Twenty years of Biomedical imaging and splines
EPFL, Lausanne, 23 March 2018

High-Level B-Splines and Positive-Definite Partition of Unity

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We will see that not all B-splines are positive functions

and

We will claim that a partition of the unity may be more efficient if "definite-positive" instead of positive

Part 1:
Various B-splines

usual **Cardinal cubic B-spline** (with regular knots)

Definition :

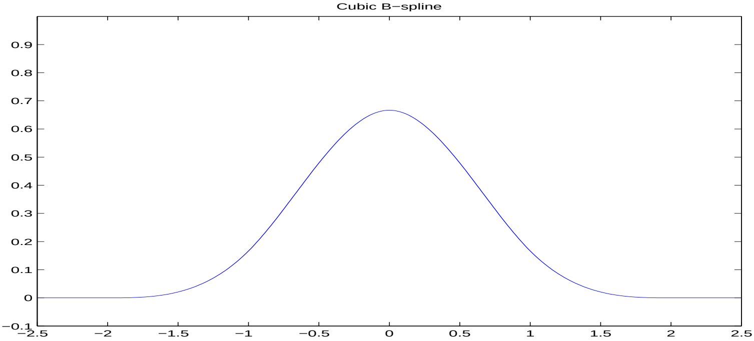
Let $\varphi = \frac{1}{12}\,|\,\blacksquare\,|^3$; then $\varphi^{(4)} = \mathrm{Dirac}$, and so $\widehat{\varphi}(\omega) = \frac{1}{\omega^4}$

Let δ_h^2 such that for any $f \in \mathcal{F}(\boldsymbol{R})$,

$$\delta_h^2 f = h^{-2}\,(f(\blacksquare - h) - 2f + f(\blacksquare + h)), \text{ and let } \delta_h^4 f = \delta_h^2(\delta_h^2 f)$$

Then $\widehat{\delta_h^2 f}(\omega) = -4h^{-2}\sin^2\frac{h\omega}{2}\,\widehat{f}(\omega)$

Now let define B, by $\boxed{B = h\,\delta_h^4\,\varphi} \quad = h\,\delta_h^4\,\mathrm{D}^{-4}\mathrm{Dirac}$



Some properties of cardinal cubic B-splines

$$\boxed{\widehat{B}(\omega) = h\,\left(\frac{2\,\sin\frac{h\omega}{2}}{h\omega}\right)^4}$$

...easy to prove : $\widehat{\varphi}(\omega) = \frac{1}{\omega^4}$,
and $\widehat{\delta_h^4 f}(\omega) = (-4\,h^{-2}\sin^2(\frac{\omega h}{2}))^2\,\widehat{f}(\omega))$

Partition of unity: $\forall x \in \boldsymbol{R},\ B(x) \geq 0$ and $\sum_{i\in\mathbb{Z}} B(x - i\,h) = 1$

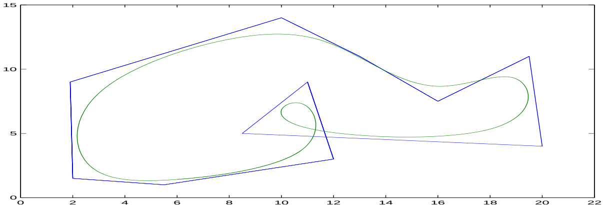
B **is a positive definite function** ("pdf") (connected to $\widehat{B}(\omega) \geq 0$) :

$$\forall n \in \mathbb{N}\ ,\ \forall(\lambda_j)_{j=1:n} \in \boldsymbol{R}^n\ ,\ \forall(x_j)_{j=1:n} \in \boldsymbol{R}^n\ ,\ \sum_{j,k=1:n} \lambda_j\,\lambda_k\,B(x_j - x_k) \geq 0$$

Multiscale equation: $B(x/h) = \sum_{j=-2:2} a_j\,B(2x/h - j)$

Riesz Basis: $c\,\|y\|_2 \leq \left\|\sum_{j\in\mathbb{Z}} y_j\,B(\,\bullet/h - h)\right\|_{L^2(\boldsymbol{R})} \leq C\,\|y\|_2$

Curve construction: $C(t) = \sum_{i=1:n} P_i\,B(t/h - i)$



Order m (degree $2m-1$) cardinal B-splines

Very similar

Let $\varphi = \frac{1}{2\,(2m-1)!}\,|\,\blacksquare\,|^{2m-1}$; then $\varphi^{(2m)} = \mathrm{Dirac}$, and so $\widehat{\varphi}(\omega) = \frac{1}{\omega^{2m}}$

Let define B_m, by $\boxed{B = h\,\delta_h^{2m}\,\varphi} \quad = h\,\delta_h^{2m}\,\mathrm{D}^{-2m}\mathrm{Dirac}$

Some properties

$$\boxed{\widehat{B}_m(\omega) = h\,\left(\frac{2\,\sin\frac{h\omega}{2}}{h\omega}\right)^{2m}}$$

(...same proof as for cubic B-spline)

Partition of unity: $\forall x \in \boldsymbol{R},\ B(x) \geq 0$ and $\sum_{i\in\mathbb{Z}} B(x - i\,h) = 1$

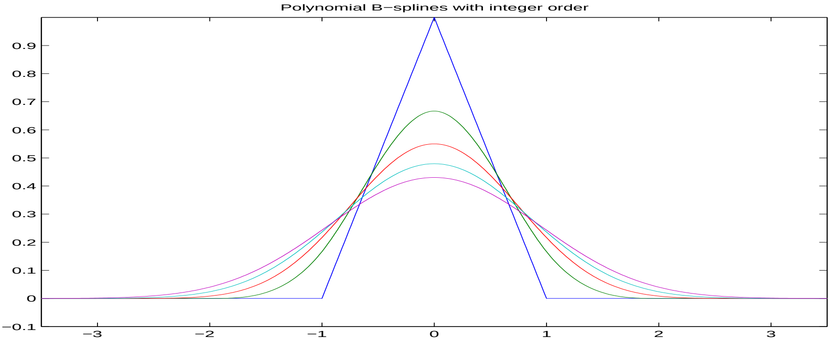
B **is a positive definite function** ("pdf") (connected to $\widehat{B}(\omega) \geq 0$) :

$$\forall n \in \mathbb{N}\ ,\ \forall(\lambda_j)_{j=1:n} \in \boldsymbol{R}^n\ ,\ \forall(x_j)_{j=1:n} \in \boldsymbol{R}^n\ ,\ \sum_{j,k=1:n} \lambda_j\,\lambda_k\,B(x_j - x_k) \geq 0$$

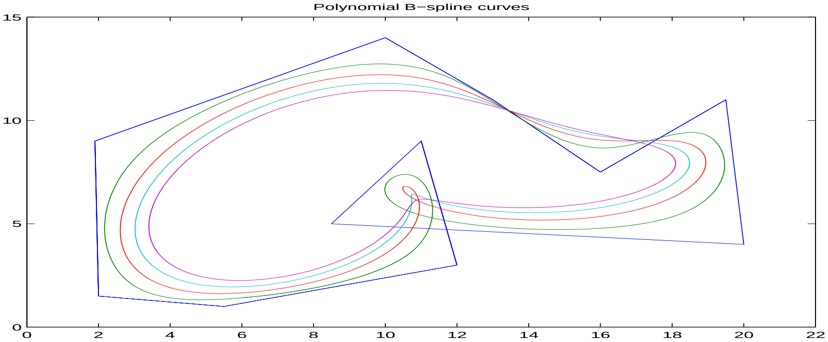
Multiscale equation: $B_m(x) = \sum_{j=-m}^{m} a_j\,B(2x - j)$

Riesz Basis: $c_m\,\|y\|_2 \leq \left\|\sum_{j\in\mathbb{Z}} y_j\,B_m(\,\blacksquare - jh)\right\|_{\mathrm{L}^2(\boldsymbol{R})} \leq C_m\,\|y\|_2$

Curve construction...



Polynomial B−splines with integer order
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−3
−2
−1
0
1
2
3



Polynomial B−spline curves
15
10
5
0
0
2
4
6
8
10
12
14
16
18
20
22

Order $\alpha \in R_+$ cardinal B-spline ... that's BIG, isn't it ?

Definition :

Let $\alpha \in \boldsymbol{R}_+$, $\alpha > 1/2$. Let φ_α be such that $\widehat{\varphi_\alpha}(\omega) = \frac{1}{\omega^{2\alpha}}$

φ_α meets $\varphi_\alpha = \begin{cases} c_\alpha |\,\blacksquare\,|^{2\alpha-1} \ln |\,\blacksquare\,|^2 & \text{if } \alpha \text{ is a half integer number} \\ c_\alpha |\,\blacksquare\,|^{2\alpha-1} & \text{else} \end{cases}$

Let now B_α be defined by $\boxed{\widehat{B_\alpha}(\omega) = h \left(\left(\frac{2 \sin \frac{h\omega}{2}}{h\omega} \right)^2 \right)^\alpha}$

Remark: $\boxed{B_\alpha = h\,\delta_h^{2\alpha}\,\varphi_\alpha} \quad = h\,\delta_h^{2\alpha}\,\mathrm{D}^{-2\alpha}\mathrm{Dirac}$

Properties :

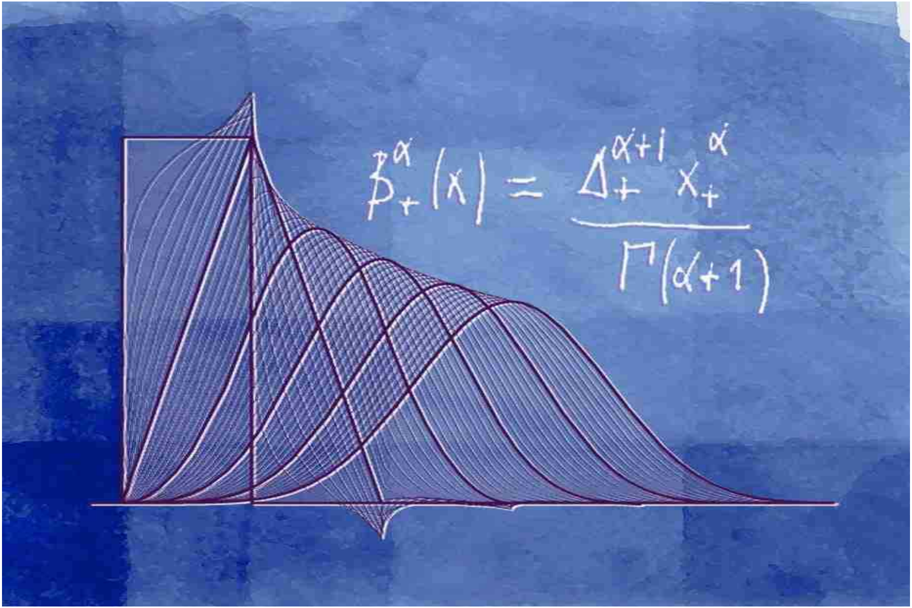
The (B_α) are positive definite functions, but most are not positive:

$$\forall n \in \mathbb{N}\,,\ \forall (\lambda_j)_{j=1:n} \in \boldsymbol{R}^n\,,\ \forall (x_j)_{j=1:n} \in \boldsymbol{R}^n\,,\ \sum_{j,k=1:n} \lambda_j\,\lambda_k\,B_\alpha(x_j - x_k) \geq 0$$

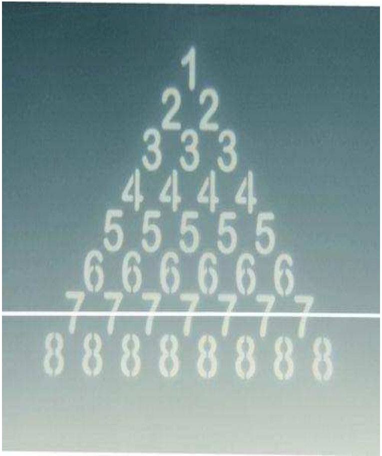
"pdf-partition of unity": $\forall x \in \boldsymbol{R},\ \sum_{i\in\mathbb{Z}} B(x - i\,h) = 1$

Riesz Basis : $c\,\|y\|_2 \leq \left\| \sum_{j\in\mathbb{Z}} y_j\,B_\alpha(\,\blacksquare - jh) \right\|_{\mathrm{L}^2(\boldsymbol{R})} \leq C\,\|y\|_2$

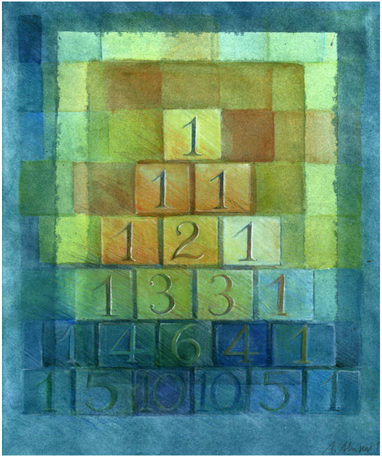
Multiscale equation: $B_\alpha(x) = \sum_{j=-m}^{m} a_j\,B_\alpha(2x - j)$



Fractional B-splines, Annette Unser



Art in Toulouse metro



BIG art, Annette Unser

(elementary) **Cardinal polyharmonic B-splines**

Definition : $m > d/2$, $m\in\mathbb{N}$ or $m\in\boldsymbol{R}_+$; "knots" $h\mathbb{Z}^d$; φ : $\Delta^m\varphi = \text{Dirac}$

$\Delta_h\varphi = h^{-2}\sum_{k=1:d}(\varphi(\bullet - h\,e_k) - 2\varphi + \varphi(\bullet + h\,e_k))$

(Δ_h: "elementary" discretisation of Δ)

$$\Delta_h^m\varphi = \Delta_h\,\Delta_h\,...\Delta_h\varphi \qquad \boxed{B = h^d\,\Delta_h^m\,\varphi} \qquad = h^d\,\Delta_h^m\,\Delta^{-m}\text{Dirac}$$

Properties: $$\boxed{\widehat{B}(\omega) = h^d\left(\frac{2\sum_{k=1:d}\sin^2\frac{h\omega_k}{2}}{\sum_{k=1:d}h\omega_k^2}\right)^m = h^d\left(\frac{\|\sin\frac{h\omega}{2}\|^2}{\|\frac{h\omega}{2}\|^2}\right)^m}$$

...easy to prove : $\Delta^m\varphi = \text{Dirac} \Longrightarrow \widehat{\varphi}(\omega) = \frac{(-1)^m}{\|\omega\|^{2m}}$, and $\widehat{\Delta_h f}(\omega) = -4\,h^{-2}\|\sin(\frac{\omega h}{2})\|^2\,f(\omega)$

$$\text{Support}(B) = \boldsymbol{R}^d \;\;;\;\; \int_{\boldsymbol{R}^d} B(\mathbf{x})\,d\mathbf{x} = \widehat{B}(0) = h^d \;\;;\;\; B(\mathbf{x}) \underset{\|\mathbf{x}\|\to\infty}{=\!=} \mathcal{O}(\|\mathbf{x}\|^{-d-2})$$

B is NOT a positive function, but B is a positive definite function:

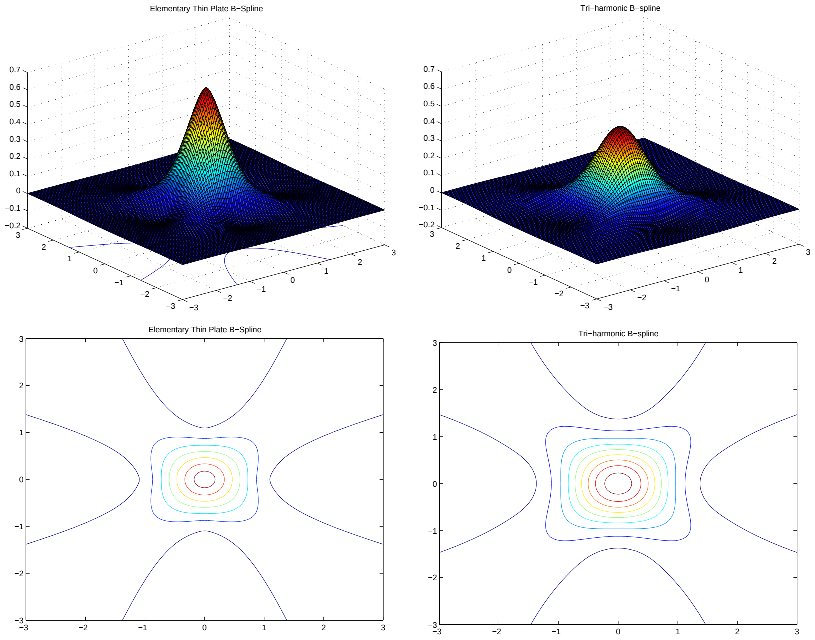
$\forall n\in\mathbb{N}$, $\forall(\lambda_j)_{j=1:n}\in\boldsymbol{R}^n$, $\forall(\mathbf{x}_j)_{j=1:n}\in(\boldsymbol{R}^d)^n$, $\sum_{j,k=1:n}\lambda_j\,\lambda_k\,B(\mathbf{x}_j - \mathbf{x}_k) \geq 0$

pdf-partition of unity: $\forall x\in\boldsymbol{R}^d$, $\sum_{i\in\mathbb{Z}} B(x - i\,h) = 1$

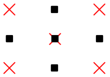
Riesz Basis : $c\,\|y\|_2 \leq \left\|\sum_{j\in\mathbb{Z}^d} y_j\,B(\bullet - j\,h)\right\|_{\mathrm{L}^2(\boldsymbol{R}^d)} \leq C\,\|y\|_2$

Multiscale equation: $B(\mathbf{x}) = \sum_{j\in\mathbb{Z}^d} a_j\,B(2\mathbf{x} - j)$, $a_j \underset{\|j\|\to\infty}{=\!=} \mathcal{O}(\|j\|^{-d-2})$

Thin Plate B-spline ($m = d = 2$) and order 3 B-spline ($d = 2, m = 3$)



Some other Thin plate B-splines $(\mathrm{m{=}d{=}2})$



$$\varphi = \frac{1}{16\pi} \,\|\bullet\|^2 \ln \|\bullet\|^2$$

$$\Delta_{h,axis}\varphi = h^{-2}\left(\varphi(\bullet - h\,e_1) + \varphi(\bullet + h\,e_1) + \varphi(\bullet - h\,e_2) + \varphi(\bullet + h\,e_2) - 4\,\varphi\right)$$

$$\Delta_{h,diag}\varphi = \frac{h^{-2}}{2}\Big(\varphi(\bullet - h(e_1+e_2)\,) + \varphi(\bullet + h(e_1+e_2)\,) + \varphi(\bullet - h(e_1-e_2)\,) + \varphi(\bullet + h(e_1-e_2)\,) - 4\,\varphi\,\Big)$$

Crossed B-spline: $B = h^2\,\Delta_{h,axis}\,\Delta_{h,diag}\,\varphi$

$\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.012$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .42$

Mixed B-spline: $B = \left(\frac{1}{3}B_{elementary} + \frac{2}{3}B_{crossed}\right) = h^2\left(\frac{1}{3}\,\Delta^2_{h,axis} + \frac{2}{3}\Delta_{axis}\Delta_{h,diag}\right)\varphi$

$\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.0017$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .50$

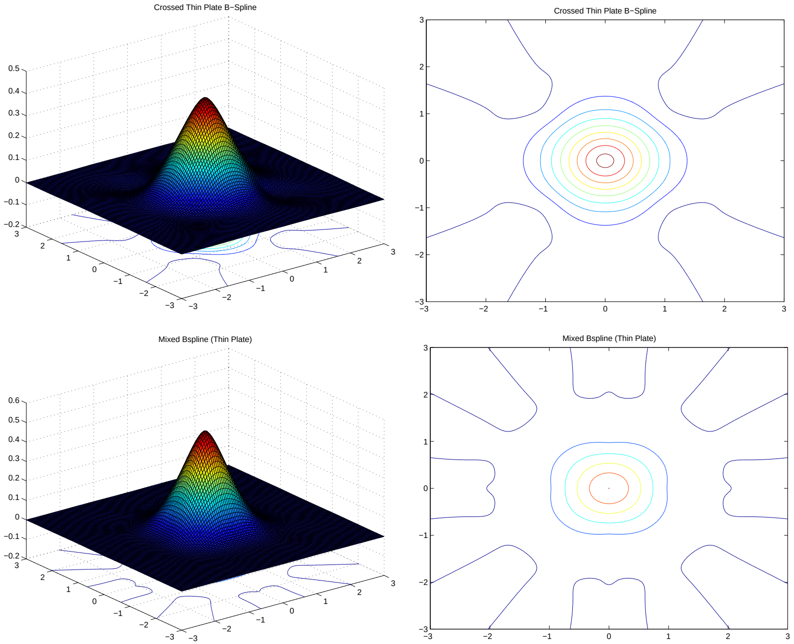
Isotropic B-spline (BIG!): $B = h^2\left(\frac{2}{3}\,\Delta_{h,axis} + \frac{1}{3}\Delta_{h,diag}\right)^2\varphi$

$\widehat{B}(\omega) \underset{\|\omega\|\to 0}{=\!=\!=} 1 + \mathcal{O}(\|\omega\|^4)$; $\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.00089$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .52$

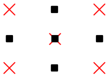
Hexagonal B-spline, or “Bees-spline” :

$$\Delta_{h,hexa}\varphi = \frac{2h^{-2}}{3}\Big(\varphi(\bullet - h\,e_1) + \varphi(\bullet + he_1) + \varphi(\bullet - h(\tfrac{e_1}{2} + \tfrac{e_2\sqrt{3}}{2})) + \varphi(\bullet + h(\tfrac{e_1}{2} + \tfrac{e_2\sqrt{3}}{2})) + \varphi(\bullet - h(\tfrac{e_1}{2} - \tfrac{e_2\sqrt{3}}{2})) + \varphi(\bullet + h(\tfrac{e_1}{2} - \tfrac{e_2\sqrt{3}}{2})) - 6\,\varphi\Big)$$

$B = \frac{h^2\sqrt{3}}{2}\,\Delta^2_{hexa}\varphi$; $\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.0041$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .56$


Crossed Thin Plate B−Spline
Crossed Thin Plate B−Spline
Mixed Bspline (Thin Plate)
Mixed Bspline (Thin Plate)

Some other Thin plate B-splines (m=d=2)



$$\varphi = \frac{1}{16\pi}\, \| \blacksquare \|^2 \ln \| \blacksquare \|^2$$

$$\Delta_{h,axis}\varphi = h^{-2}\left(\varphi(\blacksquare - h\,e_1) + \varphi(\blacksquare + h\,e_1) + \varphi(\blacksquare - h\,e_2) + \varphi(\blacksquare + h\,e_2) - 4\,\varphi\right)$$

$$\Delta_{h,diag}\varphi = \frac{h^{-2}}{2}\Big(\varphi(\blacksquare - h(e_1+e_2)\ ) + \varphi(\blacksquare + h(e_1+e_2)\ ) + \varphi(\blacksquare - h(e_1-e_2)\ ) + \varphi(\blacksquare + h(e_1-e_2)\ ) - 4\,\varphi\ \Big)$$

Crossed B-spline: $B = h^2\,\Delta_{h,axis}\,\Delta_{h,diag}\,\varphi$

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Mixed B-spline: $B = \left(\frac{1}{3}B_{elementary} + \frac{2}{3}B_{crossed}\right) = h^2\left(\frac{1}{3}\,\Delta^2_{h,axis} + \frac{2}{3}\Delta_{axis}\Delta_{h,diag}\right)\varphi$

$\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.0017$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .50$

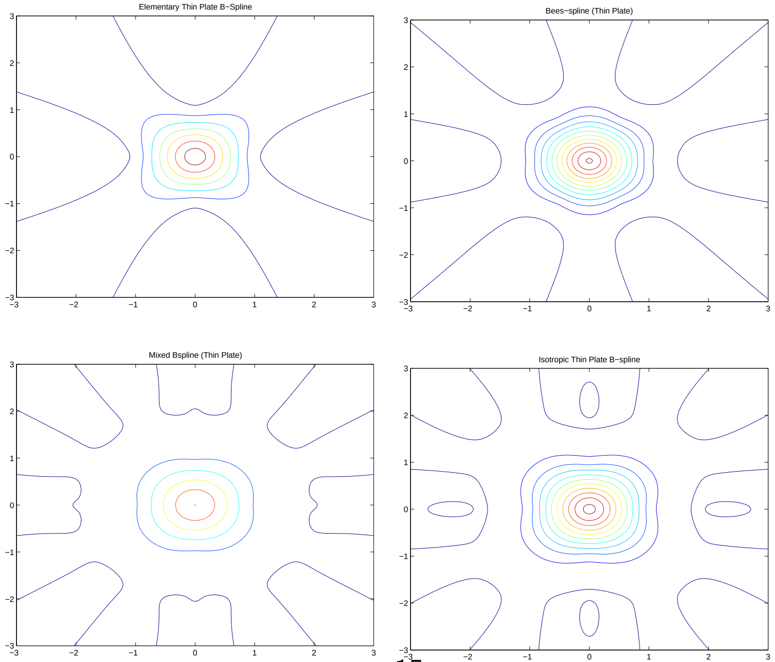
Isotropic B-spline (BIG!): $B = h^2\left(\frac{2}{3}\,\Delta_{h,axis} + \frac{1}{3}\Delta_{h,diag}\right)^2\varphi$

$\widehat{B}(\omega) \underset{\|\omega\|\to 0}{=\!=} 1 + \mathcal{O}(\|\omega\|^4)$; $\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.00089$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .52$

Hexagonal B-spline, or "Bees-spline" :

$$\Delta_{h,hexa}\varphi =$$
$$\frac{2h^{-2}}{3}\Big(\varphi(\blacksquare - h\,e_1) + \varphi(\blacksquare + he_1) + \varphi(\blacksquare - h(\tfrac{e_1}{2} + \tfrac{e_2\sqrt{3}}{2})) + \varphi(\blacksquare + h(\tfrac{e_1}{2} + \tfrac{e_2\sqrt{3}}{2})) + \varphi(\blacksquare - h(\tfrac{e_1}{2} - \tfrac{e_2\sqrt{3}}{2})) + \varphi(\blacksquare + h(\tfrac{e_1}{2} - \tfrac{e_2\sqrt{3}}{2})) - 6\,\varphi\Big)$$

$B = \frac{h^2\sqrt{3}}{2}\,\Delta^2_{hexa}\varphi$; $\min_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq -.0041$; $\max_{\mathbf{x}\in R^2} B(\mathbf{x}) \simeq .56$


Elementary Thin Plate B−Spline
Bees−spline (Thin Plate)
Mixed Bspline (Thin Plate)
Isotropic Thin Plate B−spline
−3 −2 −1 0 1 2 3
3 2 1 0 −1 −2 −3

To compute polyharmonic B-splines

Integer m :

Explicit value possible : $B(x) = h^d\, \Delta_1^m \varphi_m(\mathbf{x})$

Cancellation effect (not a too big problem, actually)

since $B(\mathbf{x}) = \sum \lambda_i\, \varphi(\mathbf{x} - \mathbf{x}_i)$ with $\sum \lambda_i = \sum \lambda_i\, \mathbf{x}_i = 0$ and $\varphi(\mathbf{x}) \xrightarrow[\|\mathbf{x}\| \to \infty]{} \infty$

(and even more : $\forall q \in \boldsymbol{P}_{m-1}$, $\sum \lambda_i\, q(\mathbf{x}_i) = 0$)

$\mathcal{O}(n)$ operations for the computation of n values of $B(\mathbf{x})$; scattered values possible.

Any m (real or integer) :

Use $\widehat{B}$ and FFT ! (**BIG** way !)

$$\widehat{B}(\omega) = \quad h^d \left(\frac{\| \sin \frac{h\omega}{2} \|^2}{\| \frac{h\omega}{2} \|^2} \right)^m = h^d \left(\frac{2 \sum_{k=1:d} \sin^2 \frac{h\omega_k}{2}}{\sum_{k=1:d} h\omega_k^2} \right)^m$$

$$\boxed{\widehat{b}_j = \widehat{B}(j/h) \qquad \text{and} \qquad b_j = B(jh) = \left(FFT(\widehat{b}) \right)_j}$$

Easy even for non-integer m. No cancellation effect

$\mathcal{O}(n \ln n)$ operations for the computation of n values of $B(\mathbf{x})$.

Values must be on a regular grid.

Please remember from this first part

There are lots of B-splines; they are obtained by applying a discretization of a differential operator (as D^4, D^{2m}, Δ^{2m}...) to the fundamental solution of it (even for real m).

Only "elementary" polynomial B-splines are positive functions

All B-splines are positive defined functions,
and form pdf-partitions of unity.

Their Fourier transform is a quite simple one

Part 2

High level B-splines

and

rate of convergence of B-spline approximation

High level cubic B-splines

Idea: Use a "better" approximation of D^2 :

$h^{-2}\,\delta_h^2 f = h^{-2}\,(f(\blacksquare - h) - 2f + f(\blacksquare + h))$ is $\boldsymbol{P}_3$-exact

$h^{-2}\,\delta_{h,5}^2 f = h^{-2}\,(\delta_h^2 f - \frac{1}{12}\,\delta_h^4 f)$ is $\boldsymbol{P}_5$-exact

As a consequence, $\boldsymbol{P}_7$-exact approximations of f^{iv} :

$$h^{-4}\,(\delta_{h,5}^2)^2 f = h^{-4}\,(\delta_h^4 f - \frac{1}{6}\,\delta_h^6 f + \frac{1}{144}\,\delta_h^8 f)$$

$$h^{-4}\,\delta_{4,h,7} f = h^{-4}\,(\delta_h^4 f - \frac{1}{6}\,\delta_h^6 f)$$

Definition: *Let* $\varphi = \frac{|\blacksquare|^3}{12}$ *(then* $D^4\varphi = \mathrm{D}irac$ *and* $\widehat{\varphi}(\omega) = \frac{1}{\omega^4}$*)*

We call level 1 cubic B-spline the functions

$$B^1 := h\,h^{-4}\,\delta_{4,h,7}\varphi = h\,h^{-4}\,\delta_{4,h,7} D^{-4}\mathrm{D}irac$$

$$C^1 := h\,h^{-4}\,(\delta_{h,5}^2)^2\varphi = h\,h^{-4}\,(\delta_{h,5}^2)^2 D^{-4}\mathrm{D}irac$$

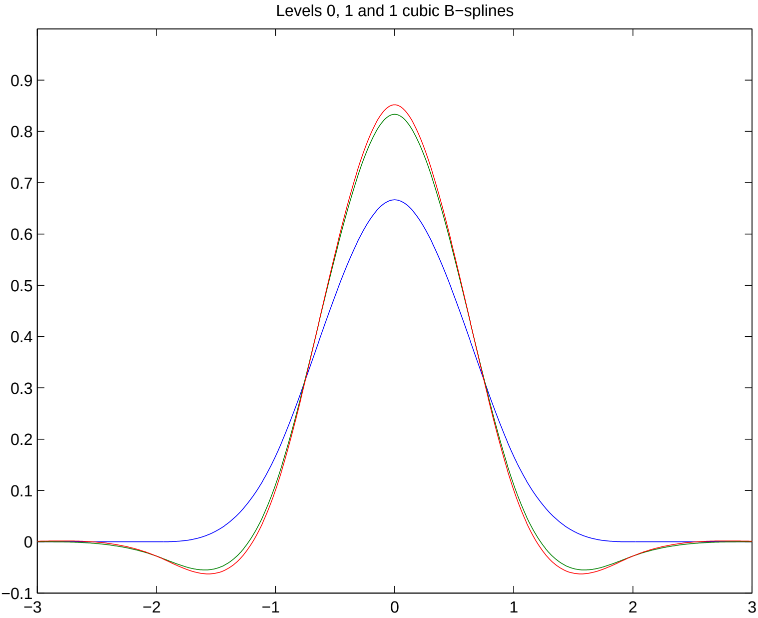
Properties: Vanishing moment: $\int_R x^2 B^1(x)dx = \int_R x^2 C^1(x)dx = 0$

Let $u(\omega) = \left(\frac{2\sin(\frac{h\omega}{2})}{h\omega}\right)^2$; $\widehat{B^1}(\omega) = h\left(u(\omega)^2 + \frac{(h\omega)^2}{6}u(\omega)^3\right) = \widehat{B}(\omega)\left(1 + \frac{(h\omega)^2}{6}u(\omega)\right)$

$\widehat{C^1}(\omega) = h\left(u(\omega)^2 + \frac{(h\omega)^2}{6}u(\omega)^3 + \frac{(h\omega)^4}{144}u(\omega)^4\right) = \widehat{B}(\omega)\left(1 + \frac{(h\omega)^2}{6}u(\omega)) + \frac{(h\omega)^4}{144}u(\omega)^2\right)$

The level 1 cubic B-splines are NOT positive functions,
but form pdf-partitions of unity

Level 1 cubic B-splines $m = 2$



Level ℓ m-harmonic B-spline : same idea!

Approximation of Δ^m

$\Delta_{h,5} f = h^{-2} \left(\delta^2_{h\,e_1} + \delta^2_{h\,e_2} - \frac{1}{6}(\delta^4_{h\,e_1} + \delta^4_{h\,e_2})\right) f$ is $\boldsymbol{P}_5$-exact

$$\left(\widehat{\Delta_{h,5}}\right)^m f(\omega) = h^{-2m} \left(\left(2(\sin^2(\tfrac{h\omega_1}{2}) + \sin^2(\tfrac{h\omega_2}{2})) + \tfrac{4}{6}(\sin^4(\tfrac{h\omega_1}{2}) + \sin^4(\tfrac{h\omega_2}{2})) \right)^2 \right)^{\frac{m}{2}} \widehat{f}(\omega)$$

Definition of level 1 m-harmonic B-spline

Let $u(\omega) = \dfrac{\left\|\sin(\frac{h\omega}{2})\right\|^2}{\left\|\frac{h\omega}{2}\right\|^2}$ *and* $v(\omega) = \dfrac{\left\|\sin(\frac{h\omega_1}{2})\right\|^4 + \left\|\sin(\frac{h\omega_2}{2})\right\|^4}{\left\|\frac{h\omega}{2}\right\|^4}$

The level 1 m-harmonic B-spline B^1_m is defined

by its Fourier transform: $\boxed{B^1_m(\omega) = h\left(u(\omega) + \frac{\|h\omega\|^2}{24} v(\omega)\right)^m}$

Some properties: Higher levels are possible (useful only for $\ell \leq m-1$)

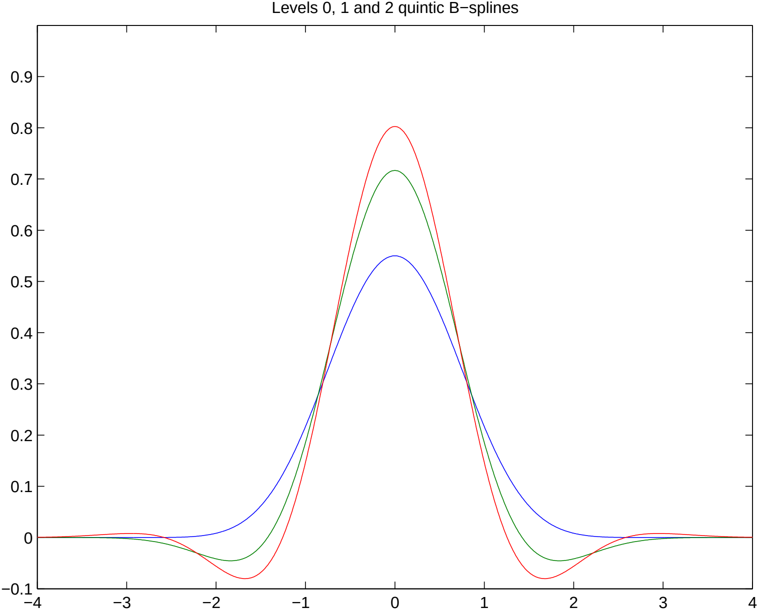
B^ℓ_m is a positive definite function (any $\ell \in \mathbb{N}$)

$$\ell \leq m-1 \Rightarrow B^\ell_m(x) \underset{\|x\|\to\infty}{=\!=} \mathcal{O}(\|x\|^{-2\ell-d-2})$$

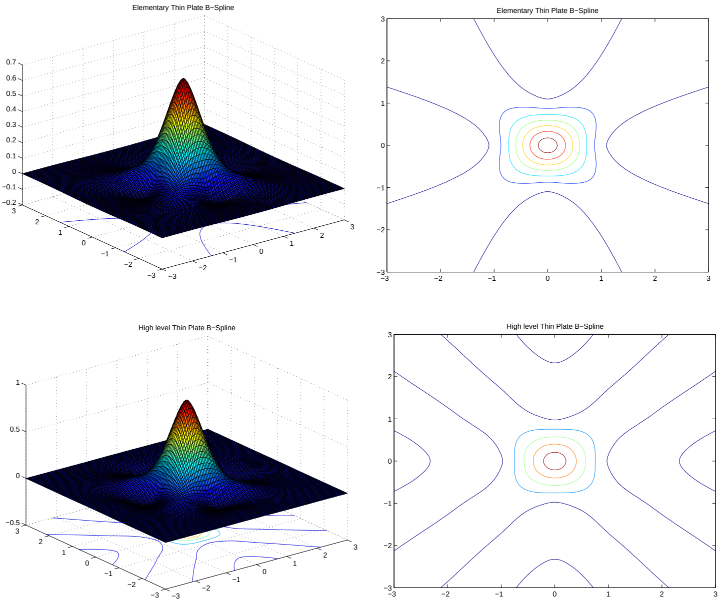
Vanishing moments : $\ell \leq m-1 \Rightarrow \int_{\boldsymbol{R}^d} \|x\|^{2\ell} B^\ell_m(x)\, dx = 0$

$\min_{x\in R^d} B(x) \simeq -.036$; $\min_{x\in R^d} B^1_2(x) \simeq -.09$; $\max_{x\in R^d} B(x) \simeq .66$; $\max_{x\in R^d} B^1_2(x) \simeq .9$

Levels 1 and 2 quintic B-splines (m=3)



Elementary and level 1 Thin Plate B-spline



B-spline approximation on a cardinal grid

Definition: $f \in \mathcal{F}(h\mathbb{Z}^d, \boldsymbol{R})$

$A_h(f) := \sum_{j\in\mathbb{Z}^d} f(jh)\, B_h(\blacksquare - jh)$; $A_h^\ell(f) := \sum_{j\in\mathbb{Z}^d} f(jh)\, B_h^1(\blacksquare - jh)$

$A_h(y) := \sum_{j\in\mathbb{Z}^d} y_j\, B_h(\blacksquare - jh)$; $A_h^\ell(f) := \sum_{j\in\mathbb{Z}^d} y_j\, B_h^1(\blacksquare - jh)$

General shape of $A_h f$ and $A_h^1 f$

A_h is $\boldsymbol{P}_1$-reproducing : $\forall f \in \boldsymbol{P}_1$, $A_h f = f$

$\ell \leq m-1 \Rightarrow A_h^\ell$ is $\boldsymbol{P}_{2\ell+1}$-reproducing : $\forall f \in \boldsymbol{P}_{2\ell+1} A_h^\ell f = f$,

$A_h f$ and $A_h^\ell f$ follow the general form of $(jh\,, f(jh))_{j\in\mathbb{Z}^d}$

Convergence: Let $f \in \mathcal{C}^\ell(\boldsymbol{R}^d)$, then:

$\|A_h f - f\|_{\infty, R^d} \underset{h\to 0}{=\!=\!=} \mathcal{O}(h^{\inf(d+2)})$;

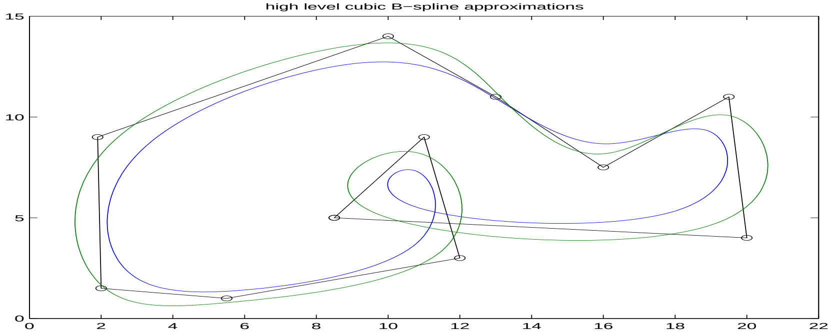
$\ell \leq m-1 \Rightarrow \|A_h^\ell f - f\|_{\infty, R^d} \underset{h\to 0}{=\!=\!=} \mathcal{O}(h^{\inf(2\ell+d)})$

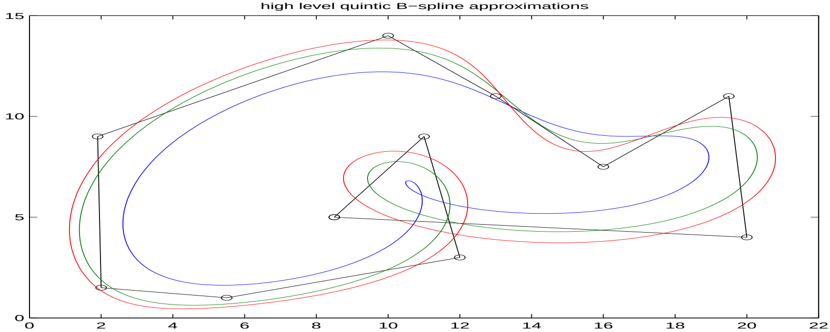
Same order of convergence for a finite sum, further than ε from the boundary, for any ε.

Consequence: $A_h^1 f$ is closer to f than $A_h f$
A_h^ℓ is closer to y than $A_h^{\ell-1} y$

Remark: Convergence rate is all the bigger that the negative part is more important

High level cubic and quintic B-spline curves





Please remember from this second part

By applying a high order discretization of a differential operator (as $\mathrm{D}^4, \mathrm{D}^{2m}, \Delta^{2m}$...) to the fundamental solution of it, we obtain "better" "B-splines" (not with minimal support) :

Higher rate of convergence of $\sum_{j \in \mathbb{Z}^d} f(jh)\, B_h(\bullet - jh)$ towards f.

Reproducing polynomial of higher degree

Higher rate of decay when $\|x\| \longrightarrow \infty$; vanishing moments

$\sum_{j \in \mathbb{Z}^d} P_j\, B_h^k(\bullet - jh)$ is closer to the control points $(P_j)_{j \in \mathbb{Z}}$ (or $(P_j)_{j=1..n}$).

Besides

All B-splines are positive defined functions,
and form pdf-partitions of unity.

Their Fourier transform is a quite simple one

Part 3

Appeal for

Positive definite partitions of unity

Use high order B-splines instead of elementary B-splines

In order the a_j in $\sum_{j\in\mathbb{Z}^d} a_j\, B_h^k(\,\centerdot - jh)$ are closer to the obtain function

In order the obtained curve $\sum_{j\in\mathbb{Z}^d} P_j\, B_h^k(\,\centerdot - jh)$ is closer to the control points

In order to accelerate convergence of any iterative method

Remarks

The Riesz relation guaranties stability

It is still possible to use Fourier transform, and so to work with fractional order splines... and so BIG researchers can use them!

Use positive definite partitions of unity instead of positive ones

"Positive definite" means, in some way "essentially positive", or "the negative parts are of small importance", and...

In many situations, having (small) negative part is a bit counter-intuitive, but it often gives better results:

Wavelets with various vanishing moments are usually more efficient than those with a small number of them.

The negative part, if not "too important", gives the way to phenomenons such as Gibbs phenomenon, which frequently need to be modellised.

My guess is that the "positive definite" condition guaranties the stability of the process. But, sorry, I did not prove it

As a conclusion...

Try and use "high level B-splines", it's high level fun (and high level efficiency)!

Forget the implicit word "positive" in "partition of unity", by replacing it by the explicit words "positive definite"

Thanks to Michael, Dimitri and all others for the various uses of (fractional order) splines, vectorial splines and more...

Thanks to the organizers of this very special day...
and of Chamonix trip!

Enjoy the end of the day, enjoy the coming end of the week...

Enjoy your life!

High level Cubic and Quintic B-spline curves