A subdivision approach to splines, exponential splines and beyond

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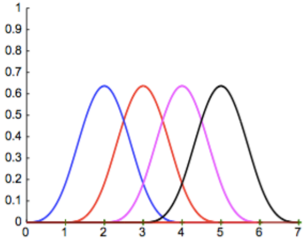
Twenty Years of Biomedical Imaging and Splines
Losanna, Switzerland, March 23, 2018

Outline

The polynomial B-spline basis

Polynomial B-splines are bell shaped compacted supported basis functions for polynomial splines that find application in many different context

- geometric modeling
- computer graphics
- curve/surface fitting
- numerical differentiation/integration
- signal/image processing
- solution of PDE also via IgA
- statistics
- ....



The polynomial cardinal B-spline basis: integer simple knots

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- recurrence : $\rightarrow B_{h,i}(t) = \frac{t-i}{i+h-1-i} B_{h-1,i}(t) + \frac{i+h-t}{i+h-(i+1)} B_{h-1,i+1}(t), \quad h = 2, ..., N$

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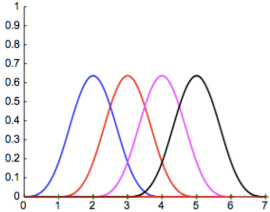
i support size $N + 1$ and non negative;
ii C^{N-1} global regularity;
iii form a partition of unity $\sum_{i \in \mathbb{Z}} B_{N,i} = 1$;
iv shifted copies of each other $B_{N,i} = B_{N,0}(\cdot - i)$;



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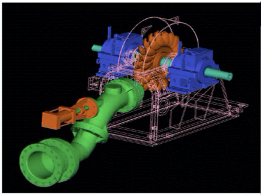
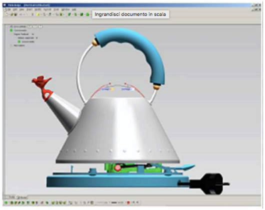
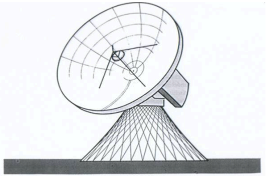
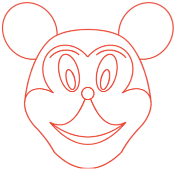
- their use to model manifolds with arbitrary topology is conceptually very complicated and extremely expensive;
- have a low approximation order. A pre-processing of the data is necessary to get higher approximation order;
- are not able reproduce geometries like conic sections which are important e.g. in geometric modeling, biomedical imaging and IgA.

Conics in geometric modeling/CAD

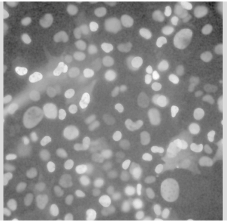


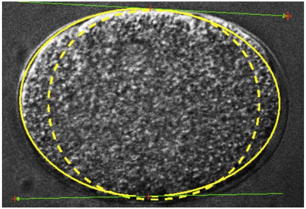






Conics in medical image processing





NURBS: non uniform rational B-splines

In geometric modeling a B-splines generalization yield in the 90s NURBS

$$s(t) = \sum_{i=1}^{n} P_i \, B_{N,i}(t), \qquad s(t) = \frac{\sum_{i=1}^{n} P_i \, B_{N,i}(t) w_i}{\sum_{i=1}^{n} B_{N,i}(t) w_i} = \sum_{i=1}^{n} P_i \, R_{N,i}(t).$$

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- need additional parameters or weights which do not have an evident geometric meaning and whose selection is often unclear;
- not exact description of transcendental curves like a helix or a cycloid;
- their rational nature make them unpleasant with respect to differentiation and integration which are crucial operators in analysis;

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- They possess all fundamental properties of algebraic B-splines (recurrence construction, minimum support, local linear independence, knot-insertion, degree elevation, which are shared by NURBS) but behave completely similar to B-splines with respect to differentiation and integration.

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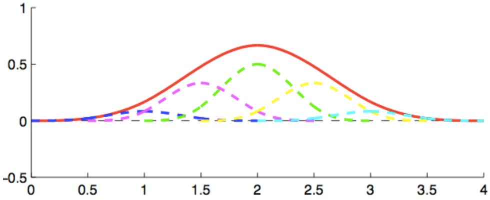
- In same instances they are also called Tchebycheffian B-splines

A subdivision approach for cardinal B-splines

Cardinal B-splines: refinement properties

An important property of polynomial cardinal B-splines is their refinability: they can be written as linear combination of shifts of dilates version of themself

$$B_3(t) = \frac{1}{8}B_3(2t) + \frac{1}{2}B_3(2t-1) + \frac{3}{4}B_3(2t-2) + \frac{1}{2}B_3(2t-3) + \frac{1}{8}B_3(2t-4)$$



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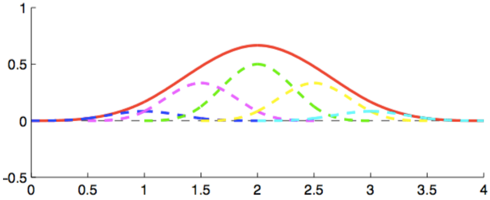

☞ Coefficients of the cubic B-spline refinement mask: $\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}$

The binary subdivision scheme for cardinal cubic splines

Using the refinability properties

$$B_3(t) = \sum_{j\in\mathbb{Z}} a_j^3 B_3(2t-j), \quad \text{where} \quad \mathbf{a}^3 = \cdots 0, \frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}, 0, \cdots$$

any cubic polynomial spline can be written as

$$s(t) = \sum_{i\in\mathbb{Z}} P_i B_3(t-i) = \sum_{i\in\mathbb{Z}} P_i \sum_{j\in\mathbb{Z}} a_j^3 B_3(2(t-i)-j), \quad \text{that is}$$

$$s(t) = \sum_{i\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{i-2j}^3 P_j \right) B_3(2t-i) = \sum_{i\in\mathbb{Z}} P_i^{(1)} B_3(2t-i)$$

where

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that is

$$s(t) = \sum_{i \in \mathbb{Z}} P_i^{(2)} B_3(4t - i), \qquad \cdots \qquad s(t) = \sum_{i \in \mathbb{Z}} P_i^{(k+1)} B_3(2^k t - i),$$

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☞ Since the support of $B_3(2^k\cdot)$ shrink for k large enough the coefficients $\mathbf{P}^{(k)}$ are a good discrete representation of s.

The binary subdivision scheme for degree N splines

More in general, since $B_N(t) = \sum_{i\in\mathbb{Z}} a_j^N B_N(2t-j)$, starting with an initial sequence of points $\mathbf{P}^{(0)}$, the iterative computation of sequence of points

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defines the so called *subdivision scheme for cardinal degree-N splines* whose limit is a degree N spline associated with the initial control points $\mathbf{P}^{(0)}$ and often denoted as $S_{\mathbf{a}^N}^{\infty}\mathbf{P}^{(0)}$.

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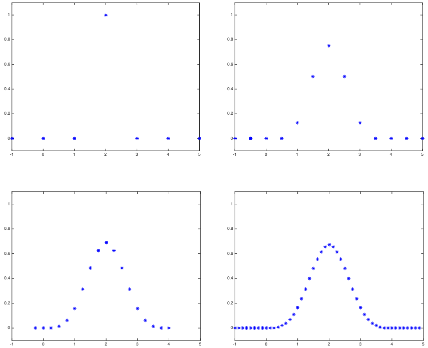
☞ Any degree-N B-splines is the *basic limit function of the corresponding subdivision scheme* when starting with the sequence $\delta = 0,\ 0,\ 1,\ 0,\ 0.$

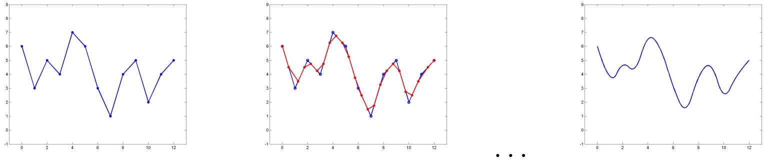
Binary subdivision scheme



Binary subdivision scheme

This subdivision idea allows us to define other type of refinable basis functions not necessarily piecewise polynomial but useful "generalization" of B-splines.

$$S_{\mathbf{a}} \quad \Leftrightarrow \quad \begin{cases} \text{Input} \quad \mathbf{P}^{(0)} \\ \text{For} \quad k = 0, 1, \cdots \\ \qquad\qquad \mathbf{P}^{(k+1)} := S_{\mathbf{a}} \mathbf{P}^{(k)} \end{cases}$$



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  - $\phi = \sum_{i\in\mathbb{Z}} a_i \phi(2\cdot - i);$
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Non-stationary or level dependent subdivision scheme

This subdivision recursion idea can also be implemented by changing the set of coefficients at each level.

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☞ In that case we have a family of basic limit functions

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$$\{\phi_m,\ m \geq 0\}, \quad \phi_m = \lim_{k \to \infty} S_{\mathbf{a}^{(k)}} \cdots S_{\mathbf{a}^{(m)}} \delta.$$

☞ They still satisfy a refinability property: $\phi_m = \sum_{i \in \mathbb{Z}} a_i^{(m)} \phi_{m+1}(2 \cdot -i).$

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In the subdivision community a similar notion to the z-trasform is used: the subdivision symbol associated with the subdivision mask .

Subdivision symbol

The symbol of a subdivision mask $\{a_i,\ i \in \mathbb{Z}\}$ is the Laurent polynomial

$$a(z) = \sum_{i \in \mathbb{Z}} a_i\, z^i, \qquad z \in \mathbb{C} \setminus \{0\}.$$

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- Many of the properties of a subdivision scheme can be easily checked using algebraic conditions on the subdivision symbols;
- This is also true for the properties of the basic limit functions

Exponential-polynomial B-splines:
a "perfect" base for the space of exponential-polynomial splines

Cardinal exponential-polynomial B-splines

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Exponential-polynomials

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$$EP_\Gamma = \text{span}\{\ x^{r_i}\ e^{\theta_i x},\ r_i = 0, \cdots, \tau_i - 1,\ i = 1, \cdots, n\}.$$

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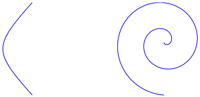
Exponential-polynomials

Let $n \in \mathbb{N}$ and let $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ with $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_i \neq \theta_j$ if $i \neq j$ and $\tau_i \in \mathbb{N}$, $i = 1, \cdots, n$.
We define the space of exponential polynomials EP_Γ

$$EP_\Gamma = \mathrm{span}\{\, x^{r_i}\, e^{\theta_i x},\ r_i = 0, \cdots, \tau_i - 1,\ i = 1, \cdots, n\}.$$

☞ EP_Γ is a linear space of dim. $N = \sum_{\ell=1}^{n} \tau_\ell$.
It contains polynomials as special case.



Cardinal Exponential B-splines and Green functions

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Cardinal exponential B-spline for $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ are defined as

Definition of exponential B-splines

$$B_{N,\Gamma} = \underbrace{B_{\theta_1}}_{\tau_1-\text{times}} * \underbrace{B_{\theta_2}}_{\tau_2-\text{times}} * \cdots * \underbrace{B_{\theta_n}}_{\tau_n-\text{times}}, \qquad B_\theta(t) = \rho_\theta(t) - e^\theta \rho_\theta(t-1).$$

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Symbols of exponential B-splines

For $\Gamma = \{(\theta_1, \tau_1), \ldots, (\theta_n, \tau_n)\}$ the symbols of exponential B-splines of order $N = \sum_{\ell=1}^{n} \tau_\ell$, are given by $B_{N,\Gamma}^{(k)}(z) = F_N^{(k)} \prod_{\ell=1}^{n} (1 + e^{\frac{\theta_\ell}{2^{k+1}}} z)^{\tau_\ell}, \quad k \geq 0.$

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- When $\Gamma = \{(0, N)\}$ the symbols reduce the symbol of polynomial B-spline of order N (degree $N-1$) given by $B_{N-1}(z) = \frac{1}{2^{N-1}}(1+z)^N$.

Exponential B-splines: four dimension ("cubic") case

Support: $[0, 4]$

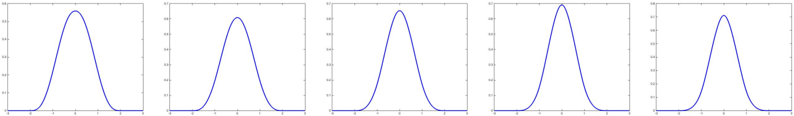
Knot vector: $\{1, 2, 3, \}, \ \mathcal{M} = (1, 1, 1)$

Pieces in the space: $\{1, t, e^{\theta t}, e^{-\theta t}\}, \ \theta \in \mathbb{R} \cup i\mathbb{R}$

$$P_{2i}^{(k+1)} = \frac{1}{4(v^{(k)}+1)} P_{i-1}^{(k)} + \frac{1+2v^{(k)}}{2(v^{(k)}+1)} P_i^{(k)} + \frac{1}{4(v^{(k)}+1)} P_{i+1}^{(k)}$$

$$P_{2i+1}^{(k+1)} = \frac{1}{2} P_i^{(k)} + \frac{1}{2} P_{i+1}^{(k)}$$

For $v^{(k)} = \frac{1}{2}\left(e^{\frac{\theta}{2^{k+1}}} + e^{\frac{-\theta}{2^{k+1}}}\right), \quad v^{(k)} = \sqrt{\frac{1 + v^{(k-1)}}{2}}, \quad k \geq 0, \quad v^{(-1)} > -1$



Basic limit functions for different values of $v^{(-1)} \in \{-0.9, \ -0.5, \ 0.5, \ 0.25, \ 0.45\}$

Cardinal Exponential B-splines

Are B-splines and Exponential B-splines enough?

Beyond B-splines and Exponential B-splines

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- They both have low approximation order;

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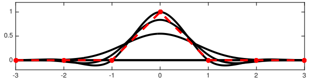
Beyond B-splines and Exponential B-splines

- They both have low approximation order;
- The approximation order of (any) subdivision scheme is strictly connected with its generation/reproduction properties;
- By subdivision we can define pseudo-splines (pol. and exp. pol.) that have higher approximation order and generalize B-splines;

- In both cases the first member of the family of pseudo-splines is a B-splines the last an interpolatory basis function
- Pseudo-splines give a wide range of choices balancing approximation order, length of the support and regularity;
- are good candidate for application in different contexts.



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Happy birthday Michael!