) > 0$
- Stationarizing effect of wavelet analysis
  - Analysis of fractional Brownian field with exponent H:

    $$\langle B_H, \psi_\gamma\left(\tfrac{\cdot - \boldsymbol{x}_0}{a}\right)\rangle \propto \langle(-\Delta)^{\frac{H}{2}+\frac{d}{4}}B_H, \psi'_{\gamma'}\left(\tfrac{\cdot - \boldsymbol{x}_0}{a}\right)\rangle = \langle W, \psi'_{\gamma'}\left(\tfrac{\cdot - \boldsymbol{x}_0}{a}\right)\rangle$$
  - Equivalent spectral noise shaping: $S_{\mathrm{wave}}(e^{j\boldsymbol{\omega}}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |\hat{\psi}'_\gamma(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2$
    $\Rightarrow$ Extent of wavelet-domain whitening depends on flatness of $S_{\mathrm{wave}}(e^{j\boldsymbol{\omega}})$
  - "Whitening" effect is the same at all scales up to a proportionality factor
    $\Rightarrow$ fractal exponent can be deduced from the log-log plot of the variance

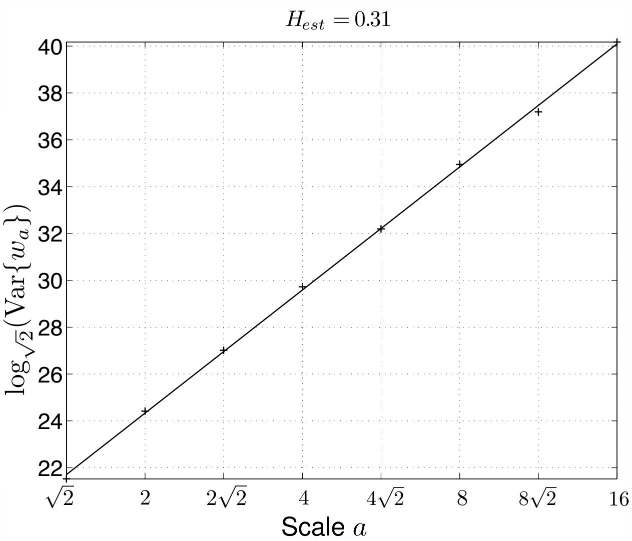
(Tafti *et al.*, *IEEE-IP* 2009)

Wavelet analysis of fractional Brownian fields

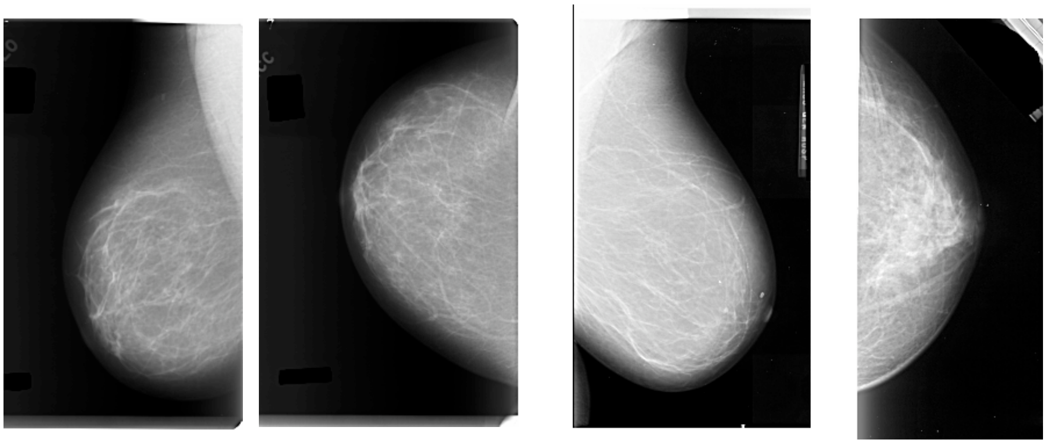
Theoretical scaling law : $\mathrm{Var}\{w_a[k]\} = \sigma_0^2 \cdot a^{(2H+d)}$

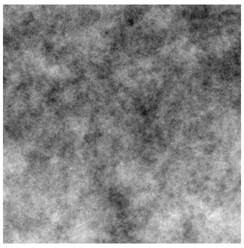


log-log plot of variance



Fractals in bioimaging: fibrous tissue



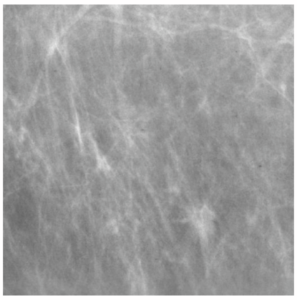


DDSM: University of Florida

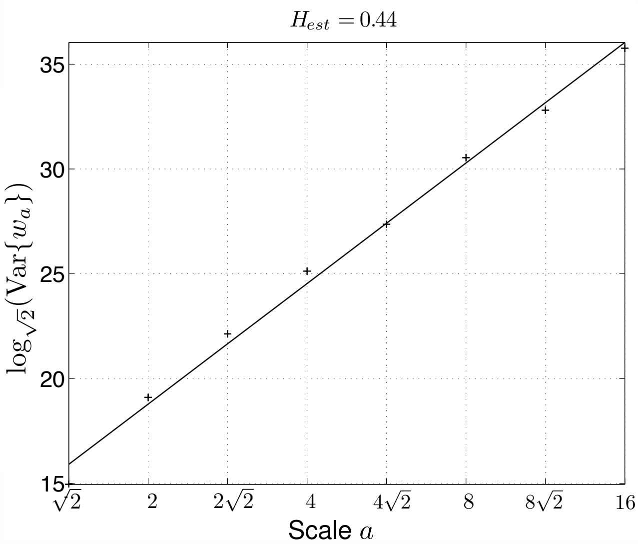
(Digital Database for Screening Mammography)

(Laine, 1993; Li et *al.*, 1997)

Wavelet analysis of mammograms



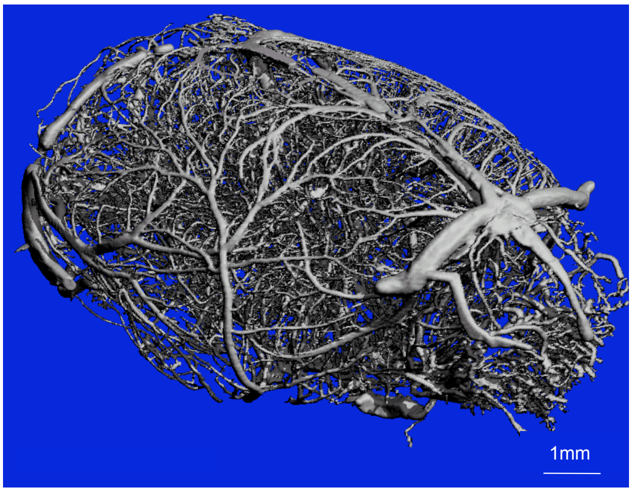
log-log plot of variance



Fractal dimension: $D = 1 + d - H = 2.56$ with $d = 2$ (topological dimension)

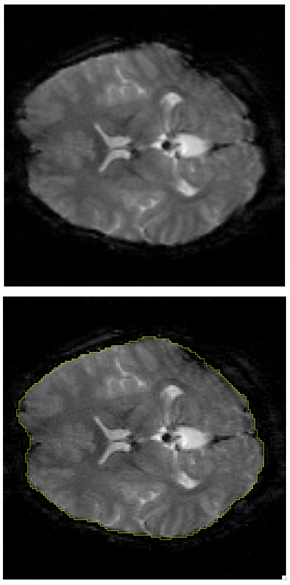
Brain as a biofractal

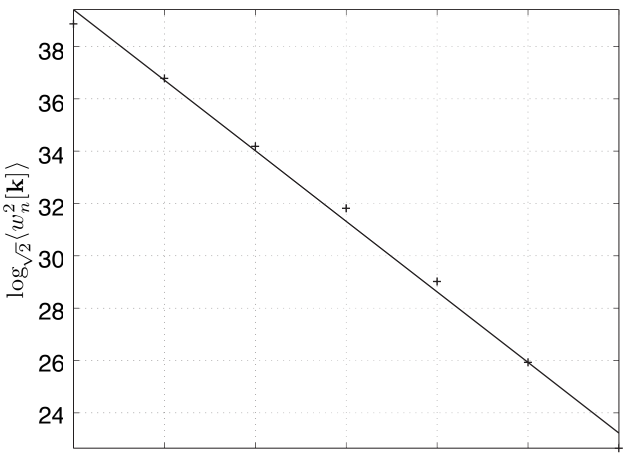
(Bullmore, 1994)



Courtesy R. Mueller ETHZ

Wavelet analysis of fMRI data

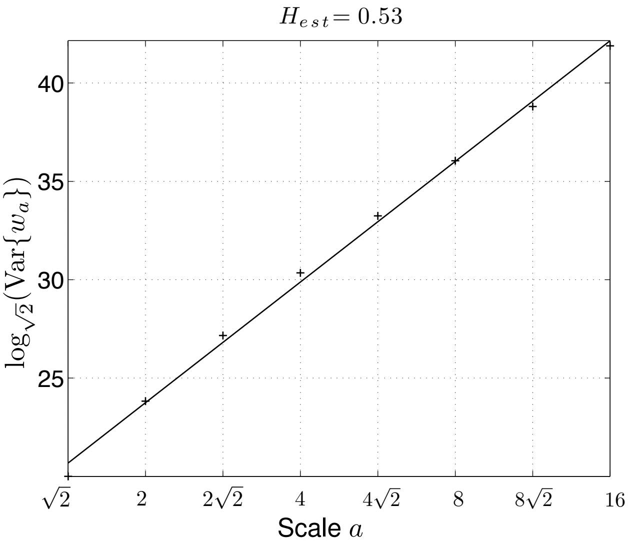




Brain: courtesy of Jan Kybic

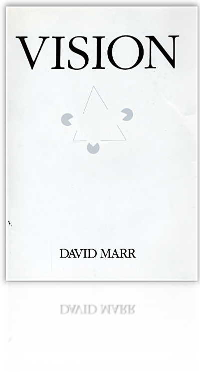
Fractal dimension: $D = 1 + d - H = 2.65$ with $d = 2$ (topological dimension)

...and some non-biomedical images...



THE MARR WAVELET

- Laplace/gradient operator
- Steerable Marr wavelets
- Wavelet primal sketch
- Directional wavelet analysis



Complex TRS-invariant operators in 2D

- **Invariance theorem**

The complete family of ***complex***, translation-, scale- and rotation-invariant 2D operators is given by the fractional complex Laplace-gradient operators

$$\mathrm{L}_{\gamma,N} = (-\Delta)^{\frac{\gamma-N}{2}} \left(\frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} \right)^N \overset{\mathcal{F}}{\longleftrightarrow} \|\boldsymbol{\omega}\|^{\gamma-N} (j\omega_1 - \omega_2)^N$$

with $N \in \mathbb{N}$ and $\gamma \geq N \in \mathbb{R}$

- Key property: steerability

$$\mathrm{L}_{\gamma,N}\{\delta\}(\mathbf{R}_\theta \boldsymbol{x}) = e^{jN\theta}\, \mathrm{L}_{\gamma,N}\{\delta\}(\boldsymbol{x})$$

Simplifying the maths: Unitary Riesz mapping

- Complex Laplace-gradient operator

$$\mathrm{L}_{\gamma,N} = (-\Delta)^{\frac{\gamma-N}{2}} \left(\frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} \right)^N = (-\Delta)^{\frac{\gamma}{2}} \mathcal{R}^N$$

where $\mathcal{R} = \mathrm{L}_{\frac{1}{2},1} \overset{\mathcal{F}}{\longleftrightarrow} \left(\frac{j\omega_1 - \omega_2}{\|\boldsymbol{\omega}\|} \right)$

- Property of Riesz operator $\mathcal{R}$
  - $\mathcal{R}$ is shift- and scale-invariant
  - $\mathcal{R}$ is rotation covariant (a.k.a. steerable)
  - $\mathcal{R}$ is unitary
    in particular, $\mathcal{R}$ will map a Laplace-like wavelet basis into a complex Marr-like wavelet basis

Complex Laplace-gradient wavelet basis

$$\psi'_\gamma(\boldsymbol{x}) = (-\Delta)^{\frac{\gamma-1}{2}} \left(\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2}\right)\phi_{2\gamma}(\mathbf{D}\boldsymbol{x}) = \mathcal{R}\psi_\gamma(\boldsymbol{x})$$

$\phi_{2\gamma}(\boldsymbol{x})$: polyharmonic spline interpolator of order $2\gamma > 1$

$\mathbf{D}$: admissible dilation matrix

- Wavelet basis functions: $\psi'_{(i,\boldsymbol{k})}(\boldsymbol{x}) = \mathcal{R}\psi_{(i,\boldsymbol{k})}(\boldsymbol{x}) = |\det(\mathbf{D})|^{i/2}\psi'_\gamma\left(\mathbf{D}^i\boldsymbol{x} - \mathbf{D}^{-1}\boldsymbol{k}\right)$
- $\{\psi'_{(i,\boldsymbol{k})}\}_{(i\in\mathbb{Z},\, \boldsymbol{k}\in\mathbb{Z}^2\backslash\mathbf{D}\mathbb{Z}^2)}$ forms a complex semi-orthogonal basis of $L_2(\mathbb{R}^2)$

$$\forall f \in L_2(\mathbb{R}^2), \quad f = \sum_{i\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^2\backslash\mathbf{D}\mathbb{Z}^2} \langle f, \psi'_{(i,\boldsymbol{k})}\rangle\, \tilde{\psi}'_{(i,\boldsymbol{k})} = \sum_{i\in\mathbb{Z}} \sum_{\boldsymbol{k}\in\mathbb{Z}^2\backslash\mathbf{D}\mathbb{Z}^2} \langle f, \tilde{\psi}'_{(i,\boldsymbol{k})}\rangle\, \psi'_{(i,\boldsymbol{k})}$$

where $\{\tilde{\psi}'_{(i,\boldsymbol{k})}\}$ is the dual wavelet basis of $\{\psi'_{(i,\boldsymbol{k})}\}$

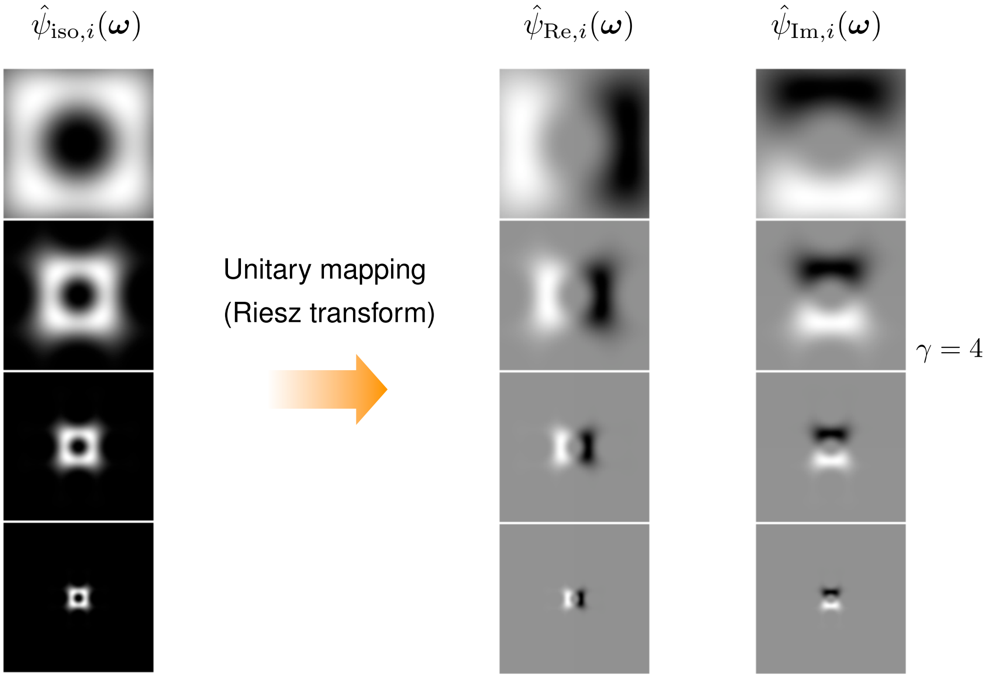
- The wavelet analysis implements a multiscale version of the Gradient-Laplace (or Marr) operator and is perfectly reversible (one-to-one transform)
- The wavelet transform has a fast filterbank algorithm

[Van De Ville-U., *IEEE-IP*, 2008]

Wavelet frequency responses

Laplacian-like / Mexican hat

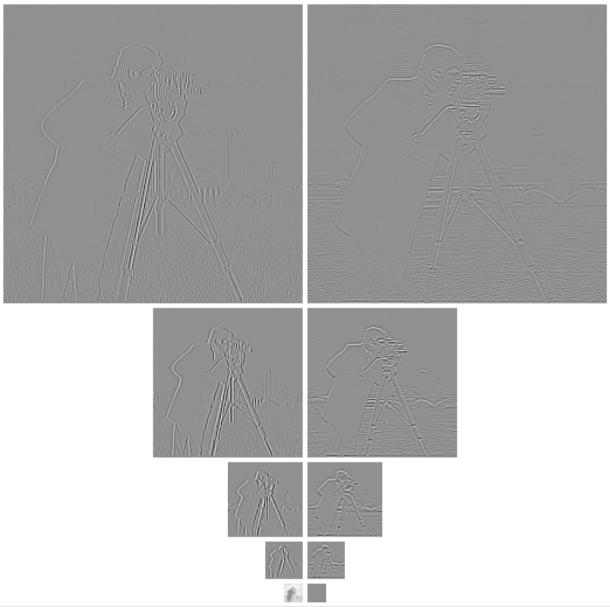
Marr pyramid (steerable)



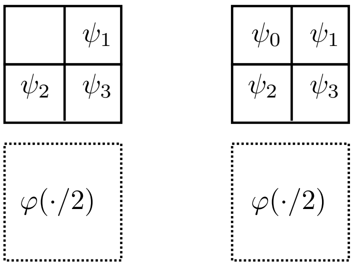
Marr wavelet pyramid

- Steerable pyramid-like decomposition: redundancy $2 \times \frac{4}{3}$

$$\psi_{\mathrm{Re}}(\boldsymbol{x}) = \frac{\partial}{\partial x}\Delta\beta_{2\gamma}(2\boldsymbol{x}) \qquad \psi_{\mathrm{Im}}(\boldsymbol{x}) = \frac{\partial}{\partial y}\Delta\beta_{2\gamma}(2\boldsymbol{x})$$



Basic dyadic sampling cell

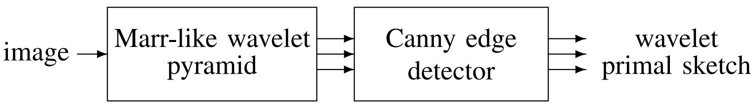


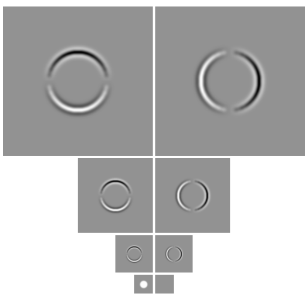
Wavelet basis

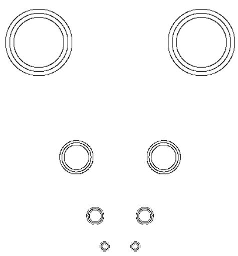
Overcomplete by 1/3

Processing in early vision - primal sketch

- Wavelet primal sketch [Van De Ville-U., IEEE-IP, 2008]
  - blurring — smoothing kernel ϕ
  - Laplacian filtering — Δ
  - zero-crossings and orientation — ∇
  - segment detection and grouping — Canny edge detection scheme

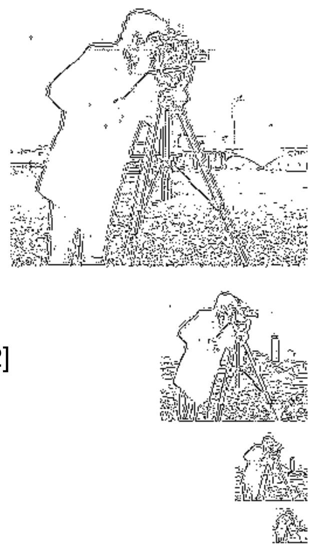






Edge detection in wavelet domain

- Edge map (using Canny's edge detector)
  - Key visual information (Marr's theory of vision)



Similar to Mallat's representation from
wavelet modulus maxima [Mallat-Zhong, 1992]

... but much less redundant !

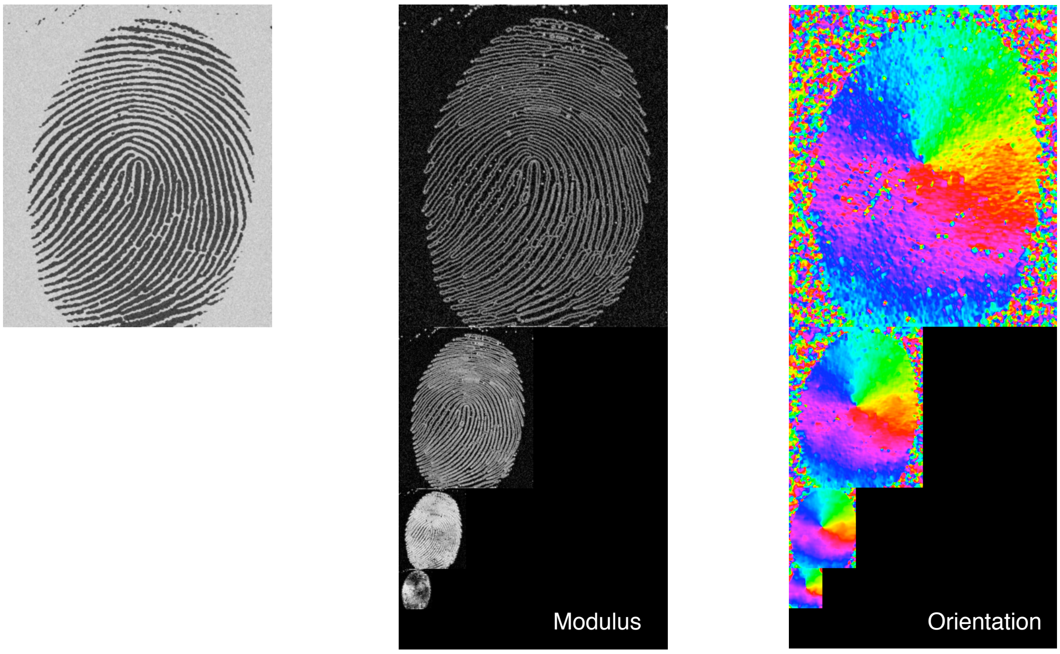
Iterative reconstruction

- Reconstruction from information on edge map only
  - Better than 30dB PSNR

31.4dB

[Van De Ville-U., IEEE-IP, 2008]

Directional wavelet analysis: Fingerprint



Wavelet-domain structure tensor

$\gamma = 2, \sigma = 2$

Marr wavelet pyramid - discussion

- Comparison against state-of-the-art

	Steerable pyramid	Complex dual-tree	Marr wavelet pyramid
Translation invariance	$++$	$++$	$++$
Steerability	$++$	$+$	$++$
Number of orientations	$2K$	6	2
Vanishing moments	no	yes, 1D	$\lceil\gamma\rceil$
Implementation	filterbank/FFT	filterbank	FFT
Decomposition type	tight frame	frame	complex frame
Redundancy	$8K/3+1$	4	$8/3$
Localization	slow decay	filterbank design	fast decay
Analytical formulas	no	no	yes
Primal sketch	-	-	yes
Gradient/structure tensor	-	-	yes

[Simoncelli, Freeman, 1995] [Kingsbury, 2001]
[Selesnick et al, 2005]

CONCLUSION

- Unifying operator-based paradigm
  - ***Operator*** identification based on ***invariance*** principles (TSR)
  - Specification of corresponding ***spline and wavelet*** families
  - Characterization of ***stochastic processes*** (fractals)
- ***Isotropic*** and ***steerable*** wavelet transforms
  - Riesz basis, analytical formulaes
  - Mildly redundant frame extension for improved TR invariance
  - Fractal and/or directional analyses
  - Fast filterbank algorithm (fully reversible)
- Marr wavelet pyramid
  - Multiresolution Marr-type analysis; wavelet primal sketch
  - Reconstruction from multiscale edge map
- Implementation will be available very soon (Matlab)

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- P.D. Tafti, D. Van De Ville, M. Unser, "Invariances, Laplacian-Like Wavelet Bases, and the Whitening of Fractal Processes," *IEEE Trans. Image Processing*, vol. 18, no. 4, pp. 689-702, April 2009.

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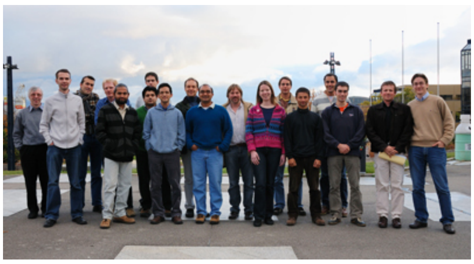

- Preprints and demos: http://bigwww.epfl.ch/