ÉCOLE POLYTECHNIQUE
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Wavelets demystified

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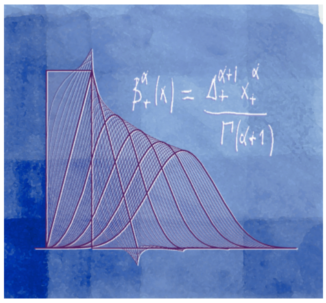
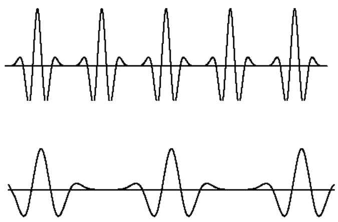


Eindhoven, June 2006

THE WAVELET TRANSFORM

- Wavelet basis functions
  - Dilation and translation of a single prototype

$$\psi_{i,k} = 2^{-i/2}\psi\left(\frac{x - 2^i k}{2^i}\right)$$



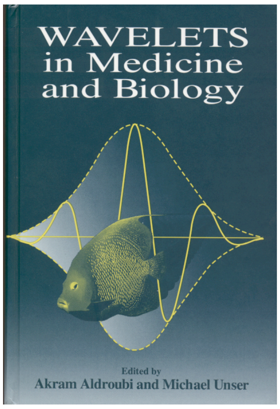
- Motivation
- From “legos” to wavelets
- Signal processing perspective
- Mathematical properties

Motivation for using wavelets

- Remarkable wavelet properties
  - Multi-scale decomposition
  - Self-similarity
  - One-to-one vs. redundant
  - Decoupling: (bi-)orthogonality
  - Vanishing moments
    - Kills polynomials
    - Sparse representation of piecewise-smooth functions
    - Multi-scale differentiation
  - Joint time-frequency localization

- New computational paradigm
  - Multi-resolution formulation
  - Filterbank algorithms: $O(N)$ complexity
  - Regularization via sparsity constraints

- Classes of problems
  - **Data compression**: JPEG2000 ...
  - **Data processing**: filtering, denoising, inverse problems
  - **Data analysis**: singularities, texture, fractals ...



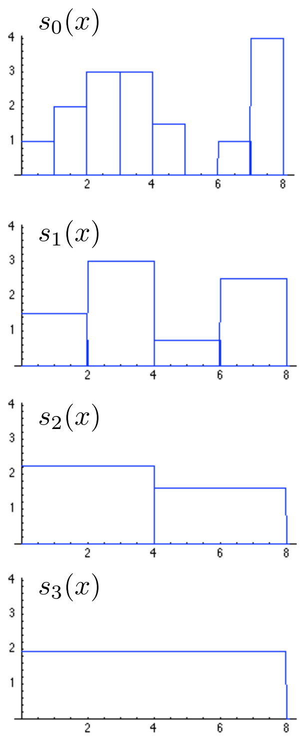
Wavelets in medical imaging: Survey 1991-1999

References
- Unser and Aldroubi, *Proc IEEE*, 1996
- Laine, *Annual Rev Biomed Eng*, 2000

- Special issue, *IEEE Trans Med Im*, 2003

Image processing task	Application / modality	Principal Authors
Image compression	• MRI • Mammograms • CT • Angiograms, etc…	Angelis 94; DeVore 95; Manduca 95; Wang 96; etc …
Filtering	*Image enhancement* • Digital radiograms • MRI • Mammograms • Lung X-rays, CT	Laine 94, 95; Lu, 94; Qian 95; Guang 97; etc …
	Denoising • MRI • Ultrasound (speckle) • SPECT	Weaver 91; Xu 94; Coifman 95; Abdel-Malek 97; Laine 98; Novak 98, 99
Feature extraction	*Detection of micro-calcifications* • Mammograms	Qian 95; Yoshida 94; Strickland 96; Dhawan 96; Baoyu 96; Heine 97; Wang 98
	Texture analysis and classification • Ultrasound • CT, MRI • Mammograms	Barman 93; Laine 94; Unser 95; Wei 95; Yung 95; Busch 97; Mojsilovic 97
	Snakes and active contours • Ultrasound	Chuang-Kuo 96
Wavelet encoding	• Magnetic resonance imaging	Weaver-Healy 92; Panych 94, 96; Geman 96; Shimizu 96; Jian 97
Image reconstruction	• Computer tomography • Limited angle data • Optical tomography • PET, SPECT	Olson 93, 94; Peyrin 94; Walnut 93; Delaney 95; Sahiner 96; Zhu 97; Kolaczyk 94; Raheja 99
Statistical data analysis	*Functional imaging* • PET • fMRI	Ruttimann 93, 94, 98; Unser 95; Feilner 99; Raz 99
Multi-scale Registration	*Motion correction* • fMRI, angiography *Multi-modality imaging* • CT, PET, MRI	Unser 93; Thévenaz 95, 98; Kybic 99
3D visualization	• CT, MRI	Gross 95, 97; Muraki 95; Kamath 98; Horbelt 99

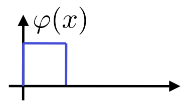
Wavelets: Haar transform revisited



Signal representation

$$s_0(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k)$$

Scaling function



Multi-scale signal representation

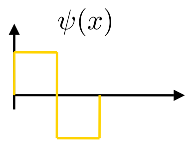
$$s_i(x) = \sum_{k\in\mathbb{Z}} c_i[k]\varphi_{i,k}(x)$$

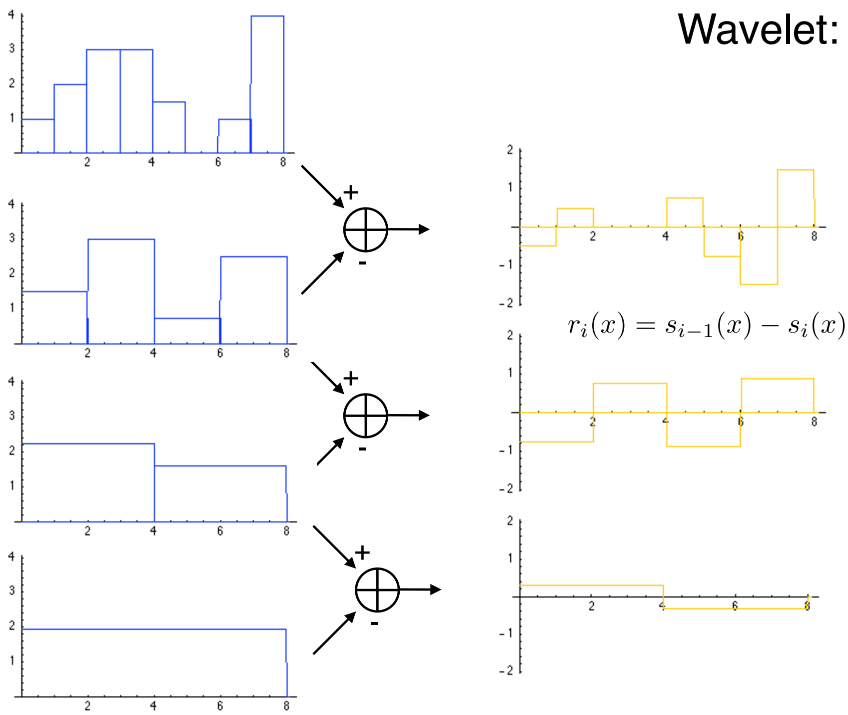
Multi-scale basis functions

$$\varphi_{i,k}(x) = \varphi\left(\frac{x-2^i k}{2^i}\right)$$

Wavelets: Haar transform revisited

Wavelet:

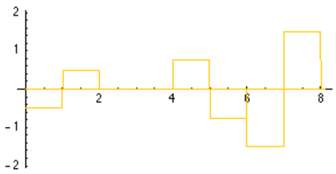




$$r_i(x) = s_{i-1}(x) - s_i(x)$$

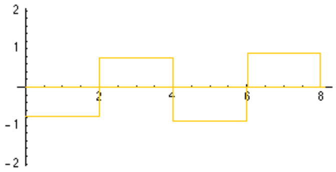
Wavelets: Haar transform revisited

$$r_1(x) = \sum_k d_1[k]\psi_{1,k}$$



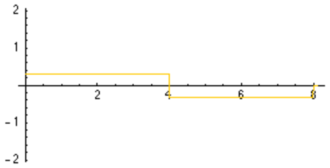
+

$$r_2(x) = \sum_k d_2[k]\psi_{2,k}$$



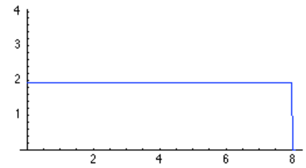
+

$$r_3(x) = \sum_k d_3[k]\psi_{3,k}$$

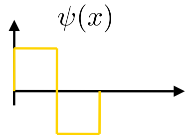


+

$$s_3(x) = \sum_k c_3[k]\varphi_{3,k}$$

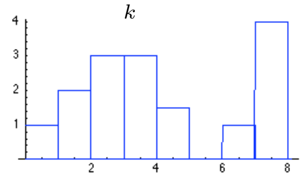


Wavelet:



=

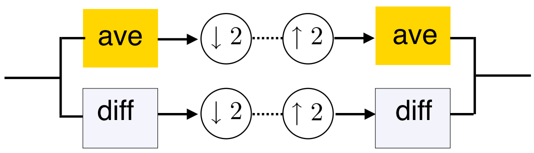
$$s(x) = \sum_k c[k]\varphi(x-k)$$



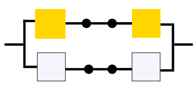
Signal processing perspective

Splitting and putting together again ...

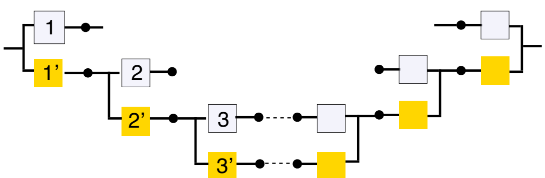
- Perfect reconstruction filterbank



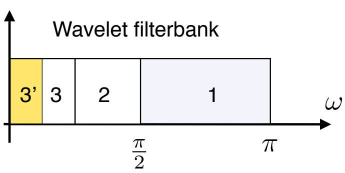
Identity operator:



- Tree-structured wavelet transform



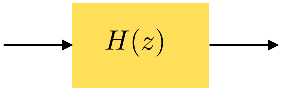
- Subband decomposition



Multi-rate operations

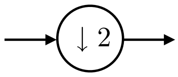
- Filtering

z-transform: $x[k] \overset{z}{\longleftrightarrow} X(z) = \sum_{k \in \mathbb{Z}} x[k] z^{-k}$



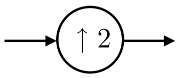
$$(h * x)[k] = \sum_{l \in \mathbb{Z}} h[l] x[k-l] \overset{z}{\longleftrightarrow} H(z) \cdot X(z)$$

- Down-sampling



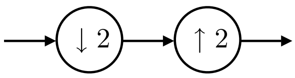
$$(x)_{\downarrow 2}[k] = x[2k] \overset{z}{\longleftrightarrow} \frac{1}{2}\left(X(z^{1/2}) + X(-z^{1/2})\right)$$

- Up-sampling



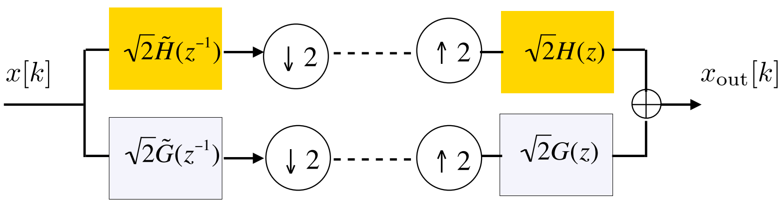
$$(x)_{\uparrow 2}[k] = \begin{cases} 0, & k \text{ odd} \\ x[l], & 2l = k \text{ even} \end{cases} \overset{z}{\longleftrightarrow} X(z^2)$$

- Down-sampling and up-sampling



$$(x)_{\downarrow 2 \uparrow 2}[k] \overset{z}{\longleftrightarrow} \frac{1}{2}\left(X(z) + X(-z)\right)$$

Perfect reconstruction filterbanks



$$X_{\text{out}}(z) = H(z)\left(\tilde{H}(z^{-1})X(z) + \tilde{H}(-z^{-1})X(-z)\right) + G(z)\left(\tilde{G}(z^{-1})X(z) + \tilde{G}(-z^{-1})X(-z)\right)$$

- Perfect reconstruction conditions

(PR-1) $\tilde{H}(z^{-1})H(z) + \tilde{G}(z^{-1})G(z) = 1$ (distortion-free)

(PR-2) $\tilde{H}(-z^{-1})H(z) + \tilde{G}(-z^{-1})G(z) = 0$ (aliasing-free)

- Wavelet transform design

Construct 4 filters such that (PR-1) and (PR-2) are satisfied.

CONSTRUCTION OF WAVELET BASES

- Scaling functions
- Multiresolution analysis
- From scaling functions to wavelets
- The lego revisited
- Fractional B-splines
- Wavelet bases of L_2

Scaling function

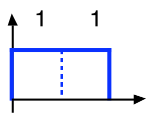
Definition: $\varphi(x)$ is an admissible scaling function of L_2 iff:

- Riesz basis condition

$$\forall c \in \ell_2, \qquad A \cdot \|c\|_{\ell_2} \leq \left\| \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) \right\|_{L_2} \leq B \cdot \|c\|_{\ell_2}$$

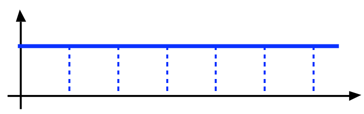
- Two-scale relation

$$\varphi(x/2) = 2\sum_{k\in\mathbb{Z}} h[k]\varphi(x-k)$$

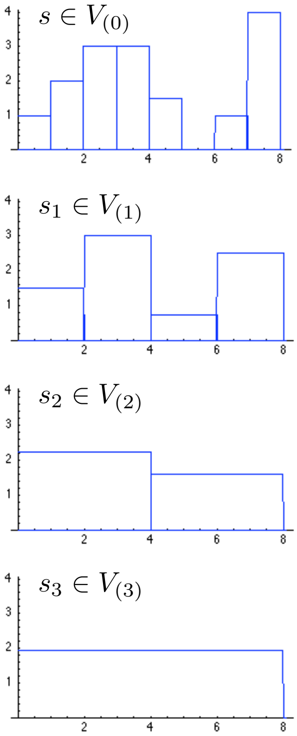


- Partition of unity

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) = 1$$



Multiresolution analysis of L_2



- Multiresolution basis functions: $\varphi_{i,k}(x) = 2^{-i/2}\varphi\left(\frac{x-2^i k}{2^i}\right)$
- Subspace at resolution i: $V_{(i)} = \text{span}\{\varphi_{i,k}\}_{k\in\mathbb{Z}}$

Two-scale relation $\Rightarrow$ $V_{(i)} \subset V_{(j)}$, for $i \geq j$

Partition of unity $\Leftrightarrow$ $\overline{\bigcup_{i\in\mathbb{Z}} V_{(i)}} = L_2(\mathbb{R})$

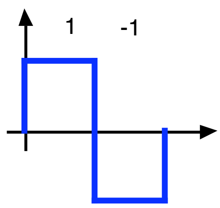
From scaling functions to wavelets

- Wavelet bases of L_2 [Mallat-Meyer, 1989]

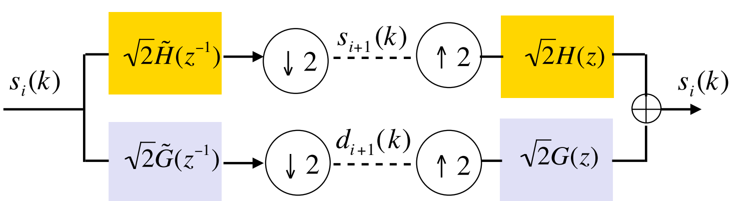
Theorem: For any given admissible scaling function of L_2, $\varphi(x)$, there exists a wavelet $\psi(x/2) = 2\sum_{k\in\mathbb{Z}} g[k]\varphi(x-k)$ such that the family of functions

$$\left\{2^{-i/2}\psi\left(\frac{x-2^i k}{2^i}\right)\right\}_{i\in\mathbb{Z},k\in\mathbb{Z}}$$

forms a Riesz basis of L_2.



- Constructive approach: perfect reconstruction filterbank



The lego revisited

- Continuous derivative

$$\mathrm{D}\{\cdot\} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad j\omega$$

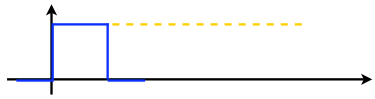
- Discrete version (finite difference)

$$\Delta_+\{\cdot\} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 1 - e^{-j\omega}$$

- Construction of the B-spline of degree 0

Step function: $x_+^0 = \mathrm{D}^{-1}\{\delta(x)\}$

$\beta_+^0(x) = x_+^0 - (x-1)_+^0 = \Delta_+^1 x_+^0$



- Fourier domain formula

$$\hat{\beta}_+^0(\omega) = \frac{1 - e^{-j\omega}}{j\omega}$$

Discrete operator (finite difference)

Continuous operator (derivative)

Beyond legos: fractional splines

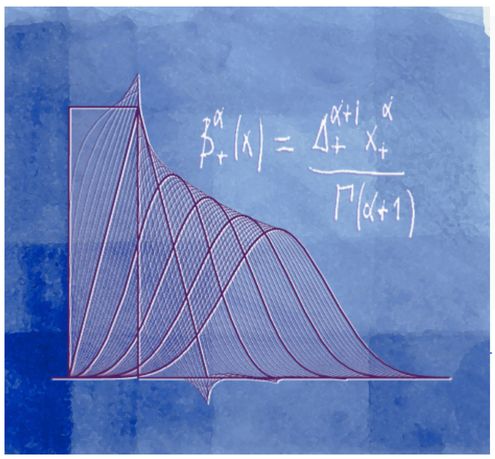
- Fractional B-splines

$$\beta_+^0(x) = \Delta_+ x_+^0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1 - e^{-j\omega}}{j\omega}$$

$$\vdots \qquad\qquad \vdots$$

$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \left(\frac{1 - e^{-j\omega}}{j\omega}\right)^{\alpha+1}$$

One-sided power function: $x_+^\alpha = \begin{cases} x^\alpha, & x \geq 0 \\ 0, & x < 0 \end{cases}$



- Properties

(Unser & Blu, *SIAM Rev*, 2000)

  - $\{\beta_+^\alpha(x-k)\}_{k\in\mathbb{Z}}$ is a valid Riesz basis for $\alpha < -\frac{1}{2}$
  - Convolution property: $\beta_+^{\alpha_1} * \beta_+^{\alpha_2} = \beta_+^{\alpha_1+\alpha_2+1}$

Binomial refinement filter

- Dilation by a factor of 2

$$\beta_+^\alpha(x/2) = 2\sum_{k\in\mathbb{Z}} h_+^\alpha[k]\beta_+^\alpha(x-k)$$

Hint for the proof:

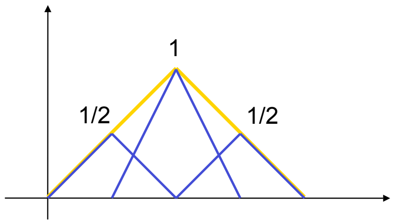
$$H_+^\alpha(e^{j\omega}) = \frac{\hat{\beta}_+^\alpha(2\omega)}{\hat{\beta}_+^\alpha(\omega)} = \left(\frac{1+e^{-j\omega}}{2}\right)^{\alpha+1}$$

- Generalized binomial filter

$$h_+^\alpha[k] = \frac{1}{2^{\alpha+1}}\binom{\alpha+1}{k} \quad \stackrel{z}{\longleftrightarrow} \quad H_+^\alpha(z) = \left(\frac{1+z^{-1}}{2}\right)^{\alpha+1}$$

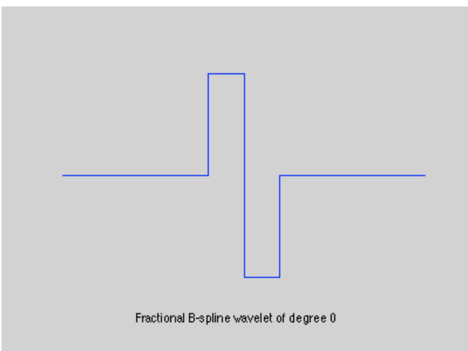
where $\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$

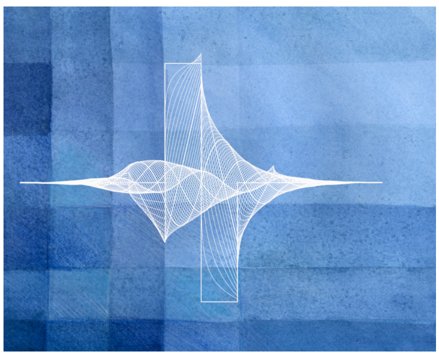
- Example: piecewise linear splines: $\alpha = 1$



Fractional B-spline wavelets

$$\psi_+^\alpha(x/2) = \sum_{k\in\mathbb{Z}} \underbrace{\frac{(-1)^k}{2^\alpha}\sum_{n\in\mathbb{N}}\binom{\alpha+1}{n}\beta_*^{2\alpha+1}(n+k-1)}_{2g[k]}\,\beta_+^\alpha(x-k)$$

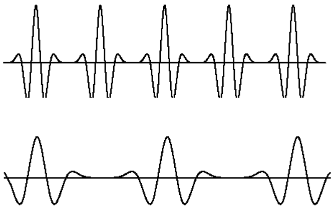




(Unser and Blu, *SIAM Review*, 2000)

- **Remarkable property**
  Each of these wavelets generates a Riesz basis of L_2

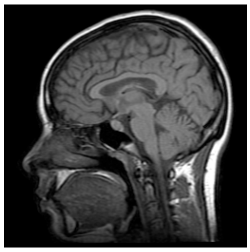
Wavelet basis of L_2



- Family of wavelet templates (basis functions)

$$\psi_{i,k} = 2^{-i/2}\psi_+^\alpha\left(\frac{x - 2^i k}{2^i}\right)$$

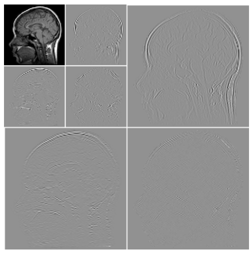
- Semi-orthogonal wavelet basis



$$\langle\tilde{\psi}_{i,k}, \psi_{j,l}\rangle = \delta_{i-j,k-l}$$

$$\forall f(x) \in L_2, \quad f(x) = \sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f, \tilde{\psi}_{i,k}\rangle\ \psi_{i,k}$$

- Orthogonal wavelet basis (Generalized Battle-Lemarié)

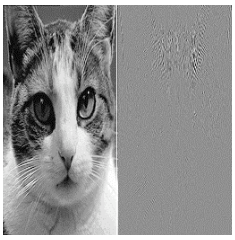


$$\langle\psi_{i,k}, \psi_{j,l}\rangle = \delta_{i-j,k-l}$$

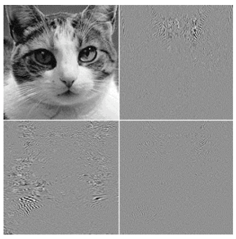
$$\forall f(x) \in L_2, \quad f(x) = \sum_{i\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f, \psi_{i,k}\rangle\ \psi_{i,k}$$

Extension to higher dimension: separability

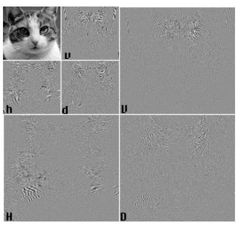
Split rows



Split columns



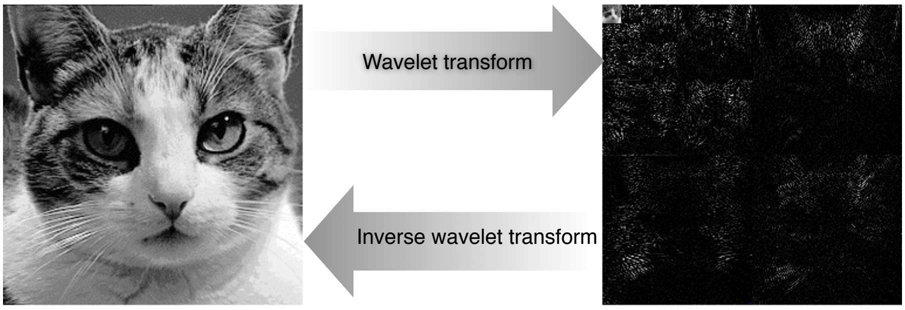
and iterate ...



Tensor-product basis functions:

$$\psi_{k_1,\cdots,k_p}(x_1,\cdots,x_p) = \psi_{k_1}(x_1) \times \psi_{k_2}(x_2)\cdots\times\psi_{k_p}(x_p)$$

2D wavelet decomposition: example



66.4 dB

0.00%

Discarding "small coefficients"

Other popular wavelet families

- Basic paradigm: PR filterbank design

  Implicit definition: solution of two-scale relation

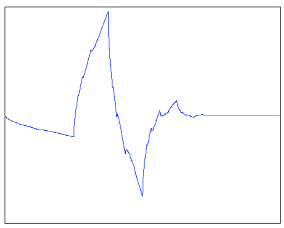
  $$\varphi(x/2) = 2\sum_{k\in\mathbb{Z}} h[k]\varphi(x-k)$$

- Daubechies wavelets

  Shortest orthogonal filter such that:

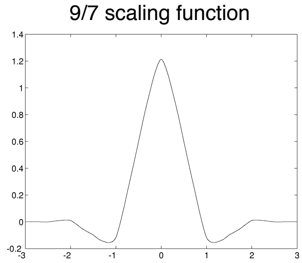
  $H(z) = (1+z^{-1})^L Q(z)$ (order constraint)

Daubechies wavelet: *L*=2



- Biorthogonal splines (Cohen-Daubechies-Feauveau)

9/7 scaling function



- JPEG2000 wavelets
  - biorthogonal 5/3: linear spline (2 vanishing moments)
  - biorthogonal 9/7: symmetric, near-orthogonal (4 vanishing moments)

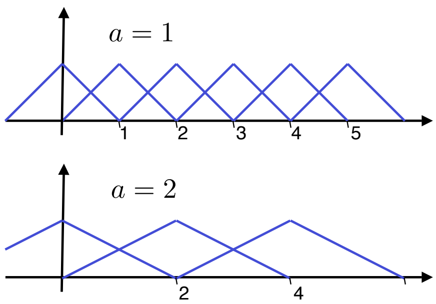
WAVELET THEORY

- Order of approximation
- Factorization theorem
- Reproduction of polynomials
- Vanishing moments
- Multi-scale differentiation
- Smoothness

Order of approximation

- General "shift-invariant" space at scale a

$$V_a(\varphi) = \left\{ s_a(x) = \sum_{k \in \mathbb{Z}} c[k]\varphi\left(\frac{x}{a} - k\right) : c \in \ell_2 \right\}$$



- Projection operator

$$\forall f \in L_2, \qquad \mathrm{P}_a f = \arg \min_{s_a \in V_a} \|f - s_a\|_{L_2}$$

- Order of approximation

Definition

A scaling/generating function φ has order of approximation γ iff.

$$\forall f \in W_2^\gamma, \qquad \|f - \mathrm{P}_a f\|_{L_2} \le C \cdot a^\gamma \cdot \|f^{(\gamma)}\|_{L_2}$$

B-spline factorization

- Factorization theorem

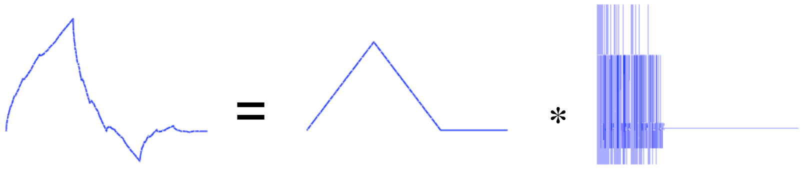
A valid scaling function $\varphi(x)$ has order of approximation γ iff

$$\varphi(x) = \left(\beta_+^\alpha * \varphi_0\right)(x)$$

where β_+^α with $\alpha = \gamma - 1$: regular, B-spline part

$\varphi_0 \in S'$: irregular, distributional part

(Unser-Blu, IEEE-SP, 2003)



- Refinement filter: general case

$$H(z) = \underbrace{\left(\frac{1+z^{-1}}{2}\right)^\gamma}_{\text{spline part}} \cdot \underbrace{Q(z)}_{\text{distributional part}} \qquad \text{with} \quad |Q(e^{j\omega})| < +\infty$$

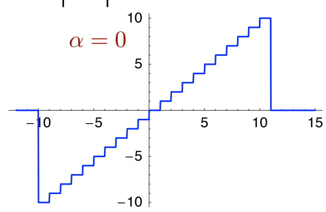
Reproduction of polynomials

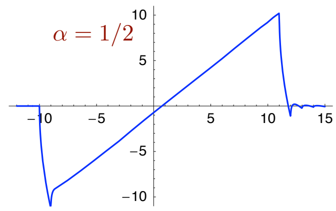
- B-splines reproduce polynomials of degree $N = \lceil \alpha \rceil$

$$\sum_{k\in\mathbb{Z}} \beta_+^\alpha(x-k) = 1$$

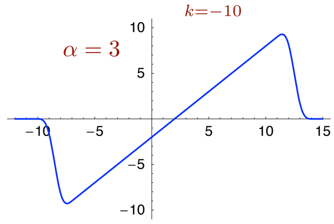
$$\vdots$$

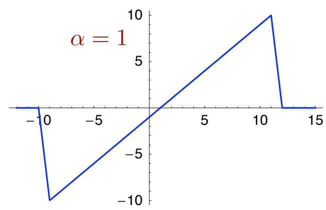
$$\sum_{k\in\mathbb{Z}} k^n \beta_+^\alpha(x-k) = x^n + a_1 x^{n-1} + \cdots + a_n$$





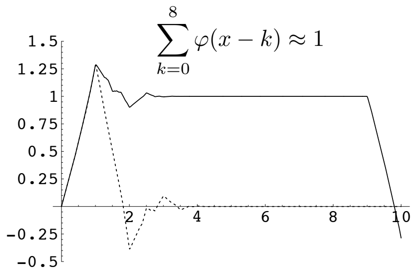
$$\sum_{k=-10}^{+10} k\, \beta_+^\alpha(x-k)$$

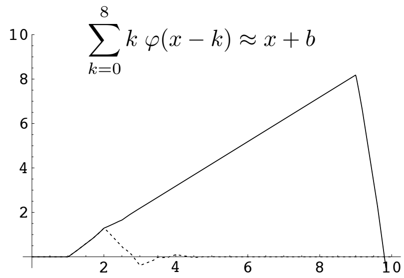




Reproduction of polynomials (Cont'd)

Proposition: If $\varphi(x) = (\beta_+^\alpha * \varphi_0)(x)$ with $\hat{\varphi}_0(0) = 1$, then $\varphi(x)$ reproduces the polynomials of degree $N = \lceil \alpha \rceil$.





Argument:

- $\varphi_0 * x^n = x^n + b_1 x^{n-1} + \cdots + b_n$
- $\sum_{k\in\mathbb{Z}} k^n \varphi(x-k) = \varphi_0 * \sum_{k\in\mathbb{Z}} k^n \beta_+^\alpha(x-k) = x^n + c_1 x^{n-1} + \cdots + c_n$

Strang-Fix (1971): Conversely, if $\varphi(x)$ reproduces the polynomials of degree N and is compactly supported, then $\varphi(x) = (\beta_+^N * \varphi_0)(x)$.

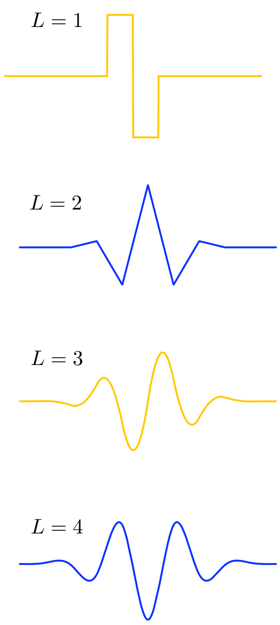
Vanishing moments

Proposition

If $\varphi(x)$ reproduces the polynomials of degree N, then the analysis wavelet $\tilde{\psi}(x)$ has $L = N + 1$ vanishing moments:

$$\int_{x\in\mathbb{R}} x^n \tilde{\psi}(x)\mathrm{d}x = 0, \quad n = 0, \cdots, N$$

B-spline wavelets:



⇒ $\tilde{\psi}$ kills all polynomials of degree $n \leq N$

$$\forall p(x) \in \pi^N, \quad \int_{x\in\mathbb{R}} p(x)\tilde{\psi}(x/a - b)\mathrm{d}x = 0$$

Argument

- Polynomial reproduction $\Leftrightarrow$ $p(x) \in \mathrm{span}\{\varphi(x-k)\}_{k\in\mathbb{Z}}$
- $\tilde{\psi}(x)$ is perpendicular to $V(\varphi)$ by construction

$\Rightarrow$ $\tilde{\psi}$ is perpendicular to $p(x)$

Differentiation and regularity

- Fractional differential operators

Derivative of order s: $\partial^s \xleftrightarrow{\mathcal{F}} (j\omega)^s$

Finite difference of order s: $\Delta_+^s \xleftrightarrow{\mathcal{F}} (1-e^{-j\omega})^s$

- B-spline differentiation formula

$$\partial^s \beta_+^\alpha(x) = \Delta_+^s \beta_+^{\alpha-s}(x)$$

Sketch of the proof:

$$\partial^s \beta_+^\alpha(x) \xleftrightarrow{\mathcal{F}} (j\omega)^s \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1} = (1-e^{-j\omega})^s \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha-s+1}$$

- Continuity and differentiability

Hölder smoothness: $\beta_+^\alpha \in \dot{C}^\alpha \Rightarrow \partial^\alpha \beta_+^\alpha$ bounded

Sobolev smoothness: $\beta_+^\alpha \in W_2^r,\ r < \alpha + \frac{1}{2} \Leftrightarrow \partial^r \beta_+^\alpha \in L_2(\mathbb{R})$

Multi-scale differentiation

- Perfect reconstruction conditions

$$\begin{pmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{pmatrix} \cdot \begin{pmatrix} \tilde{H}(z^{-1}) & \tilde{H}(-z^{-1}) \\ \tilde{G}(z^{-1}) & \tilde{G}(-z^{-1}) \end{pmatrix} = \mathbf{I}$$

Proposition: For a stable filterbank, the order constraint is equivalent to $\tilde{G}(z) = (1-z)^\gamma \cdot P(z)$ with $|P(e^{j\omega})| < +\infty$.

Theorem: Let φ and $\tilde{\varphi}$ be two valid biorthogonal scaling functions. Then, φ is of order γ (i.e., $\varphi = \beta_+^{\gamma-1} * \varphi_0$) iff $\hat{\tilde{\psi}}(\omega) = O(|\omega|^\gamma)$.

- Wavelet transform as a multi-scale differentiator

⇨ $\langle f(x), \tilde{\psi}(x-u)\rangle = \partial^\gamma \{\phi * f\}(u)$

Smoothing kernel: $\hat{\phi}(\omega) = \hat{\tilde{\psi}}^*(\omega)/(j\omega)^\gamma$

Wavelet regularity: peeling Daubechies

Theorem: If $\varphi_\gamma(x) = \beta_+^{r-1} * \varphi_{\gamma-r}(x)$ with $\varphi_{\gamma-r} \in L_p$, then $\partial^r \varphi_\gamma \in L_p$; i.e., φ_γ has r derivatives in L_p-sense.

B-spline peeling mechanism:

$$\varphi_\gamma = \beta_+^{\gamma-1} * \varphi_0 = \beta_+^{r-1} * \overbrace{(\beta_+^{\gamma-r-1} * \varphi_0)}^{\varphi_{\gamma-r}}$$

$$\Rightarrow \quad \partial^r \varphi_\gamma = (\partial^r \beta_+^{r-1}) * \varphi_{\gamma-r} = \Delta_+^r \varphi_{\gamma-r}$$

Note: $\beta_+^{-1}(x) = \delta(x)$

Theorem: If φ is a valid scaling function such that $\partial^r \varphi \in L_2$, then $\varphi(x) = \beta_+^{r-1} * \varphi_0(x)$ with $\varphi_0 \in L_2$.

Splines: The key to wavelet theory

Sobolev smoothness
$\partial^s \varphi \in L_2$

$\Downarrow \ \gamma \geq s$

B-spline factorization: $\varphi = \beta_+^{\gamma-1} * \varphi_0$ $\Leftrightarrow$ Approximation order: $\|f - P_a f\|_{L_2} = O(a^\gamma)$ $\Leftrightarrow$ Multi-scale differentiator $\hat{\tilde{\psi}}(\omega) \propto (-j\omega)^\gamma, \ \omega \to 0$

$\Downarrow \quad \Uparrow$

general case: $N < \gamma \leq N+1$

compact support: $\gamma = N+1$ (Strang-Fix)

Polynomial reproduction
degree: $N = \lceil \gamma - 1 \rceil$

$\Updownarrow$

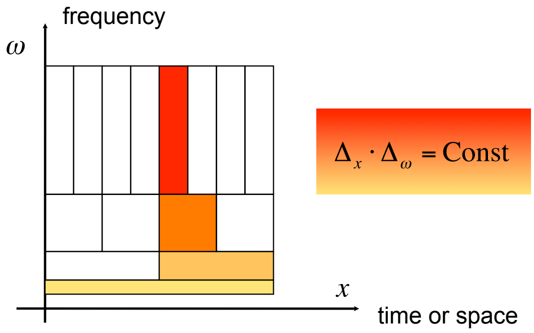
Vanishing moments:
$\int x^n \tilde{\psi}(x) \mathrm{d}x = 0, \ n = 0, \cdots, N$

(Unser and Blu, IEEE-SP, 2003)

FURTHER (ANTHROPOMORPHIC) ARGUMENTS ...

- Localization properties
- Wavelets and the uncertainty principle

Localization properties



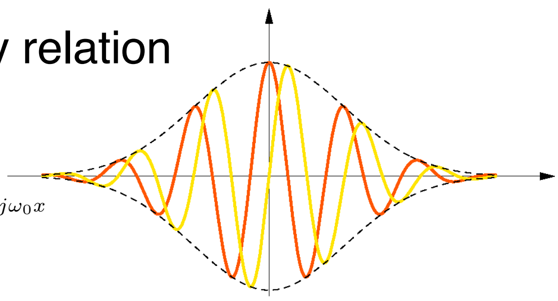
$$\Delta_x \cdot \Delta_\omega = \text{Const}$$

$$\Delta_x = \min_{x_0} \frac{\|(x - x_0)\psi(x)\|_{L_2}}{\|\psi\|_{L_2}}$$

$$\Delta_\omega = \min_{\omega_0} \frac{\|(\omega - \omega_0)\hat{\psi}(\omega)\|_{L_2}}{\|\hat{\psi}\|_{L_2}}$$

- Heisenberg's uncertainty relation

$$\Delta_x \cdot \Delta_\omega \geq \frac{1}{2}$$



with equality iff. $\psi(x) = a \cdot e^{-b(x-x_0)^2 + j\omega_0 x}$

Question: are there Gabor-like wavelet bases ?

Are there optimally localized wavelet bases ?

Theorem

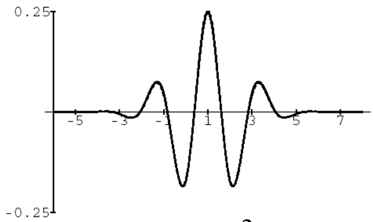
The B-spline wavelets converge (in L_p-norm) to modulated Gaussians as the degree goes to infinity :

$$\lim_{\alpha\to\infty}\left\{\beta_+^\alpha(x)\right\} = C\cdot e^{-(x-x_\alpha)^2/2\sigma_\alpha^2}$$

$$\lim_{\alpha\to\infty}\left\{\psi_+^\alpha(x)\right\} = \underbrace{C'\cdot e^{-(x-x'_\alpha)^2/2\sigma'^2_\alpha}}_{\text{Gaussian}} \times \underbrace{\cos(\omega_0 x + \theta_\alpha)}_{\text{sinusoid}}$$

$$\sigma_\alpha = \sqrt{\frac{\alpha+1}{12}}$$

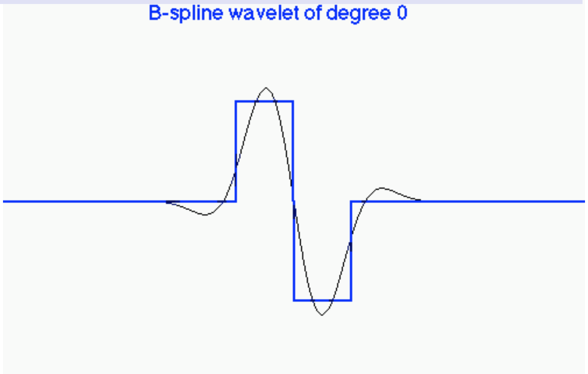
$$\sigma'_\alpha = B\cdot\sigma_\alpha \quad \text{with} \quad B \cong 2.59$$



$\alpha = 3$

Cubic B-spline wavelets:
within 2% of the uncertainty limit !

(Unser et *al*., *IEEE-IT*, 1992)



Are there wavelets in my brain ?

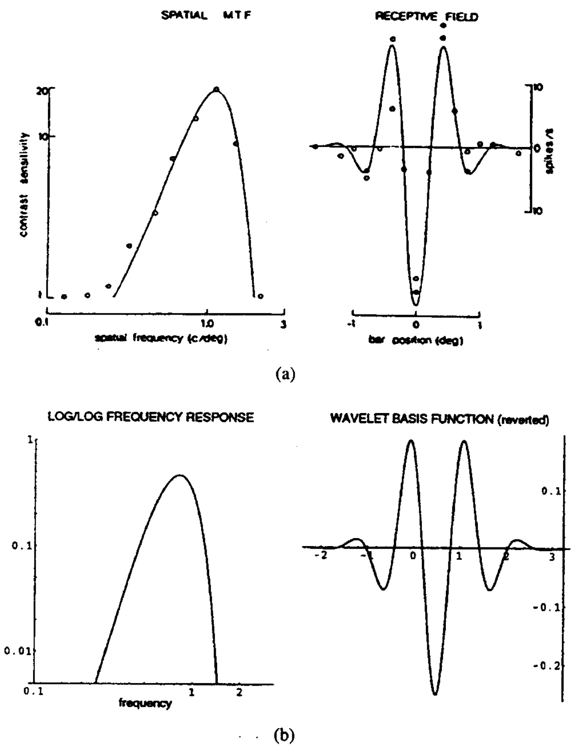


Fig. 2. Similarity between the receptive field of simple cortical cells and a wavelet basis function. (a) Response of a simple X cell from a monkey visual cortex and its fitted Gabor elementary signal [26], [67, Fig. 3]. (b) Semi-orthogonal cubic B-spline wavelet and its log-log frequency response [100].

CONCLUSION

- Important wavelet features
  - Simple, fast implementation: Mallat's filterbank algorithm
  - Mathematical properties: Riesz basis, vanishing moments, polynomial reproduction, order of approximation, ...
  - Fundamental connection between splines and wavelets
  - Simulates the organization of the primary visual system
- Many successful applications
  - Data compression
  - Filtering, denoising (non-linear)
  - Detection and feature extraction
  - Inverse problems: wavelet regularization
- Current topics in wavelet research
  - Non-separable multidimensional wavelets: isotropic vs. directional
  - Complex wavelets, Bandelets
  - Wavelet frames, ridgelets, curvelets, etc...

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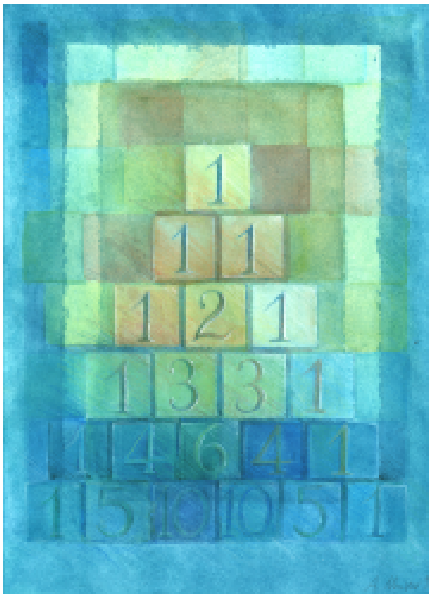

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