BIG MATHS
A PLAY IN THREE ACTS

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Twenty Years of Biomedical Imaging and Splines (TYBIS)

(and Virginie's Birthday)

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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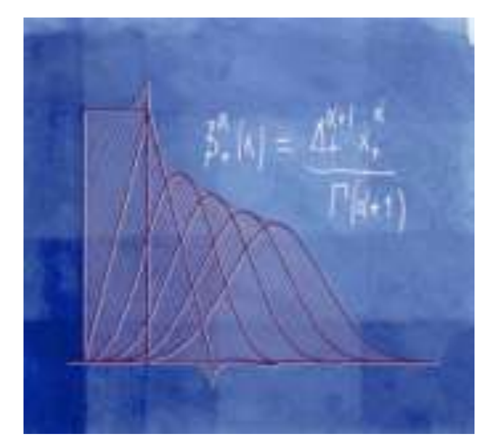


Preamble

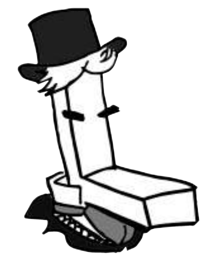
- BIG simply loves maths.
- 20 years in 20 minutes: 1/500 000
- The BIG question: How to model signals?
- We do *applied* and even *applicable* mathematics.
- “We are a continuous lab.” - M. Unser


$Ls = w$
A Play in
Three Acts
Michael Unser's
BIG Maths

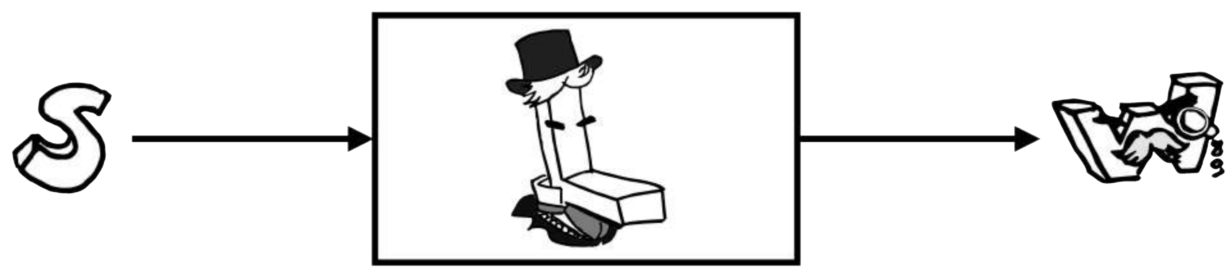
The game of the Innovation and the Operator

The characters:

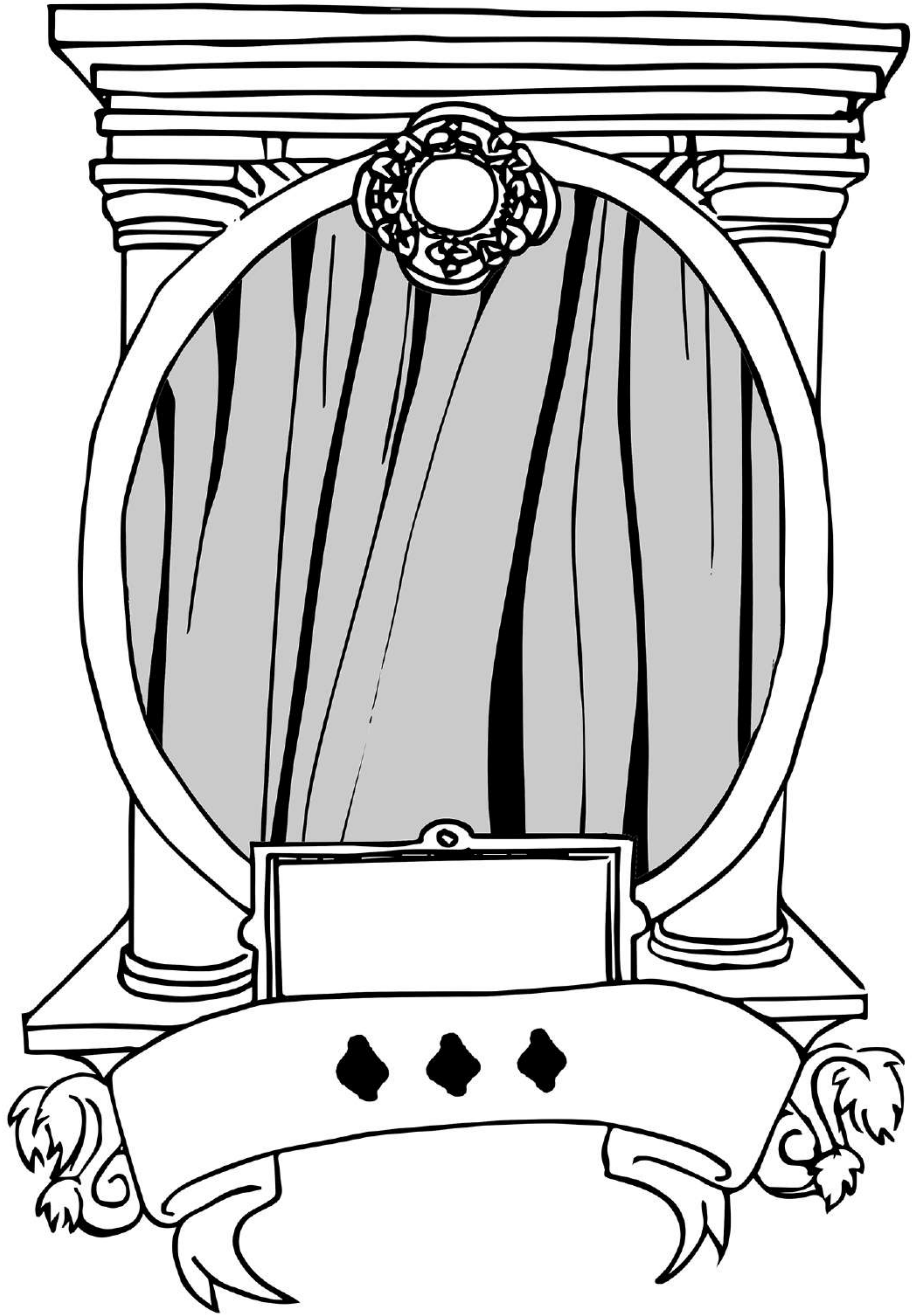
the Innovation 

the Operator 

The story of how they combine to model signals



$$\mathrm{L}s = w$$

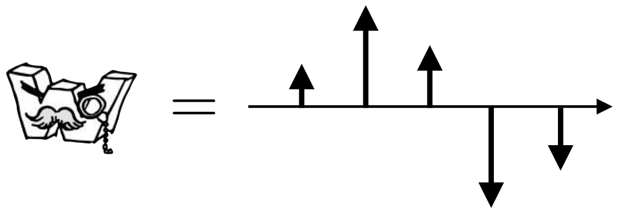


First Act
1997 - 2007
Splines are Everywhere

From Schoenberg to Unser

A function s is a uniform L-spline if

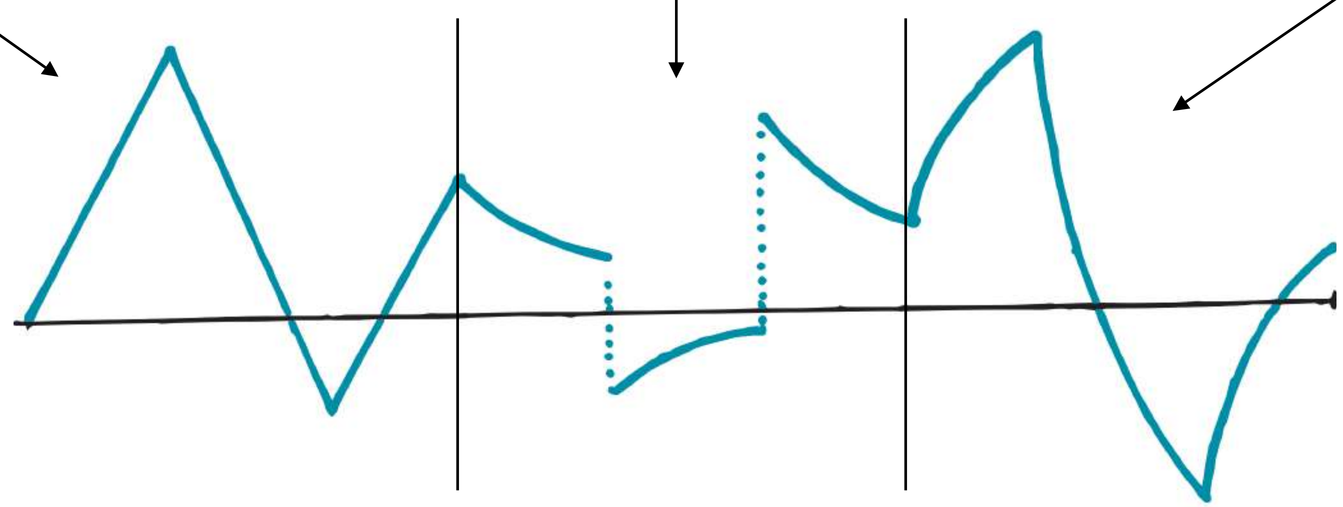
$$\mathrm{L}s = \sum_{k \in \mathbb{Z}} a_k \delta(\cdot - k)$$



$\mathrm{L} = \mathrm{D}^N$
piecewise polynomial
[S'64,U'99,BTU'01]

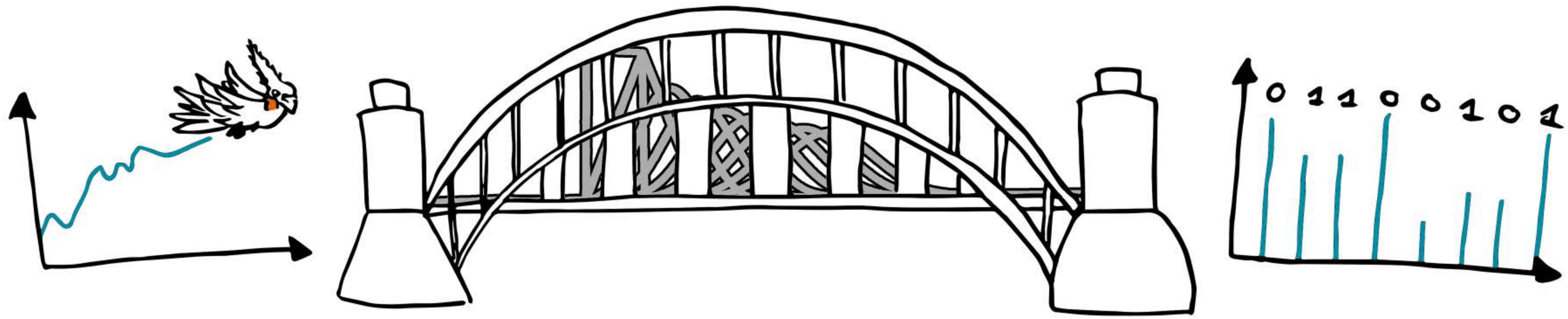
$\mathrm{L} = \mathrm{D} - \alpha \mathrm{I}$
Exponential splines
[UB'05,U'05]

$\mathrm{L} = \mathrm{D}^\gamma$
Fractional splines
[BU'00,BU'07,UB'07]

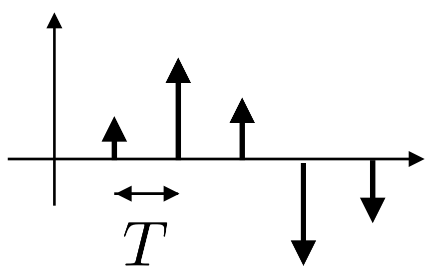


L general
"spline-admissible"
[UB'05]

"AND HERE, YOU SEE! SPLINES!" - AN ENTHUSIASTIC RESEARCHER



- Localization: $\mathrm{L}\text{-splines} = \left\{ s(t) = \sum_{k \in \mathbb{Z}} b_k \beta_L(t-k) \right\}$
- Reproduction properties: polynomials, exponentials
- Approximation power: $\|s - \mathcal{P}_T^{\mathrm{spline}}\{s\}\|_{L_2}$ when $T \to 0$



- Minimum curvature: $\min \|\mathrm{D}^2 s\|_{L_2}$ such that $s(k) = y_k$
- Statistical optimality: MMSE estimator of sampled Gaussian processes
- Minimal support properties: "The shorter, the better." - a spline aficionado

THE BLU-UNSER COLLABORATION

A textbook case: How well can you approximate with splines?

- First order question: The order of approximation?

$$\|s - \mathcal{P}_T^{\text{spline order } N}\{s\}\|_{L_2} \leq C T^N \|s^{(N)}\|_{L_2}$$

[BU'99a,BU'99b]

- Second order question: What is the constant?

$$C = \frac{1}{(2\pi)^N} \sqrt{2 \sum_{k=1}^{\infty} \frac{1}{k^{2N}}}$$

[BU'99b,UB'05]

- Comparison: splines vs. Daubechies wavelets

$$\pi = \lim_{N \to \infty} \frac{T_{\text{spline order } N}}{T_{\text{Daubechies order } N}}$$

[BU'99c]

ARE YOU SPLINE-ADMISSIBLE?

- An operator is spline-admissible if you can do splines with it.

- More precisely?

Definition 1: L is a *spline-admissible* operator of order $r > 1/2$ if and only if the following conditions are met.
1) L is a linear, shift-invariant operator with a frequency response $\hat{L}(\omega)$ such that
... $+\infty$
... unctions $\{\beta(t-k)\}_{k\in\mathbb{Z}}$ form an L_p-stable Riesz ... Specifically, the following two conditions must be ... ed for all $1 \le p \le \infty$:

$$\inf_{\|c\|_{\ell_p}=1} \left\| \sum_{k\in\mathbb{Z}} c[k]\beta(t-k) \right\|_{L_p} > 0$$

and

$$\sup_{\|c\|_{\ell_p}=1} \left\| \sum_{k\in\mathbb{Z}} c[k]\beta(t-k) \right\|_{L_p} < \infty.$$

[UB'07]

Definition 2: L is ... and only if we have ...
1) L is a line... order $r > 1$...
2) L has a well-defined inverse L^{-1} ... sponse $\rho(t)$ is a function of slow g... S'). Thus, L admits $\rho(t)$ as Green'... $\delta(t)$
3) ...
4) ...
5) ...

DEFINITION 6.2 (Spline admissibility) The Fourier-multiplier operator $L: \mathscr{X}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ with frequency response $\widehat{L}(\omega)$ is called *spline admissible* if

$$\rho_L(r) = \mathscr{F}^{-1}\left\{\frac{1}{\widehat{L}(\omega)}\right\}(r)$$

is an ordinary function ...

A linear and continuous operator L from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is *spline-admissible* if
- it is shift-invariant, meaning that

$$L\{\varphi(\cdot - x_0)\} = L\{\varphi\}(\cdot - x_0)$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$; and
- there exists a measurable function of slow growth (bounded by a polynomial) ρ_L such that

$$L\{\rho_L\} = \delta$$

with δ the Dirac delta function. The function ρ_L is a *Green's function* of L. [FUU'17]

Definition 1 (Spline-admissible operator). *A linea... $\mathcal{M}_L(\mathbb{R}^d) \supset \mathcal{S}(\mathbb{R}^d)$ is an appropriate subspace of $\mathcal{S}'(\mathbb{R}$...*
1. *it is shift-invariant;*
2. *there exists a function $\rho_L : \mathbb{R}^d \to \mathbb{R}$ of slow ... that $L\{\rho_L\} = \delta$, where δ is the Dirac impulse. ... $n_0 = \inf\{n \in \mathbb{N} : \rho_L \in L_{\infty,n}(\mathbb{R}^d)\}$.*
3. *the (growth-restricted) null space of L,*

$$\mathcal{N}_L = \{q \in L_{\infty,n_0}(\mathbb{R}^d) : L\{q\} = 0\},$$

has the finite dimension $N_0 \ge 0$.

The native space of L, $\mathcal{M}_L(\mathbb{R}^d)$, is then specified as

$$\mathcal{M}_L(\mathbb{R}^d) = \{f \in L_{\infty,n_0}(\mathbb{R}^d) : \|Lf\|_{\mathcal{M}} < \infty\}.$$

[UFW'17]

ARE YOU SPLINE-ADMISSIBLE?

- An operator is spline-admissible if you can do splines with it.
- More precisely?
- The answer:
  A. Amini, M. Unser, *A universal formula for generalized cardinal B-splines*, Applied and Computational Harmonic Analysis, in press.

Definition 2.3. Let the continuous function $\widehat{L}(\omega) = L(\mathrm{i}\omega)$ be the Fourier multiplier associated with the operator L. We call L "*spline-admissible*" if there exists $0 \leq \Omega \in \mathbb{R}$ such that

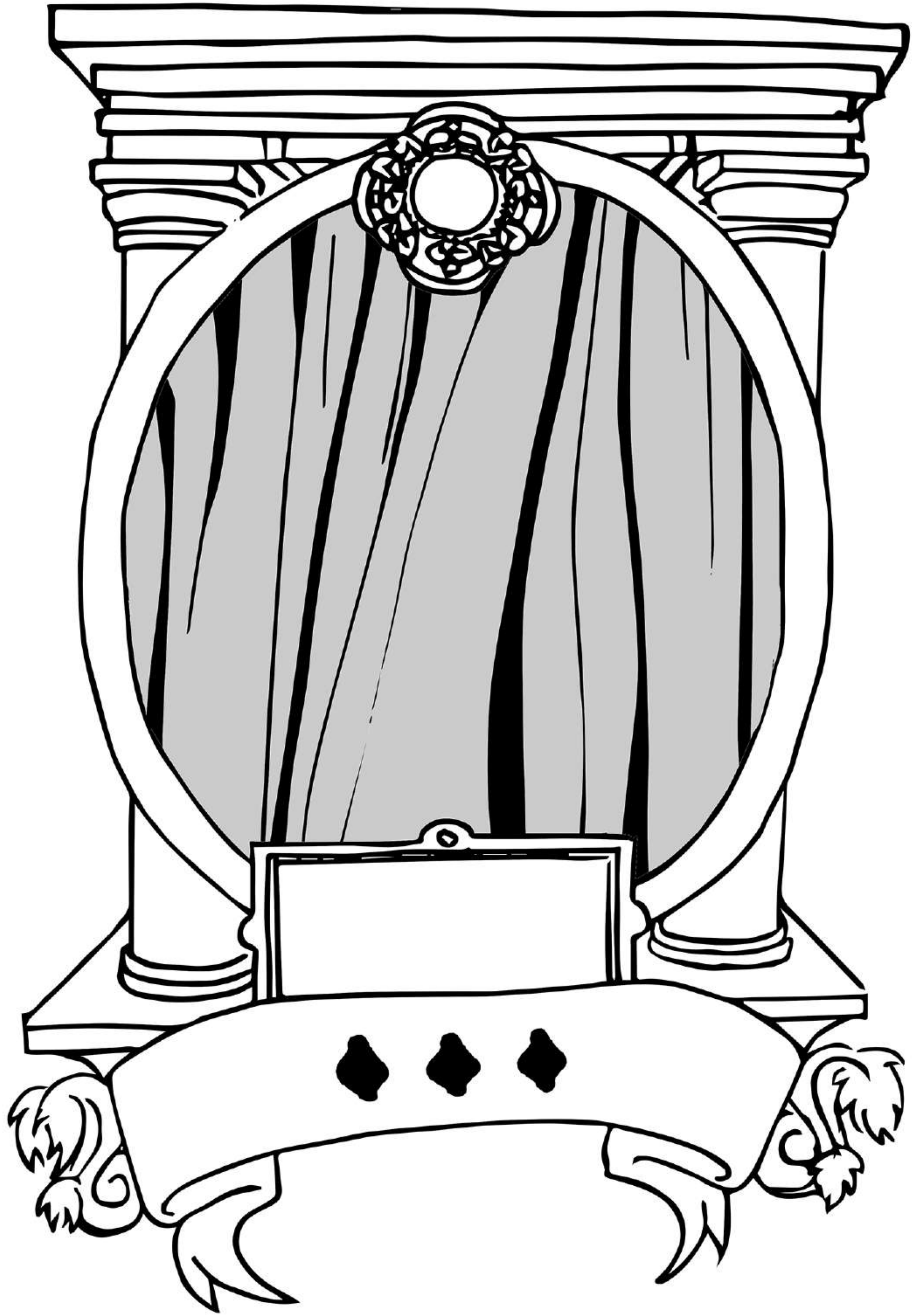
(i) $\widehat{L}(\omega)$ has a finite number of (distinct) zeros, all included in $[-\Omega, \Omega]$, and the spacing between no two of them is an integer multiple of 2π;

(ii) $\widehat{L}(\omega)$ is twice differentiable at all $|\omega| > \Omega$ and there exist constants $0 < c_1, \epsilon_1 \in \mathbb{R}$ such that

$$\forall \omega,\ |\omega| > \Omega : \quad \left| \frac{\mathrm{d}^2}{\mathrm{d}\omega^2} \log \widehat{L}(\omega) \right| \leq \frac{c_1}{|\omega|^{1+\epsilon_1}};$$

(iii) there exist constants $0 < c_2, \epsilon_2 \in \mathbb{R}$ such that

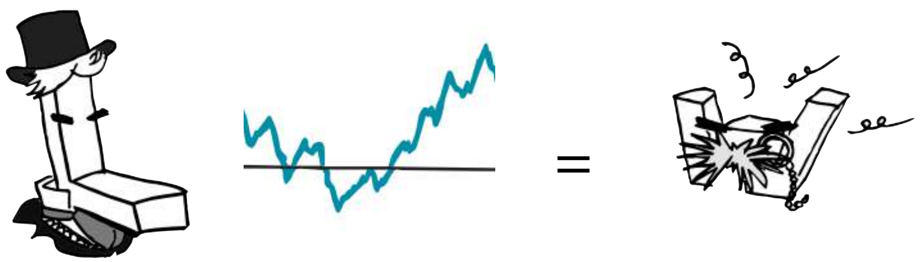
$$\forall \omega,\ |\omega| > \Omega : \quad \left| \widehat{L}(\omega) \right| \geq c_2 |\omega|^{\frac{1}{2}+\epsilon_2}.$$




Second Act
2007 - 2015
Sparse and Random

The Rise and Fall of Gaussian Models

-  = Gaussian white noise



- The Fourier-Gauss-Hilbert realm

    Fourier transforms optimally represent Gaussian processes [U'84]

    Quadratic optimization equivalent to optimal Gauss-based estimation

- But… sparsity!

    Gaussian is not sparse

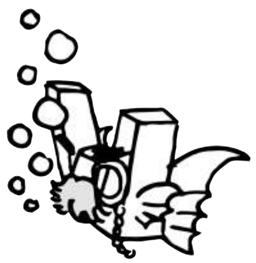
    New mathematical methods: wavelets, FRI, compressed sensing, ….

    Stochastic models for sparse signals?

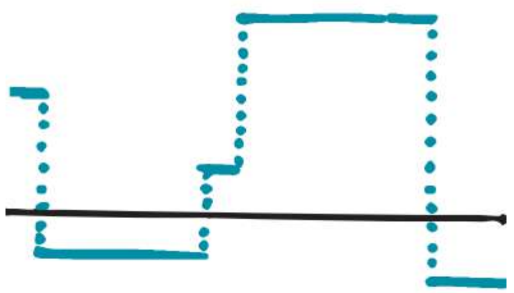
From Piecewise-Smooth to Sparse Processes

- Innovation: random, independent at every point, stationary

 = Lévy white noise

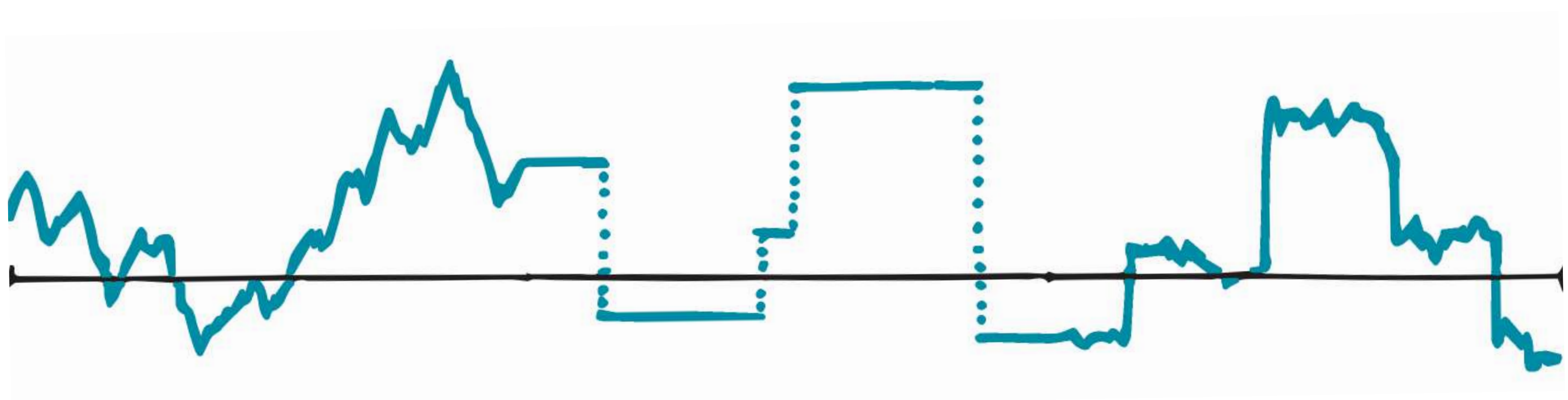
-  = Poisson white noise

  When random processes are splines [UT'11]



- **Definition:** A *sparse stochastic process* satisfies $\mathrm{L}s = w$

$\mathrm{L} = \mathrm{D}$

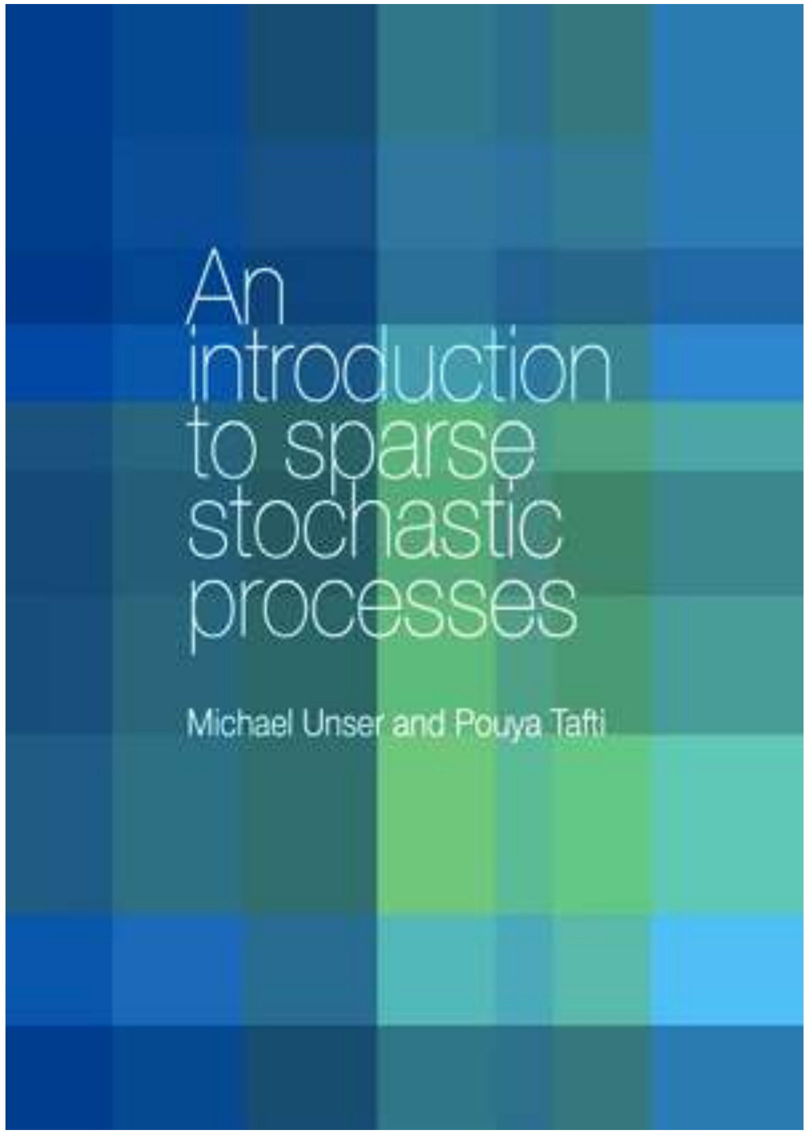


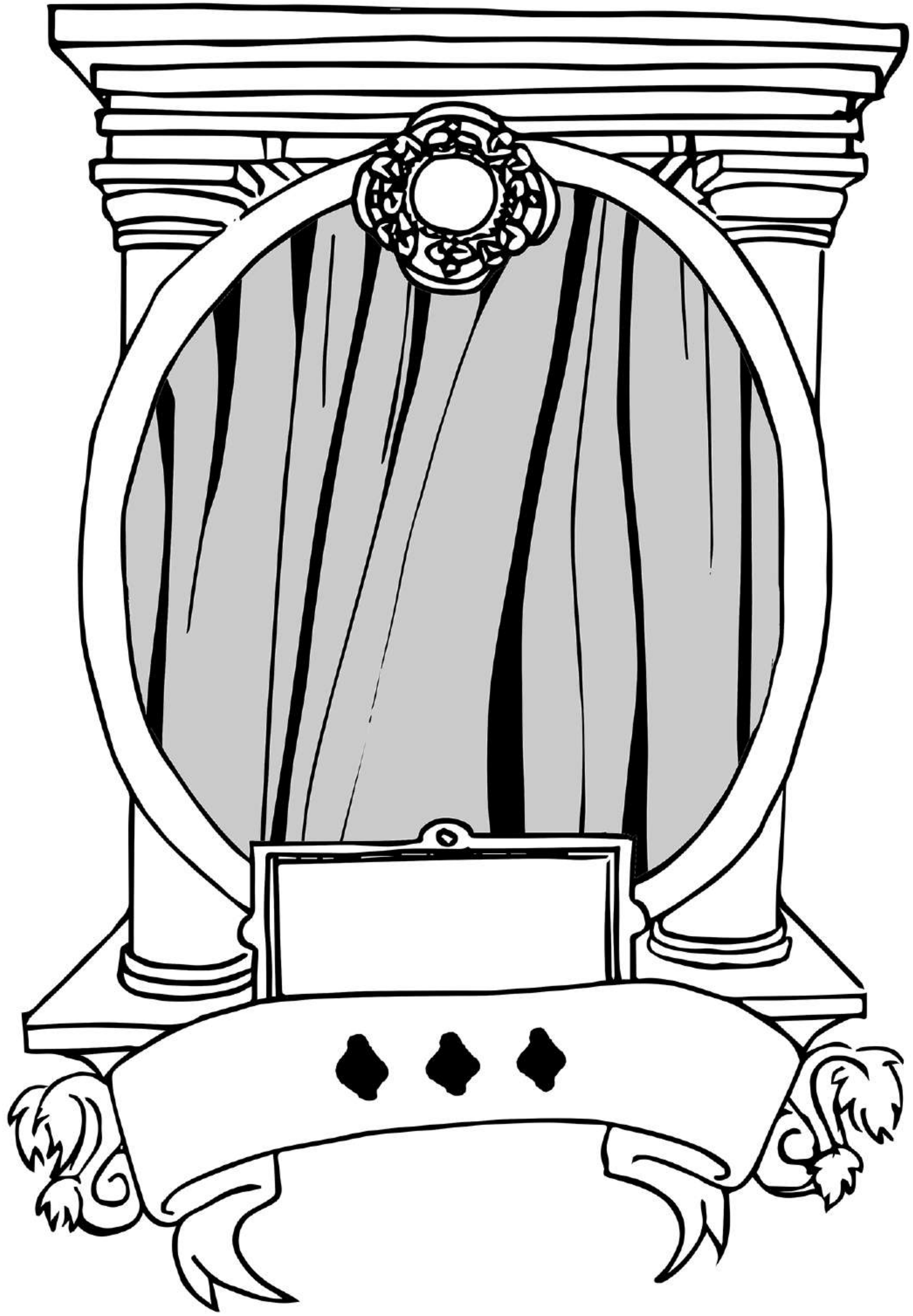
w Gauss Poisson Cauchy

Mathematics of Sparse Stochastic Processes

- *Signal processing:* probabilistic model for sparse signals
- *Mathematics:* analysis of stochastic differential equations driven by a Lévy white noise

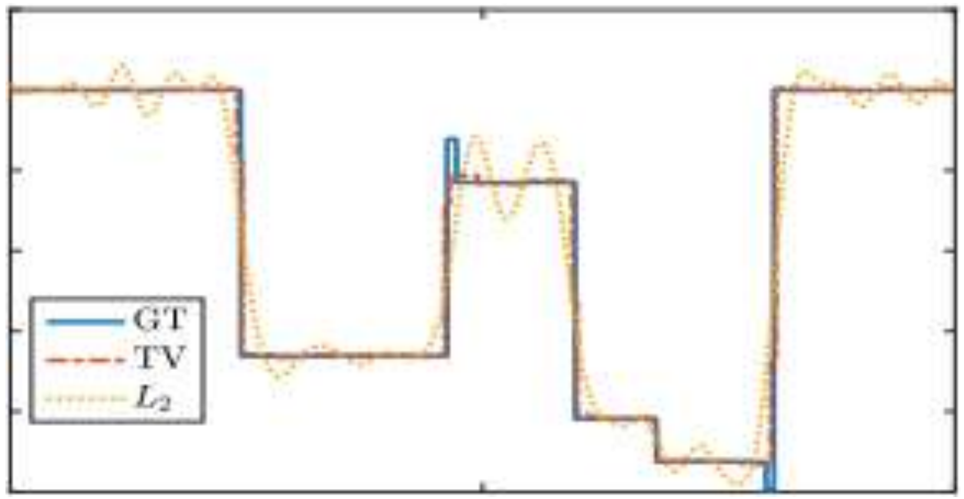
- The foundations: construct your mathematics [UT'14,FAU'14]
- Wavelets are optimal to compress sparse processes [PU'15]
- Estimation methods based on sparse priors [BKNU'13,KPAM'13]
- The mathematical meaning of sparsity [AFU'17,FUW'17]






Third Act
2015 -
The Return of the Splines

Optimization methods for sparse signals



[GFU'18]

Representer Theorem in L_2

$$\min \sum_{m=1}^{M} (y_m - s(t_m))^2 + \lambda \|\mathrm{L}s\|_{L_2}^2$$

$$\mathrm{L}^*\mathrm{L}\{s_{\text{optimal}}\} = \sum_{m=1}^{M} b_m \delta(\cdot - t_m)$$

Representer Theorem in $\mathcal{M}$ (the real L_1)

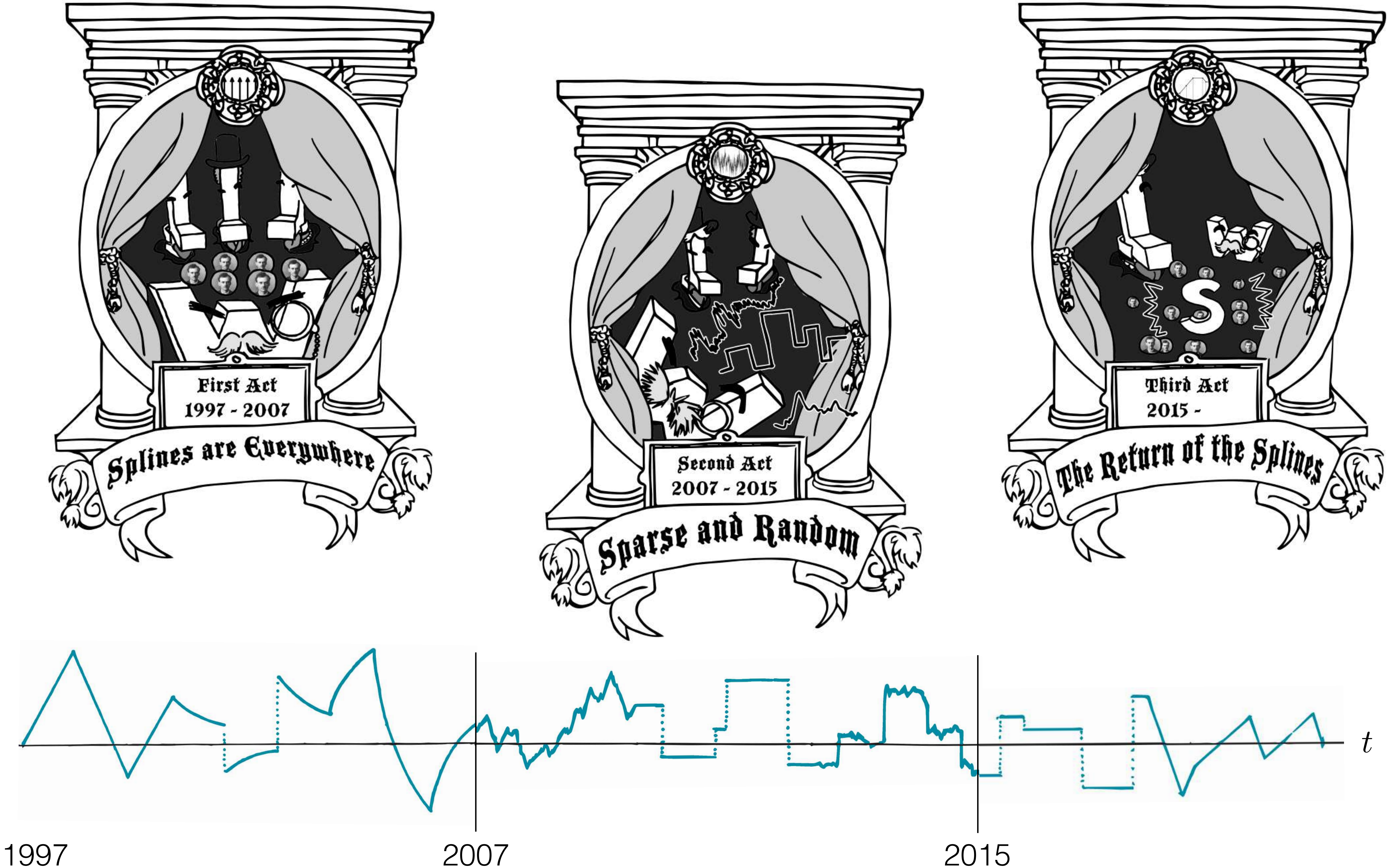
$$\min \sum_{m=1}^{M} (y_m - s(t_m))^2 + \lambda \|\mathrm{L}s\|_{\mathcal{M}}$$

$$\mathrm{L}\{s_{\text{optimal}}\} = \sum_{k=1}^{K} c_k \delta(\cdot - x_k) \qquad K \leq M$$

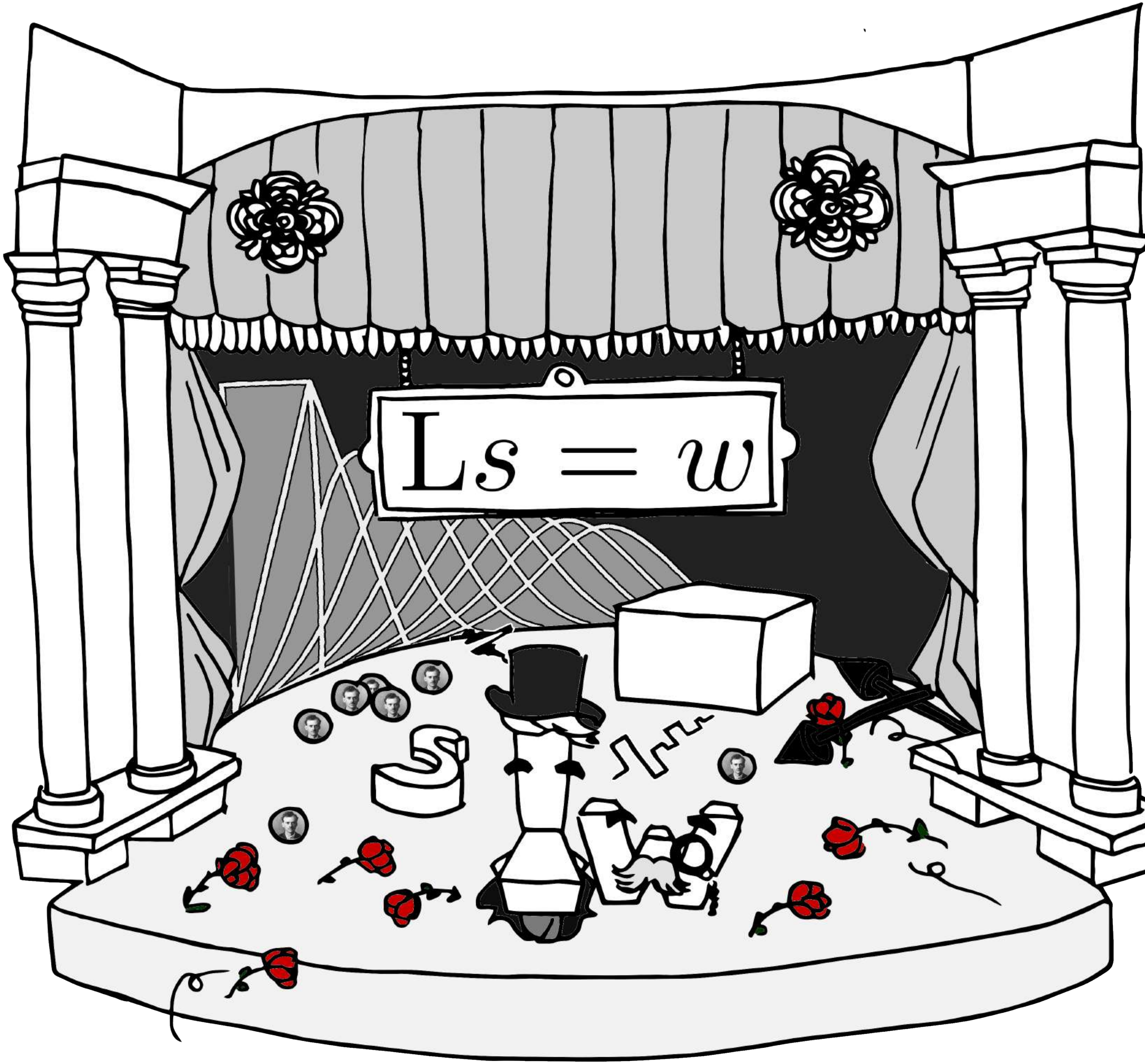
First complete mathematical framework to analyse sparse promoting regularization in continuous-domain [UFW'17,U&co'19]

New algorithms to reconstruct sparse signals
Harshit Gupta, Shayan Aziznejad, Thomas Debarre

- From the particular to the general, and back
- Original contributions to mathematics:
  approximation theory, stochastic processes, functional analysis
- Facing nowadays challenges in signal processing



A unified masterpiece



$Ls = w$