ÉCOLE POLYTECHNIQUE
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Beyond the digital divide: Ten good reasons for using splines

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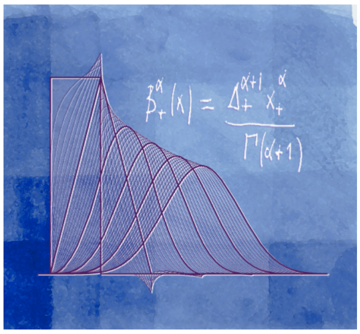
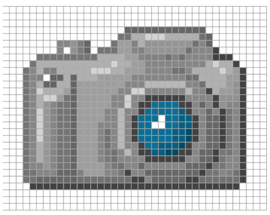


Seminars of Numerical Analysis, EPFL, May 9, 2010

The digital divide

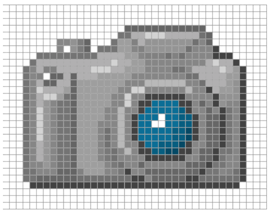
Is continuous-domain signal processing dead ?

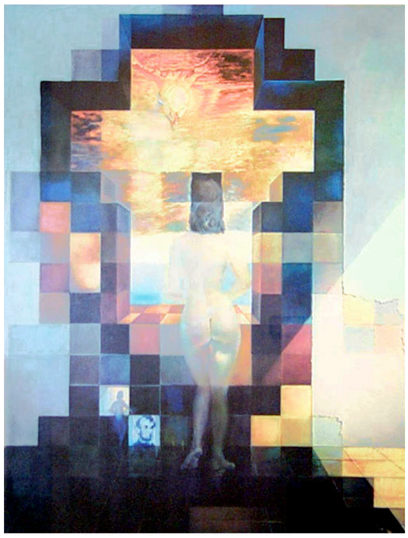


- Arguments in favor of its suppression:
  - The modern world is discrete and ruled by computers
  - Modern SP courses concentrate on digital signal processing
  - Most processing is discrete
  - Students don't like the Laplace transform...

The digital divide (Cont'd)

Are continuous mathematics obsolete ?

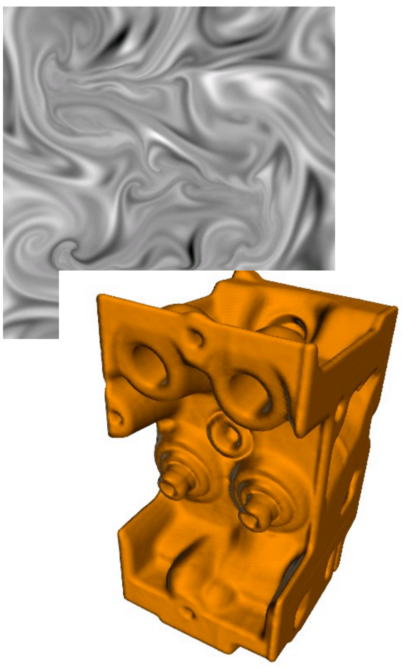


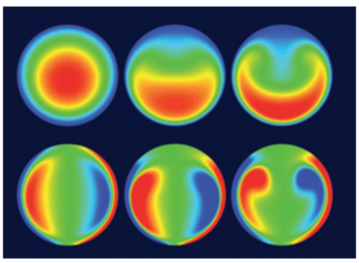


- However…
  - Most real-world objects, phenomena or signals are continuous
  - Often, the end product/goal is analog
  - Don't forget the interface: A-to-D and D-to-A
  - Many discrete algorithms require "continuous" thinking

Part of scientific computing is ...

Think analog, act digital !





OUTLINE

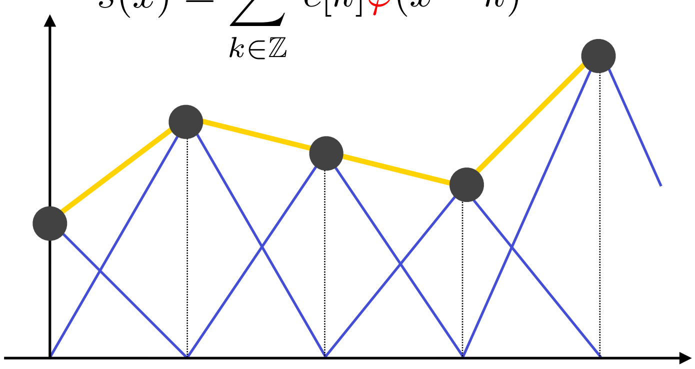
- Introduction
- Cardinal-spline formalism
- Ten+ good reasons for using B-splines
  - Computational
  - Theoretical
  - Conceptual
  - Practical
- Application examples in image processing

Basic interpolation problem

Find an interpolating function $s(x), x \in \mathbb{R}$ such that

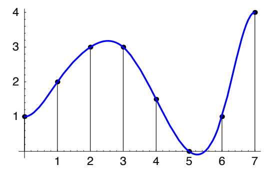
- $s(k) = f[k], \quad k \in \mathbb{Z}$
- $s(x)$ is piecewise-polynomial, continuous, ...

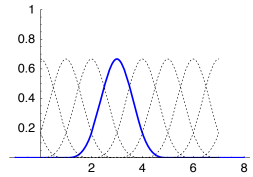
$$s(x) = \sum_{k \in \mathbb{Z}} c[k]\varphi(x-k)$$

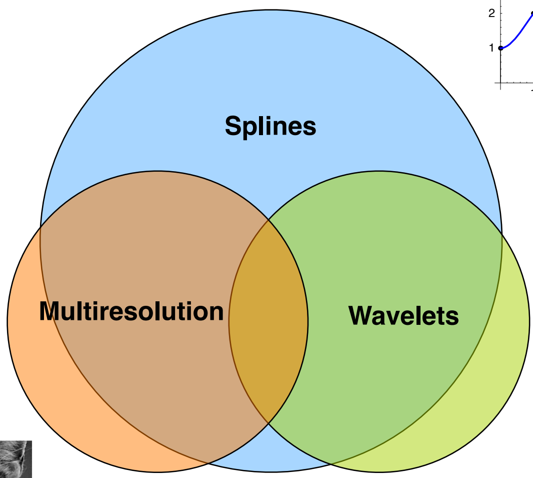


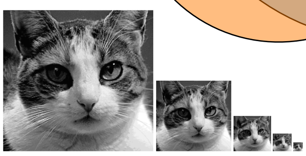
Splines: a unifying framework

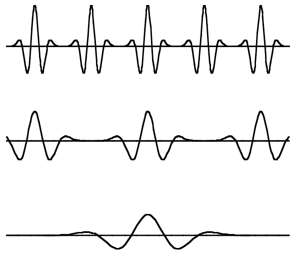
Linking the discrete and the continuous …..



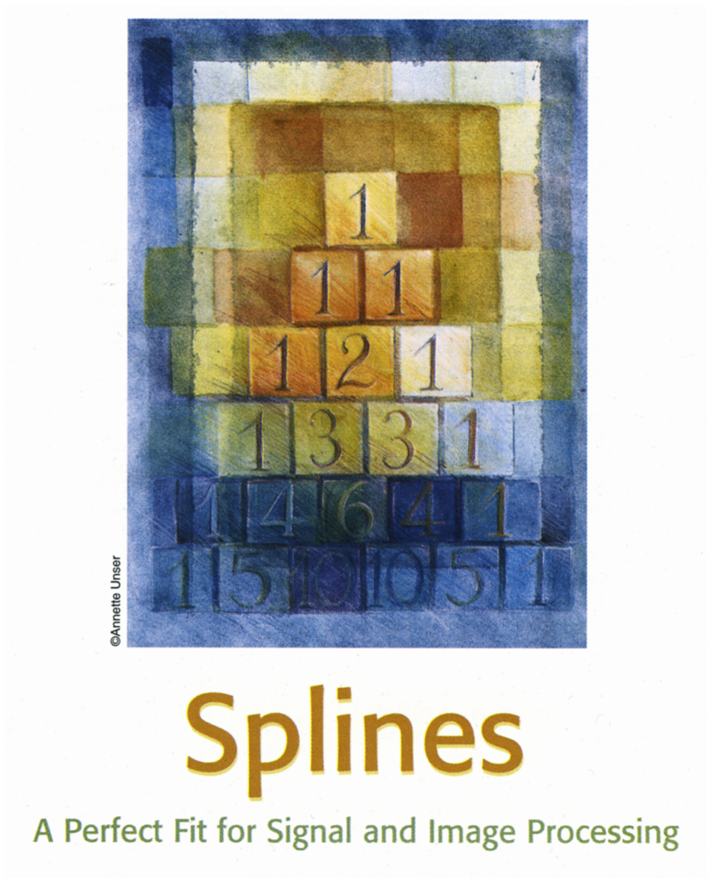








IEEE Signal Processing Magazine



November 1999

CARDINAL SPLINE FORMALISM

- Distributional definition: L-splines
- Basic atoms
- Polynomial B-splines

General concept of an *L*-spline

$\mathrm{L}\{\cdot\}$: differential operator (translation-invariant)
$\delta(\boldsymbol{x}) = \prod_{i=1}^{d} \delta(x_i)$: multidimensional Dirac distribution

Definition
The continuous-domain function $s(\boldsymbol{x})$ is a ***cardinal L-spline*** iff.

$$\mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

- Cardinality: the knots (or spline singularities) are on the (multi-)integers
  $\Rightarrow$ ideal framework for signal processing
- Generalization: includes polynomial splines as particular case $(\mathrm{L} = \frac{\mathrm{d}^N}{\mathrm{d}x^N})$

Example: piecewise-constant splines

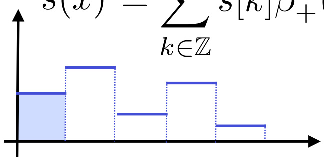
- Spline-defining operators

Continuous-domain derivative: $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \longleftrightarrow \quad j\omega$

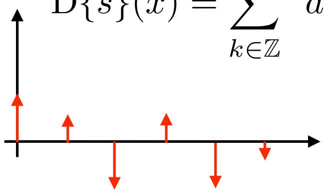
Discrete derivative: $\Delta_+\{\cdot\} \quad \longleftrightarrow \quad 1 - e^{-j\omega}$

- Piecewise-constant or D-spline

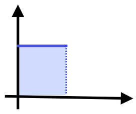
$$s(x) = \sum_{k \in \mathbb{Z}} s[k]\beta_+^0(x-k)$$



$$\mathrm{D}\{s\}(x) = \sum_{k \in \mathbb{Z}} \overbrace{a[k]}^{\Delta_+ s(k)}\, \delta(x-k)$$



- B-spline function



$$\beta_+^0(x) = \Delta_+ \mathrm{D}^{-1}\{\delta\}(x) \quad \longleftrightarrow \quad \boxed{\frac{1-e^{-j\omega}}{j\omega}}$$

Existence of a local, shift-invariant basis?

- Space of cardinal L-splines

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) : \mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \right\} \cap L_2(\mathbb{R}^d)$$

- Generalized B-spline representation

A “localized” function $\varphi(\boldsymbol{x}) \in V_{\mathrm{L}}$ is called *generalized B-spline* if it generates a Riesz basis of V_{L}; i.e., iff. there exists $(A > 0, B < \infty)$ s.t.

$$A \cdot \|c\|_{\ell_2(\mathbb{Z}^d)} \le \left\| \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \right\|_{L_2(\mathbb{R}^d)} \le B \cdot \|c\|_{\ell_2(\mathbb{Z}^d)}$$

$$\Downarrow$$

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) : \boldsymbol{x} \in \mathbb{R}^d, c \in \ell_2(\mathbb{Z}^d) \right\}$$

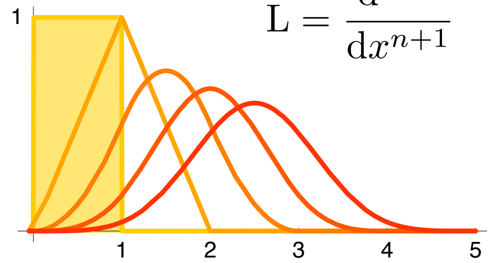
continuous-domain signal

discrete signal
(B-spline coefficients)

Polynomial B-splines

- B-spline of degree n

$$\beta_+^n(x) = \underbrace{\beta_+^0 * \beta_+^0 * \cdots * \beta_+^0}_{(n+1)\text{ times}}(x)$$



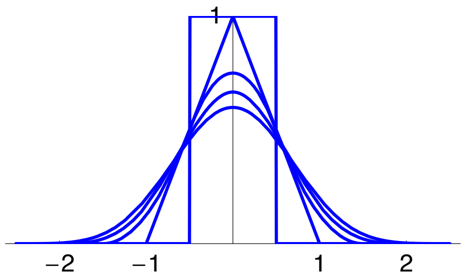
$$\mathrm{L} = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}$$

$$\beta_+^0(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise.} \end{cases}$$

- Key properties
  - Riesz basis generator for the cardinal polynomial splines
  - Shortest polynomial spline of degree n

- Symmetric B-spline

$$\beta^n(x) = \beta_+^n\left(x + \tfrac{n+1}{2}\right)$$

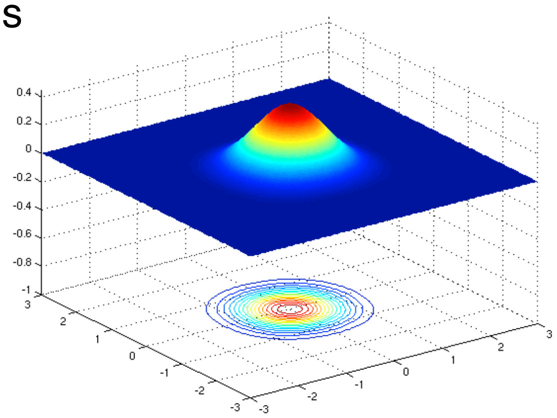


B-spline representation of images

- Symmetric, tensor-product B-splines

$$\beta^n(x_1, \cdots, x_d) = \beta^n(x_1) \times \cdots \times \beta^n(x_d)$$

$$\mathrm{L} = \frac{\partial^{dn+d}}{\partial x_1^{n+1} \cdots \partial x_d^{n+1}}$$



- Multidimensional spline function

$$s(x_1, \cdots, x_d) = \sum_{(k_1, \cdots k_d) \in \mathbb{Z}^d} c[k_1, \cdots, k_d]\, \beta^n(x_1 - k_1, \cdots, x_d - k_d)$$

continuous-space image

image array
(B-spline coefficients)

Compactly supported
basis functions

TEN REASONS FOR USING SPLINES

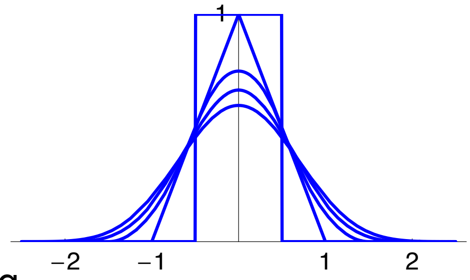
- Mathematical elegance
- Fast algorithms
- Approximation theory
- Link with *<your favorite>* theory

$$\vdots$$

1. B(eautiful) basis functions

- Polynomial B-splines (centered)

$$\beta^n(x) = \frac{\Delta^{n+1} x_+^n}{(n+1)!} = (\beta^0 * \beta^{n-1})(x)$$



- Attractive properties for image processing
  - Compact support: shortest polynomial spline of degree n
  - Symmetry
  - Positivity
  - Controlled smoothness: Hölder-continuous of order n
  - Bell-shaped (optimal space-frequency localization)

$$\beta^n(x) \rightarrow \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(\frac{-x^2}{2\sigma_n^2}\right) \quad \text{with } \sigma_n = \sqrt{\frac{n+1}{12}}$$

Reference: (Schoenberg, 1946)

2. Fast digital-filtering algorithms

All classical spline interpolation and approximation problems can be solved efficiently using recursive digital filtering

- Interpolation problem

  Given the signal samples $f[\boldsymbol{k}]$, find the B-spline coefficients $c[\boldsymbol{k}]$ such that

$$f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}} = f[\boldsymbol{k}] = \sum_{\boldsymbol{k}_1 \in \mathbb{Z}^p} c[\boldsymbol{k}_1]\varphi(\boldsymbol{k}-\boldsymbol{k}_1)$$

$\Rightarrow$ Inverse filtering solution

$c[\boldsymbol{k}] = (h_{\text{int}} * f)[\boldsymbol{k}]$ with $H_{\text{int}}(\mathbf{z}) = \dfrac{1}{B(\mathbf{z})} = \dfrac{1}{\sum_{\boldsymbol{k}\in\mathbb{Z}^p} \varphi(\boldsymbol{k})\mathbf{z}^{-\boldsymbol{k}}}$

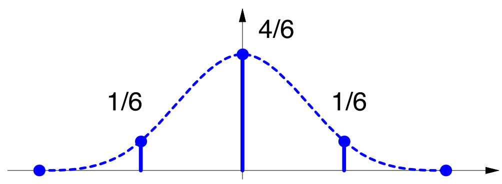
Note: $\varphi(\boldsymbol{x})$ separable $\Rightarrow$ $h_{\text{int}}[\boldsymbol{k}]$ separable

Reference: B-spline signal processing (Unser, *IEEE-SP* 1993)

Example: cubic-spline interpolation

- Cubic B-spline

$$\varphi(x) = \beta^3(x) = \begin{cases} \frac{2}{3} - \frac{1}{2}|x|^2(2-|x|), & 0 \le |x| < 1 \\ \frac{1}{6}(2-|x|)^3, & 1 \le |x| < 2 \\ 0, & \text{otherwise} \end{cases}$$



- Discrete B-spline kernel: $B(z) = \dfrac{z+4+z^{-1}}{6}$

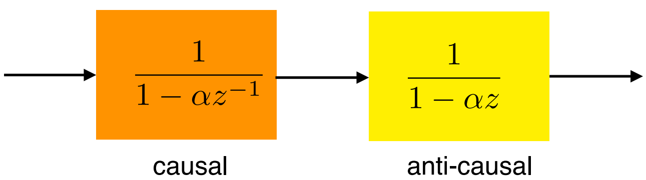
- Interpolation filter

$$\frac{6}{z+4+z^{-1}} = \frac{(1-\alpha)^2}{(1-\alpha z)(1-\alpha z^{-1})} \quad \stackrel{z}{\longleftrightarrow} \quad h_{\text{int}}[k] = \left(\frac{1-\alpha}{1+\alpha}\right)\alpha^{|k|}$$

(symmetric exponential)

$$\alpha = -2+\sqrt{3} = -0.171573$$

$\Rightarrow$ Cascade of first-order recursive filters



3. Simple manipulations

The polynomial spline family is closed with respect to differentiation

- Derivative operator

$$\mathrm{D}f(x) = \frac{df(x)}{dx}$$

- Finite-difference operator (centered)

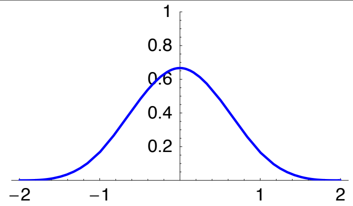
$$\Delta f(x) \triangleq f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2})$$

- Derivative of a B-spline (exact)

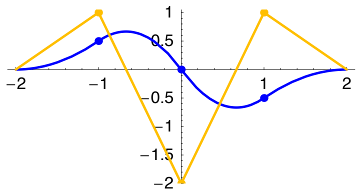
$$\mathrm{D}^m \beta^n(x) = \Delta^m \beta^{n-m}(x)$$

Discrete operator

Reduction of degree



Example: cubic B-spline



Reference: (Schoenberg, 1946)

4. Link with system theory: C-to-D converters

Exponential B-splines = the mathematical translators between continuous-time and discrete-time LSI system theories

Continuous domain
- differential equations
- circuits, analog filters
- Laplace transform:

Discrete domain
- difference equations
- digital filters
- z-transform:

zeros

$$H_{\mathrm{C}}(s) = \frac{\prod_{m=1}^{M}(s-\gamma_m)}{\prod_{n=1}^{N}(s-\alpha_n)}$$

$$H_{\mathrm{D}}(z) = \frac{1}{\prod_{n=1}^{N}(z-z_n)}$$

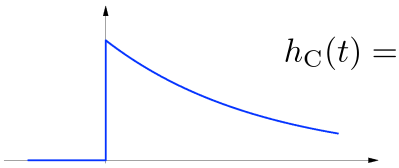
poles mapping: $z_n = e^{\alpha_n}$

Associated B-spline: $\beta_{\vec{\alpha}}(t) = \mathcal{L}^{-1}\left\{\frac{H_{\mathrm{C}}(s)}{H_{\mathrm{D}}(e^s)}\right\}(t)$

Reference: "Think analog, act digital" (Unser, *IEEE-SP* 2006)

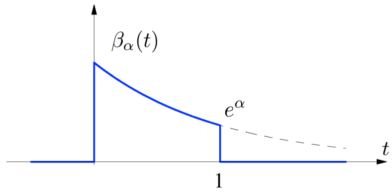
Example: 1st order system

- Continuous-time impulse response



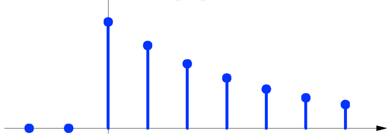
$$h_C(t) = 1_+(t)\cdot e^{\alpha t} = \begin{cases} e^{\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad H_C(s) = \frac{1}{s-\alpha}$$

1st-order exponential B-spline



- Discrete-time counterpart

$$h_D[k] = h_C(k) \quad \stackrel{z}{\longleftrightarrow} \quad H_D(z) = \frac{1}{z - e^{\alpha}}$$



Discrete-time signal

$$h_C(t) = 1_+(t)\cdot e^{\alpha t} = \sum_{k=0}^{+\infty} e^{\alpha k}\beta_\alpha(t-k) = \sum_{k\in\mathbb{Z}} h_D[k]\,\beta_\alpha(t-k)$$

Continuous-time signal

Compactly-supported basis functions

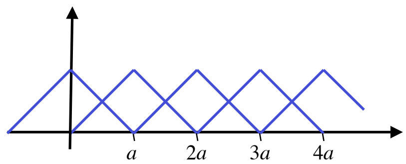
5. Best cost-performance tradeoff

Polynomial B-splines have
- maximum order of approximation for a minimum support (MOMS)
- a low asymptotic approximation constant.

This explains their superior performance in imaging applications.

- Approximation of a function at scale a

$$V_a(\varphi) = \left\{ s_a(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi\left(\frac{x}{a} - k\right) : c \in \ell_2 \right\}$$



Definition: A generating function φ has order of approximation γ iff.

$$\forall f \in W_2^\gamma, \qquad \arg\min_{s_a\in V_a} \|f - s_a\|_{L_2} \leq C_\gamma \cdot a^\gamma \cdot \|f^{(\gamma)}\|_{L_2}$$

- $\beta^n(x)$ has order of approximation $\gamma = n+1$ and $C_{\gamma,\min} = \frac{\sqrt{2\zeta(2\gamma)}}{(2\pi)^\gamma}$

Reference: (Strang-Fix, 1973; Blu-U., *IEEE-SP* 1999)

Interpolation benchmark

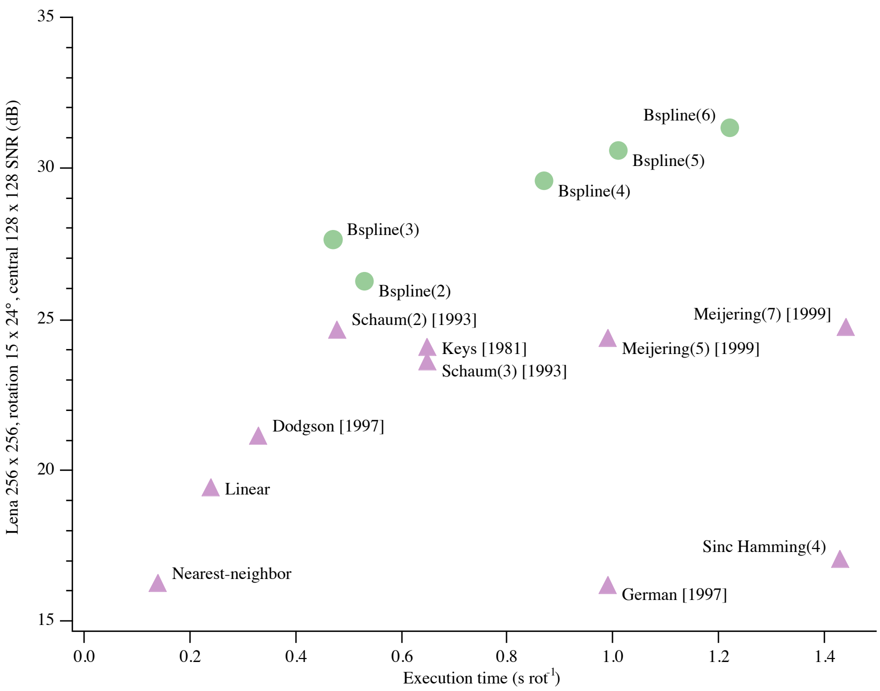
Cumulative rotation experiment: the best algorithm wins !

Bilinear

Windowed-sinc

Cubic spline

High-quality image interpolation



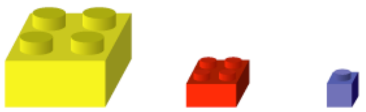
Demo

Thévenaz et al., *Handbook of Medical Image Processing*, 2000

6. Link with wavelet theory

Polynomial B-splines have remarkable dilation properties.
They play a fundamental role in wavelet theory.

- Generalized Lego™/Duplo™ relation



$$\beta_+^0(x/2) = \beta_+^0(x) + \beta_+^0(x-1)$$

B-spline dilation property: $\beta_+^n(x/2) = \sum_{k\in\mathbb{Z}} h_2^n[k]\beta_+^n(x-k)$

Binomial filter: $H_2^n(z) = \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k} z^{-k} = \frac{1}{2^n}\left(1+z^{-1}\right)^{n+1}$

B-spline factorization theorem

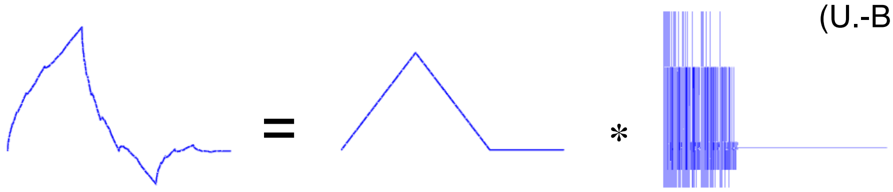
Theorem: A valid scaling function $\varphi(x)$ has order of approximation γ iff.

$$\varphi(x) = \left(\beta_+^\alpha * \varphi_0\right)(x)$$

where β_+^α with $\alpha = \gamma - 1$: regular, B-spline part

$\varphi_0 \in S'$: irregular, distributional part

(U.-Blu, *IEEE-SP*, 2003)



- Refinement filter: general case

$$H(z) = \underbrace{\left(\frac{1+z^{-1}}{2}\right)^\gamma}_{\text{spline part}} \cdot \underbrace{Q(z)}_{\text{distributional part}} \qquad \text{with} \quad |Q(e^{j\omega})| < +\infty$$

Splines: the key to wavelet theory

Sobolev smoothness
$\partial^s \varphi \in L_2$

$\Downarrow \; \gamma \geq s$

B-spline factorization:
$\varphi = \beta_+^{\gamma-1} * \varphi_0$

$\Leftrightarrow$

Approximation order:
$\|f - P_a f\|_{L_2} = O(a^\gamma)$

$\Leftrightarrow$

Multi-scale differentiator
$\hat{\tilde{\psi}}(\omega) \propto (-j\omega)^\gamma, \; \omega \to 0$

general case: $N < \gamma \leq N+1$

compact support: $\gamma = N+1$ (Strang-Fix)

$\Downarrow \quad \Uparrow$

Polynomial reproduction
degree: $N = \lceil \gamma - 1 \rceil$

$\Updownarrow$

Vanishing moments:
$$\int x^n \tilde{\psi}(x) \mathrm{d}x = 0, \; n = 0, \cdots, N$$

Reference: Wavelet theory demystified (Unser-Blu, *IEEE-SP*, 2003)

7. Link with regularization theory

Spline estimators are optimal from a variational point of view.

- Smoothing-spline estimator

Discrete, noisy input:
$f[k] = s_{\mathrm{ref}}(k) +$ noise $\longrightarrow$ Smoothing algorithm $\longrightarrow$

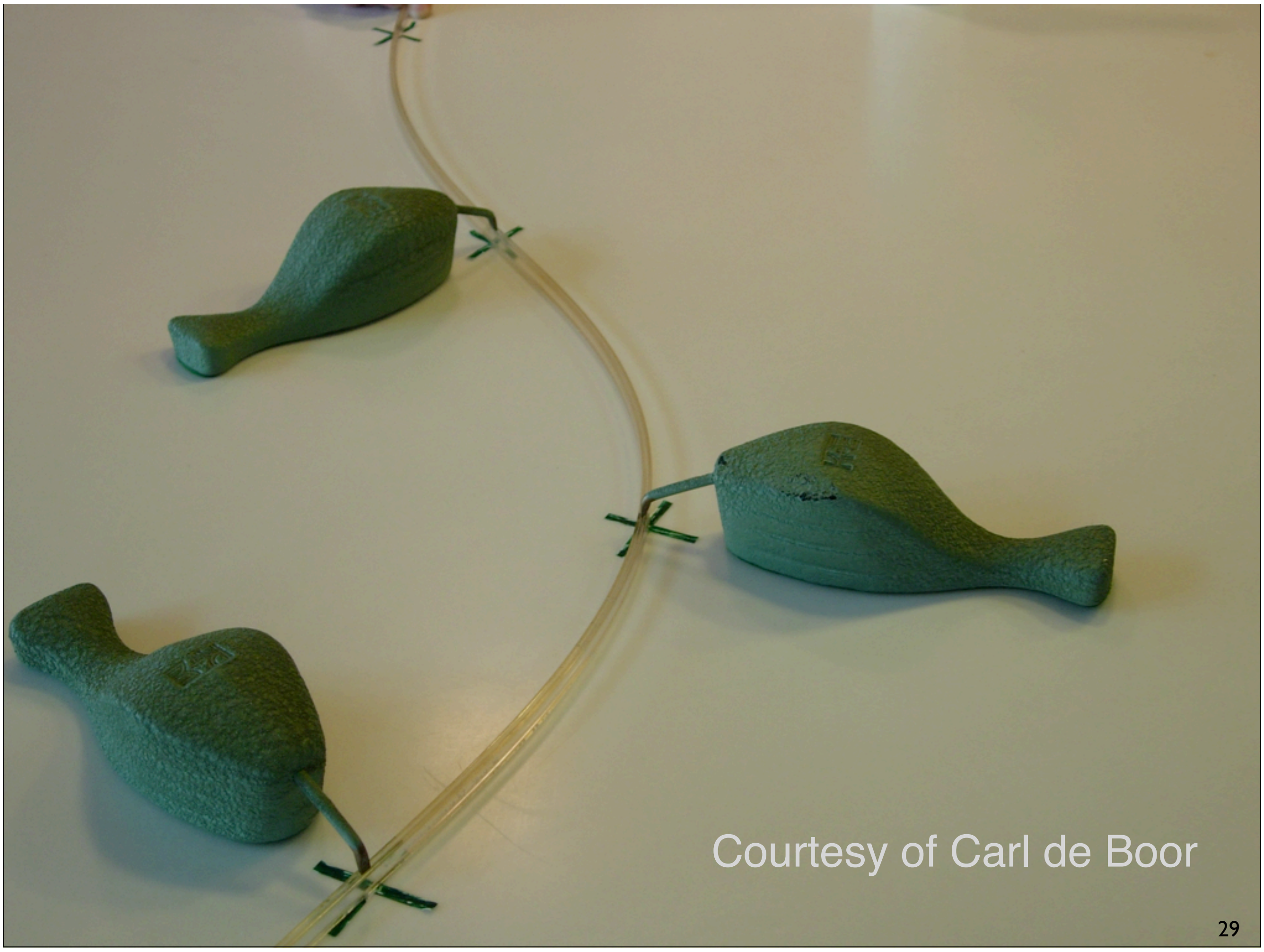
Continuous-domain estimate:
$$s(x) = \sum_{k \in \mathbb{Z}} c[k] \beta^n(x-k)$$

Theorem: The solution (among all functions) of the smoothing-spline problem

$$\min_{s(x)} \left\{ \sum_{k \in \mathbb{Z}} |f[k] - s(k)|^2 + \lambda \int_{-\infty}^{+\infty} |\mathrm{D}^m s(x)|^2 \mathrm{d}x \right\}$$

is a cardinal spline of degree $2m-1$. In addition, its coefficients $c[k] = h_\lambda * f[k]$ can be obtained by suitable digital filtering of the input samples $f[k]$.

References: theory (Schoenberg, 1964), recursive filtering algorithm (Unser, 1993)



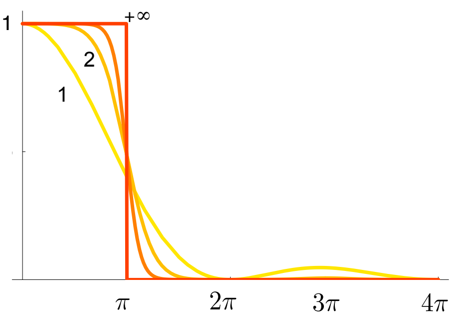
8. Link with Shannon's sampling theory

The Hilbert-space formulation of polynomial spline approximation provides an extension of Shannon's classical sampling theorem.

- Polynomial spline interpolator

Impulse response | Frequency response

$$\varphi^n_{\mathrm{int}}(x) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}^n_{\mathrm{int}}(\omega) = \underbrace{\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1}}_{\hat{\beta}^n(\omega)} H^n_{\mathrm{int}}(e^{j\omega})$$



- Asymptotic property

The cardinal-spline interpolators converge to the sinc interpolator (ideal filter) as the degree goes to infinity:

$$\lim_{n\to\infty} \varphi^n_{\mathrm{int}}(x) = \mathrm{sinc}(x), \qquad \lim_{n\to\infty} \hat{\varphi}^n_{\mathrm{int}}(\omega) = \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

(in all L_p-norms)

References: (Schoenberg, 1973; Unser, *Proc. IEEE*, 2000)

Splines and stochastic processes

Splines are in direct correspondence with stochastic processes (stationary or fractals) that are solution of the same partial differential equation, but with a random driving term.

Defining operator equation: $\mathrm{L}\{s(\cdot)\}(\boldsymbol{x}) = r(\boldsymbol{x})$

- Specific driving terms
  - $r(\boldsymbol{x}) = \delta(x) \quad \Rightarrow \quad s(\boldsymbol{x}) = \mathrm{L}^{-1}\{\delta\}(\boldsymbol{x})$: Green function
  - $r(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \quad \Rightarrow \quad s(\boldsymbol{x})$: Cardinal L-spline
  - $r(\boldsymbol{x})$: white noise $\quad \Rightarrow \quad s(\boldsymbol{x})$: generalized stochastic process

non-empty null space of L, boundary conditions

References: stationary proc. (Unser, *IEEE-SP* 2006), fractals (Blu, *IEEE-SP* 2007)

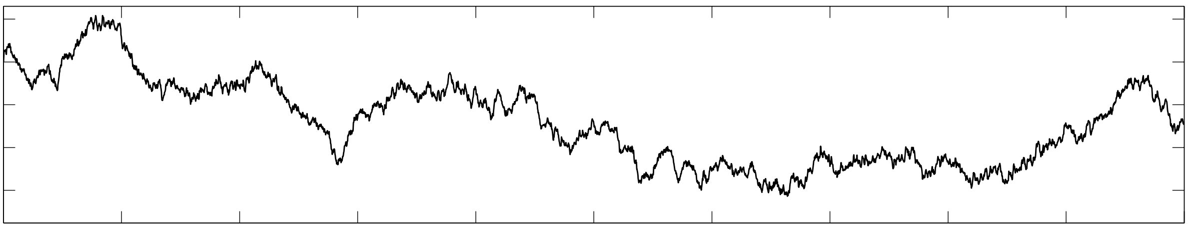
Example: Brownian motion synthesis

$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \Rightarrow \quad \mathrm{L}^{-1}$: integrator

$r(x) = w(x) \longrightarrow \boxed{\mathrm{L}^{-1}\{\cdot\}} \longrightarrow s(x)$

white Gaussian noise

Brownian motion



(Wiener, 1926)

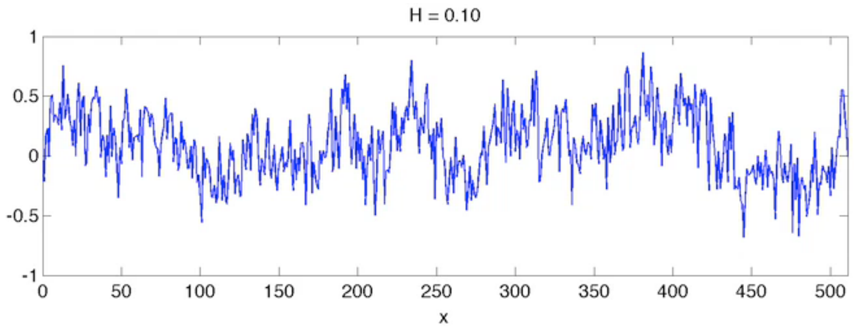
Example: going fractional (fBm)

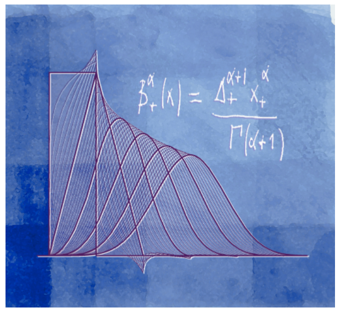
$$\mathrm{L} \quad \xleftrightarrow{\mathcal{F}} \quad (j\omega)^{H+\frac{1}{2}} \quad \Rightarrow \quad \mathrm{L}^{-1}\text{: fractional integrator}$$

$$r(x) = w(x) \longrightarrow \boxed{\mathrm{L}^{-1}\{\cdot\}} \longrightarrow s(x)$$

white Gaussian noise

fractional Brownian motion





fractional B-splines (2000)

(Mandelbrot, 1968)

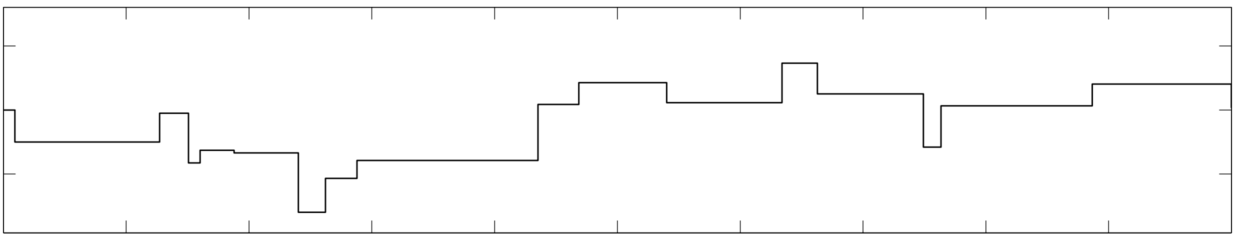
Example: Compound Poisson process (sparse)

$$\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \Rightarrow \quad \mathrm{L}^{-1}\text{: integrator}$$

$$r(x) = \sum_k a_k \delta(x - x_k) \longrightarrow \boxed{\mathrm{L}^{-1}\{\cdot\}} \longrightarrow s(x)$$

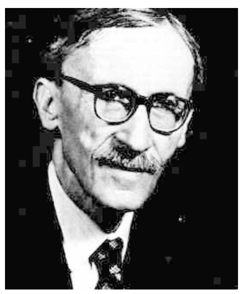
random stream of Diracs

Compound Poisson process



Random jumps with rate λ (Poisson point process)

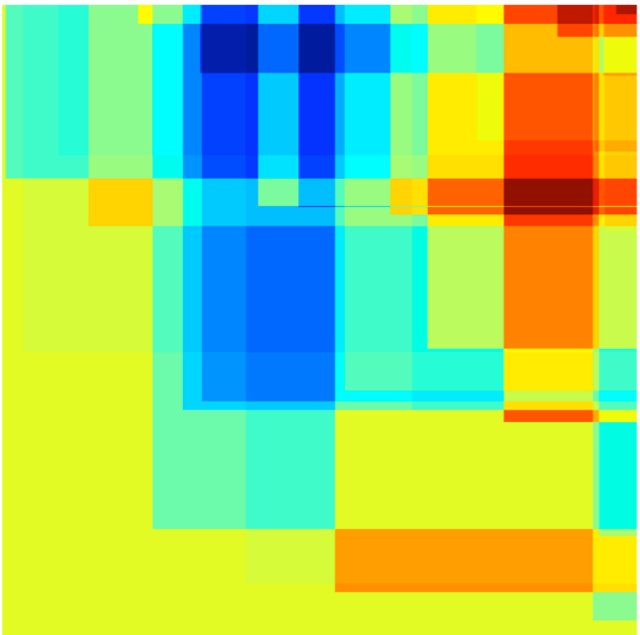
Jump size distribution: $a \backsim \mathrm{d}P(a)$



(Paul Lévy, 1934)

2D generalization: the Mondrian process

$$\mathrm{L} = \mathrm{D}_x \mathrm{D}_y \quad \xleftrightarrow{\mathcal{F}} \quad (j\omega_x)(j\omega_y)$$



$\lambda = 30$

10. Link with estimation theory

Smoothing splines are minimum-mean-square-error estimators (e.g., hybrid Wiener filters) for a corresponding class of stochastic processes (stationary and fractal)

- Measurement model: $f[\boldsymbol{k}] = s(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}} + n[\boldsymbol{k}]$
- $s(\boldsymbol{x})$: realization of a Gaussian stationary or fractal (fBm) process s.t.

  $E\left[\mathrm{L}s(\boldsymbol{x}_1) \cdot \mathrm{L}s(\boldsymbol{x}_2)\right] = \sigma_0^2\, \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ (whitening operator L)
- $n[\boldsymbol{k}]$: white Gaussian noise with variance σ^2
- MMSE spline estimator of signal $s(\boldsymbol{x})$:

$$E\left[s(\boldsymbol{x})|f\right] = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (h_\lambda * f)[\boldsymbol{k}]\, \varphi_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x} - \boldsymbol{k})$$

$\varphi_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x})$: $\mathrm{L}^*\mathrm{L}$-spline generator
$h_\lambda[\boldsymbol{k}]$: smoothing spline filter
$\lambda = \sigma^2/\sigma_0^2$: regularization factor

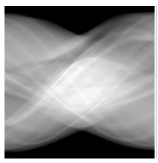
References: stationary proc. (Unser, *IEEE-SP* 2006), fBm (Blu, *IEEE-SP* 2007)

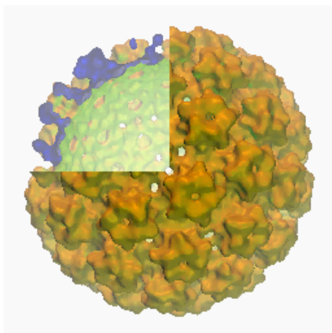
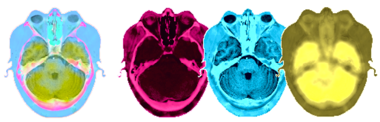
... ADDITIONAL ONES ...

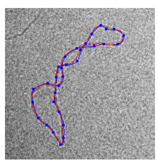
- Attractive Hilbert-space framework for continuous/discrete signal and image processing
- Splines are "π times" better than Daubechies wavelets
- Polynomial splines can be extended to fractional (and even complex) exponents
- Scale invariance and link with fractals (polynomial and fractional splines)
- Generalized (non-stationary) wavelet bases
- ...

Splines and biomedical imaging

Image processing task	Specific operation	Imaging modality
Tomographic reconstruction	• Filtered backprojection • Fourier reconstruction • Iterative techniques • 3D + time	Commercial CT (X-rays) EM PET, SPECT Dynamic CT, SPECT, PET
Sampling grid conversion	• Polar-to-cartesian coordinates • Spiral sampling • k-space sampling • Scan conversion	Ultrasound (endovascular) Spiral CT, MRI MRI
Visualization	*2D operations* • Zooming, panning, rotation • Re-sizing, scaling	All
	• Stereo imaging • Range, topography	Fundus camera OCT
	3D operations • Re-slicing • Max. intensity projection • Simulated X-ray projection	CT, MRI, MRA
	Surface/volume rendering • Iso-surface ray tracing • Gradient-based shading • Stereogram	CT MRI
Geometrical correction	• Wide-angle lenses • Projective mapping • Aspect ratio, tilt • Magnetic field distortions	Endoscopy C-Arm fluoroscopy Dental X-rays MRI
Registration	• Motion compensation • Image subtraction • Mosaicking • Correlation-averaging • Patient positioning • Retrospective comparisons • Multi-modality imaging • Stereotactic normalization • Brain warping	fMRI, fundus camera DSA Endoscopy, fundus camera, EM microscopy Surgery, radiotherapy CT/PET/MRI
Feature detection	• Contours • Ridges • Differential geometry	All
	Contour extraction • Snakes and active contours	MRI, Microscopy (cytology)

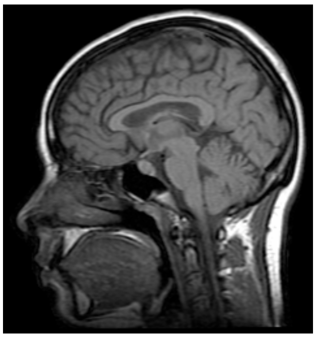


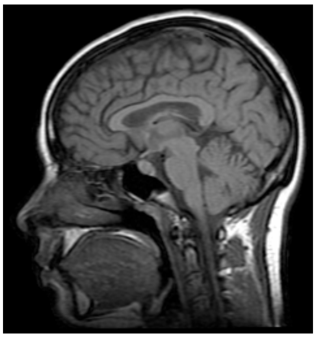





Spline approximation: LS resizing

Approximation at arbitrary scales: differential approach using splines





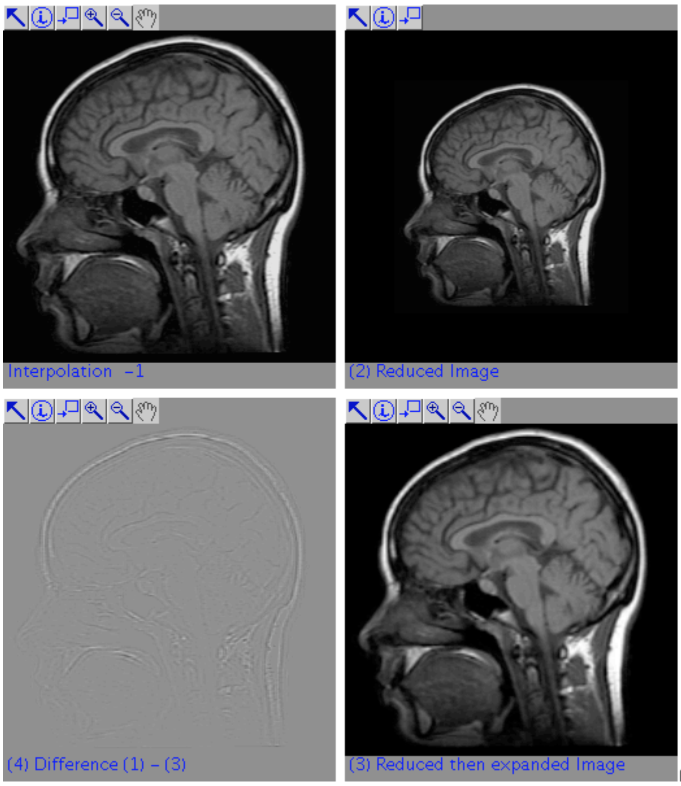
$$a = 1 \rightarrow 10$$

Minimum error approximation (orthogonal projection)

$$f_a(x) = \arg\min_{c_a} \| f(x) - \sum_{k\in\mathbb{Z}} c_a[k]\beta^n(x/a - k)\|^2_{L_2(\mathbb{R})}$$

Application: image resizing

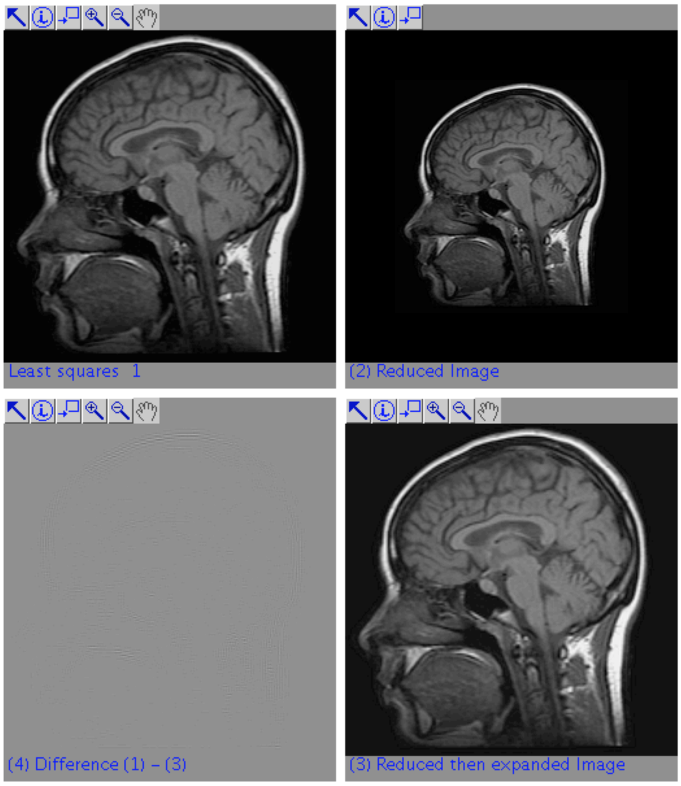
- Resizing algorithm
  - Interpolation
  - Linear splines
  - scaling= 70%



SNR=22.94 dB

Application: image resizing (LS)

- Resizing algorithm
  - Orthogonal projector
  - Linear splines
  - scaling= 70%



SNR=28.359 dB

+ 5.419 dB

Exit

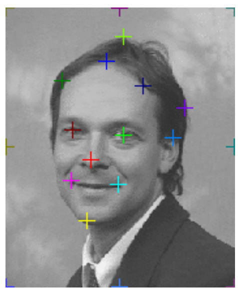
(Munoz et al., *IEEE Trans. Imag. Proc*, 2001)

Elastic registration problem

Find a diffeomorphism (warping): $\boldsymbol{x} \rightarrow \mathbf{g}(\boldsymbol{x})$ such that $f_{\mathrm{S}}(\,\mathbf{g}(\boldsymbol{x})\,) \approx f_{\mathrm{T}}(\boldsymbol{x})$

- $f_{\mathrm{S}}(\boldsymbol{x})$: source image
- $f_{\mathrm{T}}(\boldsymbol{x})$: target image (or reference)
- $\mathbf{g}(\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta})$: parametric deformation map

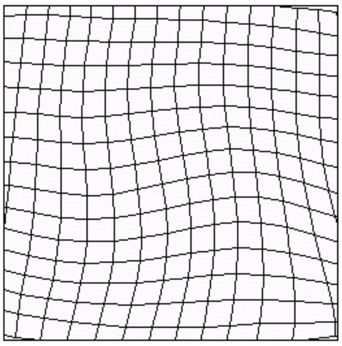
- Problem constraints
  - Similarity measure to compare images
  - Smooth deformation field (regularization)
  - Parametric model (for better efficiency)
  - Optional specification of landmarks: $\boldsymbol{x}_{\mathrm{S}}^{(n)} \rightarrow \boldsymbol{x}_{\mathrm{T}}^{(n)}$



Cubic-spline deformation map

Transformed image: $f_S\left(\mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}_h)\right)$

Deformation map: $\mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}_h) = \begin{pmatrix} g_1(\boldsymbol{x}) \\ g_2(\boldsymbol{x}) \end{pmatrix} = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \begin{pmatrix} c_1[\boldsymbol{k}] \\ c_2[\boldsymbol{k}] \end{pmatrix} \beta^3\left(\frac{\boldsymbol{x}}{h} - \boldsymbol{k}\right)$



- Parametric model (control points)
  $\boldsymbol{\Theta}_h = (\cdots, c_1[k,l], c_2[k,l], \cdots)$
- Resolution controlled by mesh size h
- Smooth deformation (cubic splines)
- Rich variety of spatial mappings, including rigid body, affine, etc.

Registration as an optimization problem

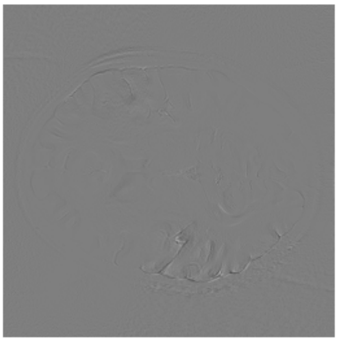
$f_S(\boldsymbol{x}) \rightarrow f_S\left(\mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}_{\mathrm{opt}})\right)$ where $\boldsymbol{\Theta}_{\mathrm{opt}} = \arg\min_{\boldsymbol{\Theta}} \{E_{\mathrm{reg}}(f_S, f_T, \boldsymbol{\Theta})\}$

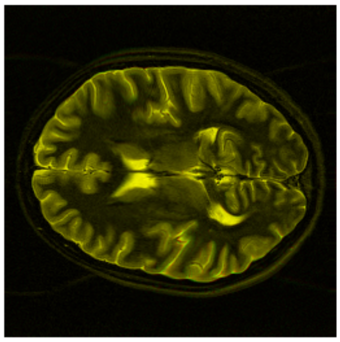
$$E_{\mathrm{reg}}(f_S, f_T, \boldsymbol{\Theta}) = E_{\mathrm{image}}(f_S, f_T, \boldsymbol{\Theta}) + E_{\mathrm{rough}}(\boldsymbol{\Theta}) + E_{\mathrm{landmark}}(\boldsymbol{\Theta})$$

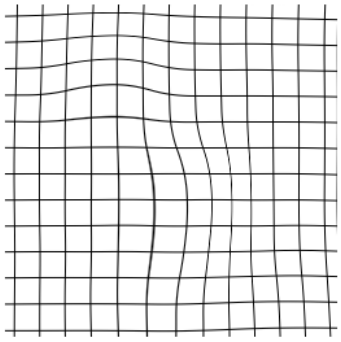
- Least-squares similarity criterion
  $$E_{\mathrm{image}}(f_S, f_T, \boldsymbol{\Theta}) = \sum_{\boldsymbol{k}} \left|f_S(\mathbf{g}(\boldsymbol{k}|\boldsymbol{\Theta})) - f_T[\boldsymbol{k}]\right|^2$$
- Vector-spline roughness penalty
  $$E_{\mathrm{rough}}(\boldsymbol{\Theta}) = \lambda_{\mathrm{div}} \left\|\boldsymbol{\nabla}\ \mathrm{div}\ \mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta})\right\|^2_{L_2(\mathbb{R}^2)} + \lambda_{\mathrm{rot}} \left\|\boldsymbol{\nabla}\ \mathrm{rot}\ \mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta})\right\|^2_{L_2(\mathbb{R}^2)}$$
- Landmark contraints: $\boldsymbol{x}_S^{(n)} \rightarrow \boldsymbol{x}_T^{(n)}$
  $$E_{\mathrm{landmark}}(\boldsymbol{\Theta}) = \frac{\lambda}{N} \sum_{n=1}^{N} \left\|\mathbf{g}(\mathbf{x}_S^{(n)}|\boldsymbol{\Theta}) - \mathbf{x}_T^{(n)}\right\|^2$$

UnwarpJ: Implementation details

- Continuous image representation
  - cubic splines
- Consistent implementation
  - analytical derivatives
  - multilevel B-spline discretization
- Quasi-Newton optimization
  - exact gradient of criterion
- Full multiresolution strategy
  - coarse-to-fine on images
  - coarse-to-fine on deformation







CONCLUSION

- B-splines are attractive computationally
  - Simple to manipulate; smooth and well-behaved
  - Fast recursive filtering algorithms ($O(1)$ per sample)
  - Multiresolution properties (pyramid, multigrid, wavelets)
- Splines: a unifying conceptual framework
  - Approximation theory
  - Link with wavelet theory
  - Signals and systems, sampling theory
  - Stochastic processes; regularization and estimation theories
- Practical Hilbert-space framework (SP counterpart of FE) for continuous/discrete image processing
  - "Think analog, act digital"
  - Toolbox: digital filters, convolution operators
  - Flexibility: piecewise-constant to bandlimited

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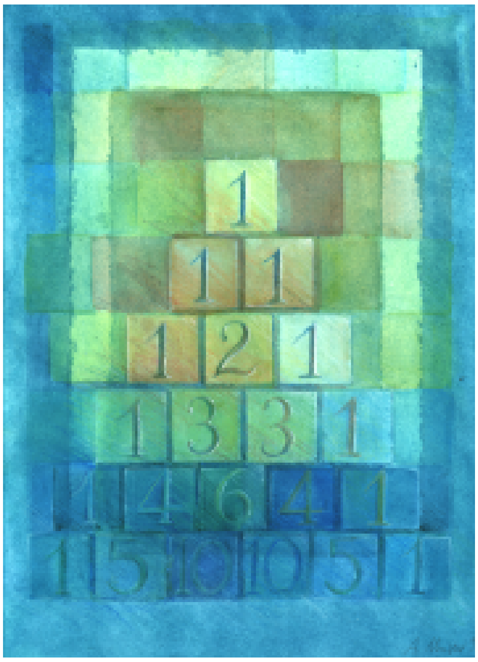
- Prof. Dimitri Van De Ville
- Annette Unser, Artist

+ many other researchers, and graduate students



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- Preprints and demos: http://bigwww.epfl.ch/