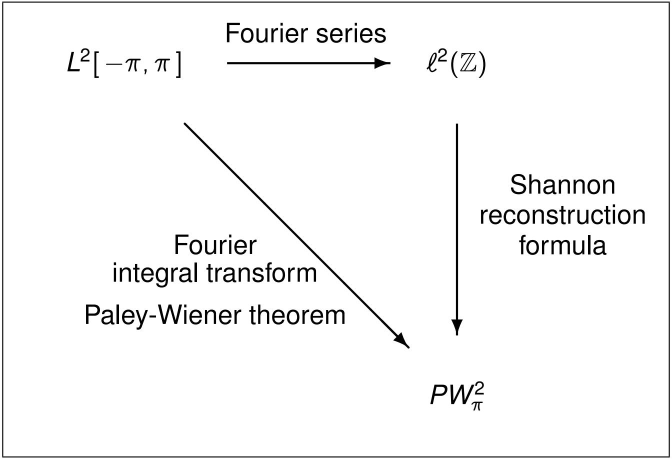
THE commutative diagram of signal processing

Brigitte Forster

Universität Passau, Germany

TYBIS – Twenty Years of Biomedical Imaging and Splines

THE commutative diagram of signal processing



Outline

1. THE commutative diagram of signal processing
2. How about irregular sampling?
3. How about L^p-spaces, $1 < p < \infty$?
4. How about less restrictive growth conditions?
5. How about multidimensional extensions?

1. The commutative diagram of signal processing

Paley–Wiener Theorem

Let $f : \mathbb{R} \to \mathbb{C}$. Then the following are equivalent:

(i) $f \in L^2(\mathbb{R})$ and f can be extended to an entire function of exponential type $\pi > 0$; i.e.,

$$|f(z)| \leq \text{const.}\, e^{\pi|z|} \quad \text{for all } z \in \mathbb{C}.$$

(ii) There exists a function $F \in L^2([-\pi, \pi])$ such that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) e^{izt}\, dt \quad \text{for all } z \in \mathbb{R}.$$

1. The commutative diagram of signal processing

Paley–Wiener space

$f \in PW_\pi^2$, iff $f \in L^2(\mathbb{R})$ and there exists a function $F \in L^2([-\pi, \pi])$ such that

$$f(z) = \int_{-\pi}^{\pi} F(t) e^{izt}\, dt \quad \text{for all } z \in \mathbb{R}.$$

I.E., $f : \mathbb{R} \to \mathbb{C}$ satisfies (ii).

"Band-limited signals of finite energy."

Bernstein space

$B_2([\,-\pi, \pi\,])$ consists entire functions of exponential type at most π which are square-integrable on the real axis, i.e., (i).

Paley-Wiener theorem sais: $\boxed{PW_\pi^2 \cong B_2([\,-\pi, \pi\,])}$

1. The commutative diagram of signal processing

The other two transforms:

Fourier transform

$$\begin{aligned} \widehat{\ } \quad &: \quad L^2([-\pi,\pi]) \to l^2(\mathbb{Z}), \\ \widehat{F}(n) \quad &= \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)e^{-int}\,dt \quad \forall n \in \mathbb{Z}. \end{aligned}$$

Sampling theorem – Shannon reconstruction formula

Every $f \in PW_\pi^2(\mathbb{R})$ can be reconstructed from its samples $\{f(k), k \in \mathbb{Z}\}$ via the cardinal series

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin\pi(t-k)}{\pi(t-k)} \quad \forall t \in \mathbb{Z}.$$

$$\widehat{F}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}.$$

How about irregular sampling?

Replace sin by a **sine-type function** S.

S is an an entire function of exponential type satisfying

$$0 < c < |S(z)| e^{-\pi|\mathrm{Im}\, z|} < C < \infty \quad (\text{when } |\mathrm{Im}\, z| > K)$$

for some constants $c, C, K > 0$.
Λ set of zeros of S.

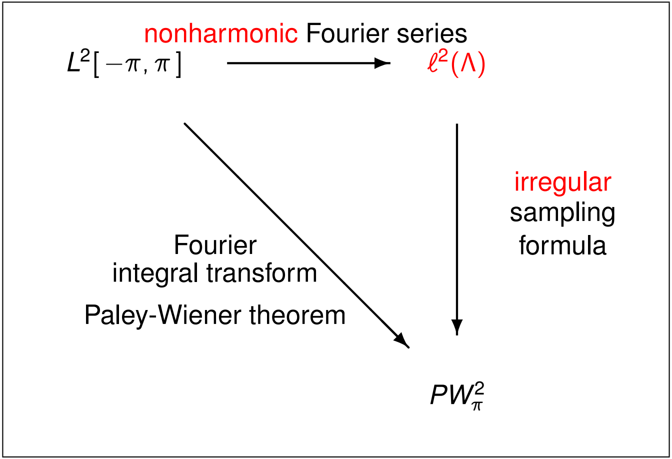
When Λ is a separated set, then

$$f(t) = S(t) \sum_{\lambda \in \Lambda} f(\lambda) \frac{1}{(t-\lambda) S'(\lambda)}.$$

Convergence as for the sampling theorem for $f \in PW_\pi^2$.

[Lewin & Ljubarskii, 1975]

2. Irregular sampling and nonharmonic Fourier series



3. Case $1 < p < \infty$

How about $L^p(\mathbb{R})$ spaces?

- The Hausdorff-Young / Hardy / Titchmarsh / Babenko-Beckner Theorem

$$\|\mathcal{F}(f)\|_q \leq \text{const.}\|f\|_p$$

holds for $1 < p \leq 2$,

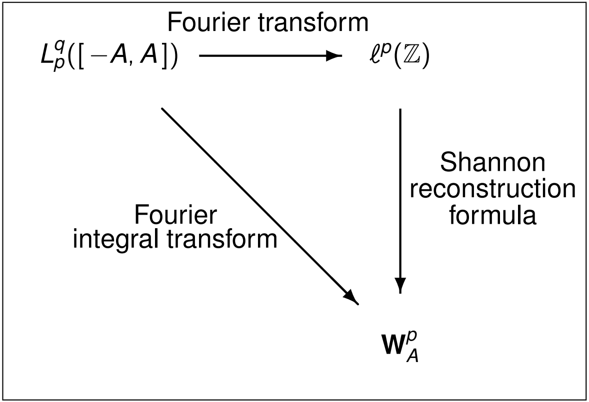
and **fails** for $p > 2$. ($\frac{1}{p} + \frac{1}{q} = 1$).

- **But there is hope:**

Denote $L^q_p([-A, A])$ the class of functions F with

  - $F \in L^q([-A, A])$
  - with Fourier coefficients in ℓ^p, $\frac{1}{p} + \frac{1}{q} = 1$.

3. Case $1 < p < 2$



[Maergoiz, 2006. All mappings are isomorphisms.]

4. How about less restrictive growth conditions?

- Entire function of exponential type at most τ:

$$|f(z)| \leq \text{const.}\, e^{\tau|z|}.$$

- **Indicator function** of f

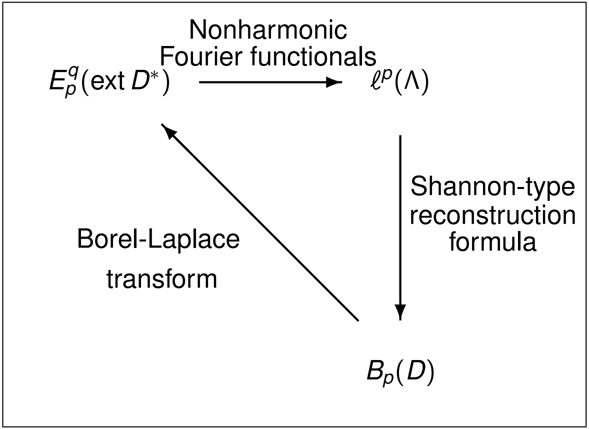
$$h_f(\theta) := \limsup_{r\to\infty} \frac{1}{r} \ln |f(re^{i\theta})|.$$

  represents the **growth exponent in direction θ.**
- **Indicator function** is the supporting function of a convex set D

$$k_K(\theta) := \max_{z\in K} \mathrm{Re}\,(ze^{-i\theta}) = h_f(\theta).$$

Note: The analytic growth of these functions is related to the geometric shape of some convex set!

4. Less restrictive growth conditions



(Lewin/Ljubarskii 1975 ($p = 2$), Sedletskii 1978 ($1 < p < \infty$), F./Semmler 2015 (continuity for $1 < p < 2$), F.)

5. Multidimensional Extension

How to extend to higher dimensions?

Which Fourier transform should we choose?

- Tensoring – no interaction between coordinates
- For color image processing: Inserting the action of quaternions. [Sangwine, Bülow/Sommer, . . .]
- General approach: Clifford algebras – Cauchy-Riemann-type interaction [Brackx et al.]

Main ingredient: entire functions $\rightarrow$ monogenic functions.

$\Rightarrow$ Clifford-Fourier transform, Clifford-sampling, Clifford band-limited functions, . . .

Is there a commutative diagram of signal processing for the multivariate Clifford setting?

Clifford algebras

- Start with $e_1, \ldots, e_n$ orthonormal basis of $\mathbb{R}^n$.
- **Just-do-it multiplication in $\mathbb{R}^n$:**

$$e_j e_k = -e_k e_j \quad \text{and} \quad e_j^2 = -1 \quad (\text{and} \quad \overline{e_j} = -e_j).$$

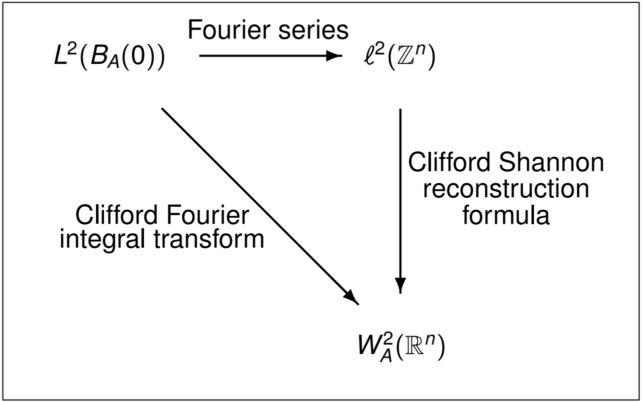
  yields a Clifford algebra $\mathbb{R}_n$ or $\mathbb{C}_n$ of dimension 2^n.

- Basis elements of the Clifford algebra:

$$e_{a_1,\ldots,a_d} := e_{a_1} \cdots e_{a_d}$$

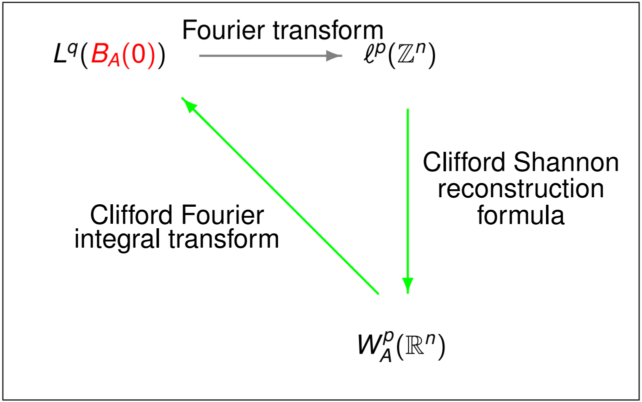
  with $\{a_1, \ldots, a_d\} \subset \mathcal{P}(\{1, \ldots, n\})$, $a_1 < a_2 < \ldots < a_d$.

5. Multidimensional Extension



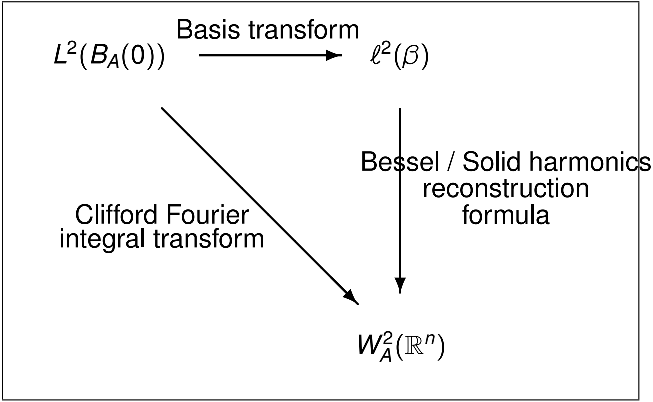
[Kou/Qian 2002; Franklin/Hogan/Larkin 2017]

5. Multidimensional Extension



$(1 < p < 2$, F./Hogan)

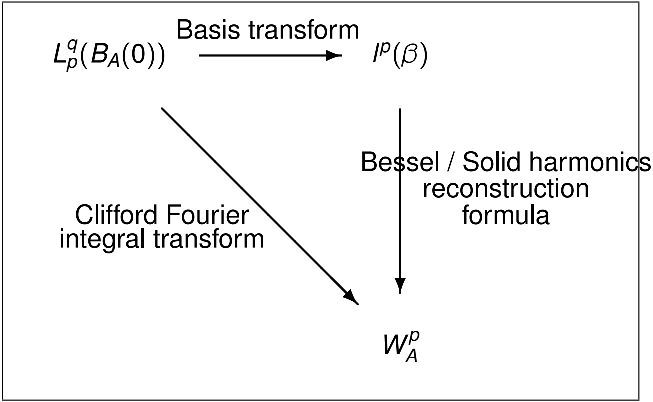
5. Multidimensional Extension



β set of scaled zeros of Bessel functions.
(Sampling: Kou, Qian, Sommen 2007; Basis: F. / Hogan)

5. Multidimensional Extension

Conjecture:

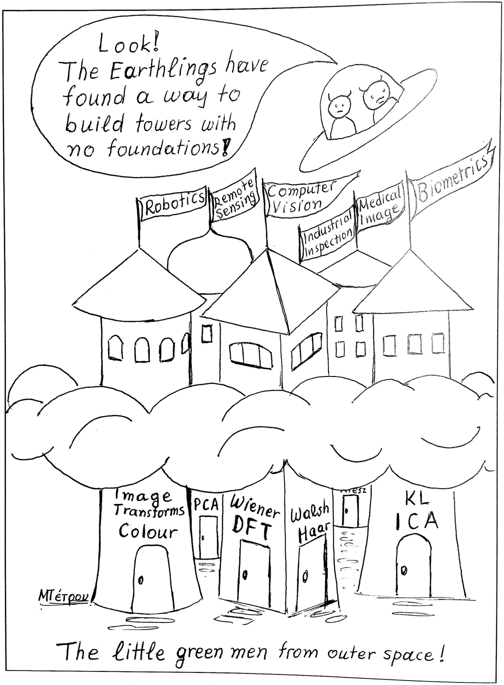


β set of scaled zeros of Bessel functions.
$1 < p < 2$.

Conclusions

- THE commutative diagram of signal processing has many aspects
- For $1 < p < 2$, ℓ^p-summability of Fourier coefficients required
- Multivariate extension via Clifford-valued functions
- Case $1 < p < 2$ needs further characterization of the Fourier series pre-image

- Open question: Directional growth for monogenic functions
- Another open question: Irregular sampling for monogenic functions



The little green men from outer space!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